Existence and Extension of Invariant Feature Associated with Self-Similar Patterns

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Abstract: A computational scheme is presented for detecting invariant feature associated with self-similar patterns. The patterns to be detected are assumed to be generated through successive application of a set of contraction mappings. By identifying the mappings with generated attractors, a conditional probability is represented on a diffusion system that yields a discrete invariant set. This invariant set can be expanded to generate associative attractors. The detection scheme is verified through computer simulation.

Keywords: Invariant Feature; Self-Similarity; Conditional Distribution; Imaging Process; Pattern Coding;

1. Introductory Remarks

Despite continuous deformation, visual information is articulated into a set of "alphabet of patterns" maybe consisting of simple figures [3] or fractal patterns [9]. The articulation of imagery plays a crucial role in various decision processes including interactive design [2], [5] and propositional scene analysis [6], [10]. However, the image articulation is intensively computational process. In articulating observed imagery, the alphabet should be defined based on the pattern to be generated. By this self-reference, articulation process yields explosion of potential decisions. Deterministic scheme enumerates this potential explosion through successive search. A practical approach to image articulation, thus, is the introduction of non-determinism.

Let a "program" be coded in terms of mappings \( \mu_i \), \( i=1,2,3,... \) from a bounded image plane \( \Omega \subset \mathbb{R}^2 \) into itself. By this program, a sequence of patterns \( \{ \Xi_t \} \) is specified through explo rative expansion:

\[
\omega_{t+1} \in \cup_{i=1}^m \mu_i(\Xi_t), \Xi_1 = (\omega_0, \in \Omega, t=1).
\]

The imaging process (1a) well generates ordered-by-results patterns, if exist. In this paper, the existence of computable information is presented for identifying the imaging process (1) through probabilistic evaluation.

2. Stochastics in Morphological Computation Process

Let the class of "visibles" be restricted to \( \mathcal{F} = \mathcal{B}(\Omega) \), the Borel field of subsets of \( \Omega \), and consider the imaging process (1) as an \( \Omega \)-valued dynamical system. This imaging dynamics generates a "cascade of exploration" \( \omega_{j+1} = \mu_i(\omega_j), \omega_j, \omega_{j+1} \in \Omega \) where \( \{ \mu_i \} \) denotes the totality of \( \Omega \rightarrow \Omega \) mappings (Fig. 1(a)). This cascade model well generates the sequence \( \{ \omega_j \in \Omega \} \) through a Monte Carlo simulation with given random sequence \( \mu^t \in \{ \mu_i \}, t=1,2,... \). Following this cascade model, however, the mapping sequence \( \{ \mu^t \} \omega_{j+1} = \mu^t(\omega_j), \omega_j, \omega_{j+1} \in \Lambda \) is not associative with \( \mathcal{F} \)-event \( \Lambda \in \mathcal{F} \). This implies that the imaging process is not \( \mathcal{F} \)-measurable as a fixed mapping sequence. Thus, the imaging process can not be coded in terms of the mapping sequence.

For overcoming this coding difficulty, consider the following non-deterministic representation of imaging process:

\[
\begin{align*}
\{ \mu^t \} : & \mathcal{F}(\omega_t^1 = \mu^1(\omega_0), \omega_0, \omega_{t+1} \in \Lambda \\
= & \{ \omega_{t+1}^1 = \mu^t(\omega_t^1), \omega_t^1, \omega_{t+1} \in \Lambda \},
\end{align*}
\]

where \( \nu \) is a fixed set of contraction mappings, \( \mathcal{M} \). In representation (2), the decision \( \{ \omega_{t+1}^1 = \mu^t(\omega_t^1), \omega_t^1, \omega_{t+1} \in \Lambda \} \) simulates an "avalanche of exploration" in image field (Fig.1(b)). If the imaging process (1) converges to an \( \mathcal{F} \)-measurable attractor, the representation (2) well specifies an \( \mathcal{F} \)-measurable event \( \nu \). Let \( \mathcal{M} \) be the totality of contraction mappings \( \Omega \rightarrow \Omega \) and define \( \mathcal{N} = \mathcal{B}(\mathcal{M}) \). Noticing that a fixed set of contraction mappings generates an attractor [4], the representation (2) of the imaging process (1) can be non-deterministically coded on the alphabet \( \mathcal{M} \). Thus, by restricting the "programming language" \( \{ \mu_i \} \) into the class \( \mathcal{M} \), we can identify generated image \( \Lambda \) with uniquely generated attractor \( \Xi \) of the contraction code \( \nu = \{ \mu_i \} \), i.e.,

\[
\Lambda = \Xi = \lim_{t \to \infty} \Xi_t.
\]

This implies that the generation rule (1) induces an \( \mathcal{F} \)-measurable coding system \( \mathcal{M} \), \( \mathcal{N} \) through the following equivalence:

\[
\Lambda \in \mathcal{F} \Rightarrow \Xi \in \mathcal{F} \Leftrightarrow \nu \in \mathcal{N}.
\]

Define \( dP(\omega) = d\omega / \int_\Omega d\omega \) and consider the evaluation of the program code \( \nu \) on the probability space \( (\Omega, \mathcal{F}, P) \). If a pattern \( \Lambda \)
has smooth boundary, the information conveyed by a pattern $\Lambda$ can be described by the following distributed parameter system:

$$\Delta \varphi(x) = p \varphi(x), x \in (\lambda - \partial \Lambda), \quad (5a)$$

$$\varphi(x) = 0, x \in \partial \Lambda \lambda, \quad (5b)$$

$$\varphi(x') = 1, x' \in \partial \Lambda, \quad (5c)$$

where $p$ is a positive constant. In this system, observed contour pattern specifies the boundary condition (5c) and diffuses inwards the domain through local interaction (5a). Since

$$\Delta \varphi(x) \geq 0, \text{a.e. in } \Omega,$$

for $p > 0$, the distribution $\varphi(x)$ is subharmonic in $\Lambda$, i.e.,

$$0 \leq \varphi(\omega \Lambda) \leq \varphi(\omega e) < \infty, \text{a.e.} \quad (6)$$

Thus, in the context of image analysis, the distribution $\varphi(x)$ yields a probabilistic evaluation of a "pixel" conditioned by the boundary $\partial \Lambda$ [7]. Furthermore, the distributed parameter system (5) yields a point set $\tilde{\Theta} = \{\theta e \in \lambda | \varphi(\theta) = 0, \varphi(\theta) > 0\}$, designated by feature pattern. Since $\varphi(x) = 0$, a.e. in $\Lambda$, the feature pattern is discrete and, by definition, is uniquely determined in terms of the domain $\Lambda$. This implies that an $\mathcal{H}$ measurable imaging process for generating visibles in $\mathcal{H}$, if exists, has a $\mathcal{H}$ measurable representation.

Then, for observables $\tilde{\Theta}$ well defined by

$$\tilde{\Theta} = \{\theta e \in \lambda | \varphi(\theta) = 0, \varphi(\theta) > 0\},$$

we can specify the following

Existence Theorem Suppose that $\Xi$ is the attractor of a contraction code $e \mathcal{H}$ and assume that the associated smoothed region $\Xi$ yields the feature pattern $\tilde{\Theta}$ satisfying the following condition:

$$\tilde{\Theta} \cap (\Xi - \partial \Xi) \neq \emptyset,$$

(12a)

$$\bigg[ \bigcup_{\mu \in \mu(\tilde{\Theta})} \Theta (\Xi - \partial \Xi) \bigg] \cap (\Xi - \partial \Xi) = \emptyset.$$  

Then there exists a subset of $\tilde{\Theta}$, designated by the invariant feature, satisfying the following condition:

$$\Theta = \{\theta e \in \lambda | \bigcup_{\nu} \mu(\Theta) \delta \}, \quad (13)$$

for some constant $\delta > 0$. In Eq. (13), $\mu(\Theta) \delta$ denotes the region in $\Omega$ given by

$$\bigcup_{\mu(\Theta) \delta} = \bigcup_{\Theta e} \bigcup_{\theta e} \theta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta 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The distributed parameter system (5) well generates feature pattern \( \Theta \) with no a priori information on the imaging process of the patterns \( \Lambda \). Suppose that the feature pattern \( \Theta \) is observed so as to yield an invariant subset with respect to a contraction code \( \nu \). Then the convergence of successive contraction process implies that the discrete information \( \Theta \) can be expanded to an associated attractor. For this imaging process, a priori estimate of this associative pattern is given in the following

Extension Theorem Suppose that the observation \( \Theta = \{ \Theta \in \Lambda | \varphi(\Theta) = 0, \varphi(\Theta) > 0 \} \) is sufficient in the following sense:

\[
\{ \nu(\Theta) \} \cap \Omega \neq \emptyset.
\]  

(15)

Then, the attractor \( \Sigma \) generated by a contraction code \( \nu = \{ \mu \in M \} \) is associative in \( \Omega \), i.e., there exists an open set \( \Lambda_\nu \) satisfying the following condition:

\[
\Lambda_\nu \cap \Omega \neq \emptyset.
\]  

(16a)

\[
\nu(\Sigma) = \Sigma \subset \Lambda_\nu.
\]  

(16b)

Remarks By the self-reference, the expansion of the attractor should be determined at the completion of imaging process. This implies that the verification of detected code through the invariant test requires considerable computation. This expansion theorem asserts the location of invariant feature yields an effective guess for the distribution of attractor points. For instance, the expansion of invariant feature provides the minimal size of expected attractor.

Fig. 3 Associative Extension

4. Continuation Transformation of Feature Patterns
By "a system of coordinate transforms" \( \nu = \{ \mu \in M \} \), the observable \( \Theta \) is transferred in a neighborhood of \( \Theta \) in \( \Omega \). This suggests that the domain transformation \( \nu(\Lambda_\nu(\Theta)) \to \Sigma \) yields the range \( \nu(\Theta) \) covering \( \Theta \). However, the distribution \( \varphi(\omega) \) conditioned by the boundary \( \partial \mu(\Sigma^\mu) \) is not identical with the restriction of \( \varphi(\omega) \) into the domain \( \mu(\Sigma^\mu) \) due to the disparity of boundary condition (Fig.4). In this section, the existence of the trajectory of observables \( \Theta \) under a "boundary preserving contraction process" is investigated.

First, we notice the continuity of contraction mapping, i.e.,

Proposition 1 Suppose that \( \mu \) is a contraction mapping of \( \Omega \) into itself. Then, for any \( \delta > 0 \), we have

\[
\mu[B_\delta(\omega)] \subset B_\delta(\mu(\omega)),
\]  

(17)

where \( B_\delta(\omega) \) is a closed ball around of radius \( \delta \).

Consider a simple mapping in \( \Sigma^\mu \). Noting that the continuity of function \( \varphi \) is preserved by continuous mapping, by proposition 1, we have

Proposition 2 Let \( \mu \) be a contraction mapping of \( \Sigma^\mu \) into itself and suppose that \( \Theta_\mu \) is feature points associated with reduced region \( \Sigma_\mu = \mu(\Sigma^\mu) \), i.e.,

\[
\Theta_\mu = \{ \Theta \in \Sigma_\mu | \varphi(\Theta) = 0, \varphi(\Theta) > 0 \},
\]  

(18)

\[
\Delta \varphi(\mu(\omega)) = p \varphi(\mu(\omega)), \omega \in \Sigma_\mu.
\]  

(19a)

\[
\varphi(\mu(\omega)) = 0, \omega \in \Omega \cup \{ \mu(\Sigma^\mu) \},
\]  

(19b)

\[
\varphi(\mu(\omega)) = 1, \omega' \in \mu(\Omega) \setminus \mu(\Sigma^\mu).
\]  

(19c)

Then

\[
\varphi(\mu(\omega)) = 0, \text{ for any } \Theta \in \Theta_\Theta.
\]  

(20)

For tracing the trajectory under boundary preserving contractions, define \( \partial = \mu(\partial \Sigma^\mu) \) and consider the solution \( \{ \varphi_i | i=1,2,\ldots \} \) to the following system:

\[
\Delta \varphi_i(\omega) = p \varphi_i(\omega), \omega \in \Sigma_\mu.
\]  

(21a)

\[
\varphi_i(\omega) = 0, \omega \in \Omega \cup \{ \mu(\Sigma^\mu) \},
\]  

(21b)

\[
\varphi_i(\omega) = 1, \omega' \in \partial_i.
\]  

(21c)

For this system, let the feature distribution \( \nu(\Theta_\Theta) \) be transferred to the feature pattern associated with a "boundary preserving map" through continuation process (Fig.5). In this process, first, the domain is transferred with associated boundary, as in the sys-
tem (19), then mapped boundary is vanished smoothly. This continuation process is described by the following one parameter group of distributed parametersystem:

\[
\Delta \phi^\alpha(\omega) = \rho \phi^\alpha(\omega), \quad \omega \in \mathbb{E}^\alpha - \mathbb{E}^1,
\]

(22a)

\[
\phi^\alpha(\omega) = (1-\alpha) + \alpha \phi(\omega^0), \quad \omega^0 \in \mathbb{E}^1, \quad \mathbb{E}^\alpha \subset \mathbb{E}^1,
\]

(22b)

\[
\phi^\alpha(\omega) = 0, \quad \omega \in \Omega - \left[ \mathbb{E}^\alpha \cup \mathbb{E}^0 \right],
\]

(22c)

\[
\phi^\alpha(\omega') = 1, \quad \omega' \in \mathbb{E}^0,
\]

(22d)

where \(\omega \in [0,1]\). Obviously the solution \(\phi^\alpha\) is a homotopy, i.e.,

\[
\phi^\alpha(\omega): \phi(\omega) \rightarrow \phi(\omega) \text{ for } \alpha:0 \rightarrow 1 \text{ and } \omega \in \mu_1(\mathbb{E}^1).
\]

(23)

By the continuity of the solution \(\phi^\alpha\) with respect to boundary value \(\phi^\alpha(\omega^0)\), the variation of the gradient field \(\nabla \phi^\alpha\) is evaluated as follows:

\[
\frac{dV \phi^\alpha}{d\alpha} = \frac{\partial \nabla \phi^\alpha}{\partial \alpha} + \left[ \Delta \phi^\alpha \right] \frac{d\phi^\alpha}{d\alpha},
\]

where \(\frac{d\phi^\alpha}{d\alpha} : \nabla \phi^\alpha(\omega^0) = 0 \text{ and } \phi^\alpha(\omega^0) > 0\).

(24)

By letting

\[
\frac{dV \phi^\alpha}{d\alpha} = 0, \text{ at } \omega^0 \in \mathbb{E}^1 \subset \mathbb{E}^1,
\]

(25)

we have a formal description of a curve \(Q\) that governs the movement of the feature point \(\theta^\alpha\) as follows:

\[
Q: \frac{d\theta^\alpha}{d\alpha} = \left(-\frac{1}{\rho \phi^\alpha}\right) \nabla \phi^\alpha, \quad \text{for } 0 \leq \alpha \leq 1.
\]

(26)

Noticing the existence of unique curve \(Q\), we have the following

Proposition 3 The feature point \(\theta^\alpha\), \(0 \leq \alpha \leq 1\), lies in a curve \(Q\).

Proof: From Eq. (27), it follows that

\[
\left| \frac{d\theta^\alpha - \theta^0}{d\alpha} \right| \leq \frac{1}{\rho \phi^\alpha(\omega) \inf_{\omega \in \mu_1(\mathbb{E})} \phi^\alpha(\omega)}.
\]

(27)

(32)

Since \(\phi^\alpha(\omega)\) is positive definite, i.e.,

\[
\inf_{\omega \in \mu_1(\mathbb{E})} \phi^\alpha(\omega) \geq \frac{1}{C_0},
\]

(34)

for some constant \(C_0\), it follows that

\[
\left| \frac{d\theta^\alpha - \theta^0}{d\alpha} \right| \leq \frac{C_0}{\rho \inf_{\omega \in \mu_1(\mathbb{E})} \phi^\alpha(\omega)} \sup_{\omega \in \mu_1(\mathbb{E})} \left| \nabla \phi^\alpha(\omega) - \nabla \phi(\omega^0) \right|.
\]

(35)

Noticing that the solution to the elliptic equation is infinitely differentiable and continuously dependent on the boundary value, we have

\[
\sup_{\omega \in \partial \mu_1(\mathbb{E})} \left| \nabla \phi^\alpha(\omega) - \nabla \phi(\omega^0) \right| \leq C \sup_{\omega \in \partial \mu_1(\mathbb{E})} \left| 1 - \phi(\omega^0) \right|.
\]

(36a)

\[
Q \subset \mathbb{E}^\alpha \neq \emptyset.
\]

(29)

Define

\[
\bar{\theta}^\alpha = \{ \theta^\alpha \in \mathbb{E}^\alpha \cap Q \mid \nabla \phi^\alpha = 0, \phi^\alpha > 0 \},
\]

(30)

Then, by the maximality of the feature pattern \(\bar{\theta}^\alpha\), i.e.,

\[
\nabla \phi^\alpha \neq 0 \text{ for any } \omega \in \Omega - \bar{\theta}^\alpha.
\]

(31)

we have \(\bar{\theta}^\alpha \in Q\) and \(\bar{\theta}^\alpha \in \Omega - Q\), as was to be proved.
\[ |1 - \varphi(\omega^n)| \leq C_2 \inf_{\omega' \in \partial \Omega} |\omega' - \omega^n|, \quad \omega' \in \partial \Omega \] (36b)

for constants \( C_1 \) and \( C_2 \). Noticing
\[ \inf_{\omega' \in \partial \Omega} |\omega' - \omega^n| < \varepsilon, \] (37)

for arbitrary \( \omega' \in \partial \Omega \), the combination of (33), (35), and (36) yields the following evaluation
\[ |q_1 - q_0| \leq \frac{C}{\rho} C_1 C_2 \varepsilon, \] (38)
as was to be proved.

5. Proof of Theorems
5.1 Proof of Existence Theorem

Define \( \Theta = \bar{\Omega} \cap (\varepsilon, \varepsilon) \) and apply the \( v-Q \) transformation specified by the mappings \( v \) and homotopic path \( Q \). By assumption (12a) \( \Theta \) is non-empty and by definition of attractor we have
\[ \mu_1(\Theta) \subset \Xi. \] (40)

Furthermore, by proposition 2 combined with proposition 4, the mapping image \( \mu_1(\Theta) \) generates a set of feature points, if exists, in associated \( \varepsilon \)-neighbourhood
\[ \varepsilon B(\mu_1(\Theta)) = \{ \omega \in \varepsilon \mid |\omega - \theta| < \varepsilon, \theta \in \varepsilon B(\mu(\Theta)) \}. \] (41)
The existence of non-vanishing homotopic points is ensured by the self-similarity of the domain \( \Xi \): under the separation condition (12b), non-empty set \( \Theta \subset \Xi \) must be generated via this \( v-Q \) process. In other words, for each feature point \( \theta \in \Theta \), there exists a non-vanishing end point of homotopic path \( \Theta \in \Theta \), such that \( \theta \in \varepsilon B(\mu_1(\Theta)) \). The remainder of this proof is devoted to \( \varphi \)-evaluation of \( \varepsilon \)-neighbourhood. Since \( \Delta \mu(\omega) > 0 \) for arbitrary \( \omega \in \varepsilon \), the region \( \Theta > 0 \) is non-empty for any \( \delta > 0 \) and \( \delta \in \Theta \). Let \( \delta \) be so small that
\[ \Theta > 0 \] (43)
for any \( \bar{\Theta} \subset \Theta \) and select the smoothing parameter \( \varepsilon > 0 \) so that the following condition is satisfied:
\[ \varepsilon B(\mu_1(\Theta)) \subset \Theta > 0, \] (42)
for arbitrary \( \bar{\Theta} \subset \Theta \). For such "slightly smoothed pattern" \( \varepsilon \), associated feature points \( \Theta \) is transferred into \( \varepsilon B(\mu_1(\Theta)) \) via the \( v-Q \) process. In addition, the combination of the expansion estimation (42) with the condition (12b) implies the separation
\[ \varepsilon B(\Theta_1) \cap \varepsilon B(\Theta_2) = \emptyset. \] (44)

This separation combined with the regenerativity of the set \( \Theta \) yields that
\[ \Theta \subset \varepsilon B(\mu_1(\Theta)) \] (45)
as was to be proved.

5.2 Proof of Extension Theorem

Let \( \omega^* \) be the fixed point associated with the mapping \( \mu_1 \in v \), i.e., \( \mu_1(\omega^*) = \omega^* \), and select
\[ \tilde{\mu}_v = \tilde{\mu}(\mu_1), \] (46a)

where
\[ \tilde{\mu}_v = \{ \omega \in \Omega \mid |\omega - \alpha^*| \leq \max_{\xi \in A} |\alpha^*|, \alpha^* \} \subset \mu_1 \in v \}. \] (46b)

Then
\[ \Lambda = \mu_1 \subset \tilde{\mu}_v, \] (47)

by definition. Define
\[ \tilde{\Xi}_{t+1} = \left[ \mu_1(\tilde{\Xi}_t) \right]. \] (48a)
\[ \tilde{\Xi}_0 = \left[ \mu_1(\tilde{\Xi}_0) \right], \] (48b)

Noticing that
\[ \tilde{\Xi}_1 = \{ \mu_1(\omega) \in \Omega \mid |\omega - \omega^*| \leq \max_{\xi \in A} |\alpha^*|, \alpha^* \} \subset \mu_1 \in v \}. \] (49a)
\[ \tilde{\Xi}_2 = \{ \mu_1(\omega) \in \Omega \mid |\omega - \omega^*| \leq \max_{\Xi} |\alpha^*|, \alpha^* \} \subset \mu_1 \in v \}. \] (49b)

we have monotone sequence \( \tilde{\Xi}_{t+1} \), \( t=0,1,2,... \), satisfying the following condition
\[ \left[ \mu_1(\tilde{\Xi}_t) \right] \cap \Theta \subset \tilde{\Xi}_t \subset \tilde{\Xi}_0 \] (50)

Define
\[ \Xi = \lim_{t \to \infty} \tilde{\Xi}_t. \] (51)

Then the uniqueness of limit point yields
\[ \Xi = \mu_1(\tilde{\Xi}_0). \] (52)

In addition, by upper-lower evaluation (50), we have
\[ \Xi \cap \tilde{\Xi}_0 = \emptyset, \] (53)
and
The coding scheme was verified through simulation studies. Following to patterns to be observed are generated through a Monte Carlo simulation based on illustrated in Fig.7. These dictionary patterns are generated based on the set of contraction mappings, the alphabet of the following form:

\[
\begin{align*}
\{( & \text{REDUCTION reduction} \\ & \text{ROTATION rotation} \\ & \text{FLIP flip-horizontal} \\ & \text{(U horizontal-shift) (V vertical-shift)} \}.
\end{align*}
\]

as was to be proved.

6. Experiments
The coding scheme was verified through simulation studies. Patterns to be observed are generated through a Monte Carlo simulation [1] and identified within a preassigned set of attractors, called the dictionary. An example of observed pattern, called "Barnsley's Fern", is shown in Fig.6 and the dictionary is illustrated in Fig.7. These dictionary patterns are generated based on the set of contraction mappings, the alphabet of the following form:

\[
\begin{align*}
\{( & \text{REDUCTION reduction} \\ & \text{ROTATION rotation} \\ & \text{FLIP flip-horizontal} \\ & \text{(U horizontal-shift) (V vertical-shift)} \}.
\end{align*}
\]

Fig.6 Pattern to be Detected

patterns to be observed are generated through a Monte Carlo simulation based on illustrated in Fig.7. These dictionary patterns are generated based on the set of contraction mappings, the alphabet of the following form:

\[
\begin{align*}
\{( & \text{REDUCTION reduction} \\ & \text{ROTATION rotation} \\ & \text{FLIP flip-horizontal} \\ & \text{(U horizontal-shift) (V vertical-shift)} \}.
\end{align*}
\]

as was to be proved.

6. Experiments
The coding scheme was verified through simulation studies. Patterns to be observed are generated through a Monte Carlo simulation based on illustrated in Fig.7. These dictionary patterns are generated based on the set of contraction mappings, the alphabet of the following form:

\[
\begin{align*}
\{( & \text{REDUCTION reduction} \\ & \text{ROTATION rotation} \\ & \text{FLIP flip-horizontal} \\ & \text{(U horizontal-shift) (V vertical-shift)} \}.
\end{align*}
\]

as was to be proved.

7. Concluding Remarks
A computational scheme was presented for detecting invariant feature associated with self-similar patterns. The imaging process is shown to be identified with generated attractors in a stochastic sense and an invariant feature is extracted in associated conditional distribution. This invariant feature can be expanded to generate associative attractors.

References