A Computational Method of Linearization of Nonlinear Systems by Cubic Hermite Interpolation and a Nonlinear Filter

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Abstract

A computational method of the formal linearization using a piecewise cubic Hermite interpolation is proposed. We introduce a linearizing function that consists of the state variables, their squares, and the cubes. The nonlinear terms are approximated by the method of the piecewise cubic Hermite interpolation and linearized with respect to the linearizing function. Numerical computations are easily carried out with the aid of computers. A nonlinear filter is synthesized as an application of this method and is verified through numerical examples.

1. Introduction

Practical system in engineering world are all inherently nonlinear process. They are most naturally described by nonlinear differential equations. The tools of linear analysis are quite well developed, while those being able to deal with nonlinear phenomena have been feeble and a few. So one wishes to have linearizing techniques of nonlinear systems so as to apply the linear theories or tools. For the last few decades, this problem has been studied using geometric methods by many researchers (e.g.⁽¹⁾). Though many interesting results have been developed, they are, in general, not so easily applicable to practical systems. Therefore, it is necessary to study how to compute the linearization that enables easy implementation^(2, 3).

In this paper we develop a numerical computational method of the formal linearization for nonlinear systems⁽²⁾. A linearizing function consists of a sequence of the terms of the cubic polynomial, i.e. state variables, their squares and cubes, thus the dimension of the linearizing function is determined automatically. The nonlinear terms are approximated by the cubic Hermite interpolation and linearized with respect to the linearizing function. As an application of this linearization, a nonlinear filter algorithm is synthesized. With the aid of computers, we easily carry out the

numerical computation of transformation from a nonlinear system to a formal linear system. Results of the numerical examples of a initial value problem of differential equation and a state estimation problem show the validity of this linearization method.

2. Formal Linearization

2.1 Scalar System

Assume that a nonlinear scalar system is given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad (1)$$

where $x(t)$ is a state variable, the domain of $x(t)$ is $D = [\mu, \mu + \nu) \subset R (\mu \in R, \nu > 0)$, and f is a smooth nonlinear function. As a linearizing function, we introduce

$$\phi(x) = [x, x^2, x^3]^T. \quad (2)$$

Differentiating $\phi(x)$ with respect to t , we have

$$\begin{aligned} \dot{\phi}(x) &= \frac{d\phi}{dx} \dot{x} = [1, 2x, 3x^2]^T f(x) \\ &\equiv [g_1(x), g_2(x), g_3(x)]^T. \end{aligned} \quad (3)$$

We approximate each element function $g_k(x)$ of Eq.(3) by the cubic Hermite interpolation. We divide the domain D into m subdomains appropriately with considering the nonlinearity of function $f(x)$. Let D_i be the i th subdomain:

$$D_i = [\alpha_i, \alpha_i + \delta_i], (\delta_i > 0), \quad (4)$$

where $D = \bigcup_{i=1}^m D_i$. When $x \in D_i$, let $h_{ik}(x)$ be a cubic Hermite interpolation of $g_k(x)$:

$$\begin{aligned} g_k(x) \approx h_{ik}(x) &= \sum_{r=0}^3 c_{ikr} x^r \\ &= [c_{ik1}, c_{ik2}, c_{ik3}] \phi(x) + c_{ik0}. \end{aligned} \quad (5)$$

Cubic polynomial $h_{ik}(x)$ must satisfy the conditions:

$$\begin{aligned} h_{ik}(\alpha_i) &= g_k(\alpha_i), \quad h_{ik}(\alpha_i + \delta_i) = g_k(\alpha_i + \delta_i), \\ h'_{ik}(\alpha_i) &= g'_k(\alpha_i), \quad h'_{ik}(\alpha_i + \delta_i) = g'_k(\alpha_i + \delta_i), \end{aligned} \quad (6)$$

where $' \equiv \frac{d}{dx}$. Since the number of the coefficients of Eq.(5) is 4, the same as the number of the conditions of Eqs.(6), $h_{ik}(x)$ is uniquely determined by

$$\begin{bmatrix} c_{in0} \\ c_{in1} \\ c_{in2} \\ c_{in3} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_i/\delta_i & \alpha_i^2/\delta_i^2 & -\alpha_i^3/\delta_i^3 \\ 0 & 1/\delta_i & -2\alpha_i/\delta_i^2 & 3\alpha_i^2/\delta_i^3 \\ 0 & 0 & 1/\delta_i^2 & -3\alpha_i/\delta_i^3 \\ 0 & 0 & 0 & 1/\delta_i^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_k(\alpha_i) \\ g_k(\alpha_i + \delta_i) \\ \delta_i g'_k(\alpha_i) \\ \delta_i g'_k(\alpha_i + \delta_i) \end{bmatrix}. \quad (7)$$

Applying Eq.(5) to each function $g_k(x)$, $k = 1, 2, 3$, the right-hand side of Eq.(3) is expressed in approximated linear form with respect to $\phi(x)$:

$$\dot{\phi}(x) \approx A_i \phi(x) + b_i, \quad (8)$$

where

$$A_i = \begin{bmatrix} c_{i11} & c_{i12} & c_{i13} \\ c_{i21} & c_{i22} & c_{i23} \\ c_{i31} & c_{i32} & c_{i33} \end{bmatrix}, \quad b_i = \begin{bmatrix} c_{i10} \\ c_{i20} \\ c_{i30} \end{bmatrix}.$$

Thus Eq.(1) is converted into the formal linear system:

$$\dot{z}(t) = A_i z(t) + b_i, \quad z(0) = \phi(x(0)). \quad (9)$$

The inversion is obtained by Eq.(2) as follows

$$x = [1, 0, 0] \phi(x). \quad (10)$$

Therefore, the approximated value of $x(t)$ through the linearization is obtained by

$$\hat{x}(t) = [1, 0, 0] z(t). \quad (11)$$

2.2. Multi-dimensional System

Assume that a nonlinear system is given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad (12)$$

where $x = [x_1, \dots, x_n]^T$ is an n state vector, the domain of x is $D = \prod_{i=1}^n [p_i, p_i + q_i] \subset R^n (p_i \in R, q_i > 0)$, $f = [f_1, \dots, f_n]^T$ is a nonlinear vector function. As a linearizing function, we introduce

$$\begin{aligned} \phi(x) &= [T_{(100\dots 0)}(x), T_{(200\dots 0)}(x), T_{(300\dots 0)}(x), \\ &\quad T_{(010\dots 0)}(x), T_{(110\dots 00)}(x), \dots, \\ &\quad T_{(r_1 r_2 r_3 \dots r_n)}(x), \dots, T_{(333\dots 3)}(x)]^T, \end{aligned} \quad (13)$$

where

$$\begin{aligned} T_{(r_1 r_2 \dots r_n)}(x) &= \prod_{k=1}^n x_k^{r_k}, \\ r_k &= 0, 1, 2, 3 \quad (r_1 + \dots + r_n \neq 0). \end{aligned} \quad (14)$$

Differentiating $\phi(x)$ with respect to t , we have

$$\begin{aligned} \dot{\phi}(x) &= \frac{\partial \phi}{\partial x^T} \dot{x} = \frac{\partial \phi}{\partial x^T} f(x) \\ &\equiv [g_1(x), \dots, g_{4^n-1}(x)]^T \equiv G(x). \end{aligned} \quad (15)$$

We approximate each element function $g_k(x)$ of Eq.(15) by the cubic Hermite interpolation. We divide the domain D into m subdomains appropriately with considering the nonlinearity of function $f(x)$. Let D_i be the i th subdomain:

$$D_i = \prod_{k=1}^n [\alpha_{ik}, \alpha_{ik} + \delta_{ik}], (\delta_{ik} > 0), \quad (16)$$

where $D = \bigcup_{i=1}^m D_i$. When $x \in D_i$, let $h_{ik}(x)$ be a cubic Hermite interpolation of $g_k(x)$:

$$g_k(x) \approx h_{ik}(x) = \sum_{r_n=0}^3 \dots \sum_{r_1=0}^3 c_{ik[r_1 \dots r_n]} \prod_{j=1}^n x_j^{r_j}. \quad (17)$$

Cubic polynomial $h_{ik}(x)$ must satisfy the conditions:

$$h_{ik}^{(p_1 p_2 \dots p_n)}(\alpha) = g_k^{(p_1 p_2 \dots p_n)}(\alpha), \quad (18)$$

where

$$\begin{aligned} \alpha &= [\alpha_{i1} + q_1 \delta_{i1}, \alpha_{i2} + q_2 \delta_{i2}, \dots, \alpha_{in} + q_n \delta_{in}]^T, \quad q_j = 0 \text{ or } 1, \\ p_j &= 0 \text{ or } 1, \quad g_k^{(000\dots 0)}(x) = g_k(x), \quad g_k^{(100\dots 0)}(x) = \frac{\partial}{\partial x_1} g_k(x), \\ g_k^{(010\dots 0)}(x) &= \frac{\partial}{\partial x_2} g_k(x), \dots, g_k^{(110\dots 0)}(x) = \frac{\partial^2}{\partial x_1 \partial x_2} g_k(x), \dots, \\ g_k^{(111\dots 1)}(x) &= \frac{\partial^n}{\partial x_1 \dots \partial x_n} g_k(x). \end{aligned}$$

Since the number of the coefficients of Eq.(17) is 4^n , the same as the number of conditions of Eqs.(18), $h_{ik}(x)$ is uniquely determined by

$$c_i = X_i \Delta_i W^{-1} F_i. \quad (19)$$

c_i is a 4^n dimensional vector. Let c_a be the a th element of c_i , the relation between a and $r_1, \dots, r_n$ of $c_{ik[r_1 \dots r_n]}$ is

$$a = \sum_{j=1}^n (4^{j-1} r_j) + 1, \quad r_j = 0, 1, 2, \text{ or } 3.$$

F_i is a 4^n dimensional vector. Each element of F_i is

$$F_a = \left(\prod_{j=1}^n \delta_{ij}^{p_j} \right) g_k^{(p_1 \dots p_n)} ([\alpha_{i1} + q_1 \delta_{i1}, \dots, \alpha_{in} + q_n \delta_{in}]^T),$$

$$a = \sum_{j=1}^n (2^{n+j-1} p_j + 2^{j-1} q_j) + 1,$$

$$p_j = 0 \text{ or } 1, \quad q_j = 0 \text{ or } 1.$$

W is a $4^n \times 4^n$ dimensional matrix. Each element of W is

$$W_{ab} = \prod_{j=1}^n r_j^{p_j} q_j^{r_j - p_j},$$

$$a = \sum_{j=1}^n (2^{n+j-1} p_j + 2^{j-1} q_j) + 1,$$

$$b = \sum_{j=1}^n (4^{j-1} r_j) + 1,$$

$$p_j = 0 \text{ or } 1, \quad q_j = 0 \text{ or } 1, \quad r_j = 0, 1, 2, \text{ or } 3.$$

Δ_i is a $4^n \times 4^n$ dimensional diagonal matrix. Each diagonal element of Δ_i is

$$\Delta_{aa} = \left(\prod_{j=1}^n \delta_{ij}^{r_j} \right)^{-1},$$

$$a = \sum_{j=1}^n (4^{j-1} r_j) + 1, \quad r_j = 0, 1, 2, \text{ or } 3.$$

X_i is a $4^n \times 4^n$ dimensional matrix:

$$X_i = M(\alpha_{in}) \otimes \dots \otimes M(\alpha_{i2}) \otimes M(\alpha_{i1}),$$

where $\otimes$ denotes tensor product. For any matrix B , $M(y) \otimes B$ is defined by

$$M(y) \otimes B = \begin{bmatrix} B & -yB & y^2B & -y^3B \\ 0 & B & -2yB & 3y^2B \\ 0 & 0 & B & -3yB \\ 0 & 0 & 0 & B \end{bmatrix}.$$

Applying Eq.(17) to each function $g_k(x)$, $1 \leq k \leq 4^n - 1$, the right-hand side of Eq.(15) is expressed in approximated linear form with respect to $\phi(x)$,

$$\dot{\phi}(x) \approx A_i \phi(x) + b_i, \quad (20)$$

where

$$A_i = \begin{bmatrix} c_{i1}[10\dots 0] & c_{i1}[20\dots 0] & \dots & c_{i1}[33\dots 3] \\ c_{i2}[10\dots 0] & c_{i2}[20\dots 0] & \dots & c_{i2}[33\dots 3] \\ \vdots & \vdots & \ddots & \vdots \\ c_{i4^n-1}[10\dots 0] & c_{i4^n-1}[20\dots 0] & \dots & c_{i4^n-1}[33\dots 3] \end{bmatrix},$$

$$b_i = [c_{i1}[00\dots 0], c_{i2}[00\dots 0], \dots, c_{i4^n-1}[00\dots 0]]^T.$$

Thus Eq.(12) is converted into the linear system

$$\dot{z}(t) = A_i z(t) + b_i, \quad z(0) = \phi(x_0). \quad (21)$$

The inversion is obtained by Eq.(13) as follows

$$x = K \phi(x), \quad (22)$$

where K is an $n \times (4^n - 1)$ dimensional constant matrix and its element k_{ij} is

$$k_{ij} = \begin{cases} 1 & \text{if } j = 4^{i-1} \quad (1 \leq i \leq n), \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

Therefore the approximated value of $x(t)$ through the linearization is obtained by

$$\hat{x}(t) = K z(t). \quad (24)$$

3. Error bound

Let $\|\cdot\|$ be the maximum norm and $g(x)$ be one of element functions of Eq.(15). Here we introduce an operator Ψ_i^j which approximates a function $g(x)$ by the cubic Hermite interpolation with respect to x_j on subdomain D_i such that

$$\Psi_i^j : g(x) \rightarrow \sum_{k=0}^3 c_{ik}(\bar{x}_j) x_j^k, \quad (25)$$

where

$$\begin{bmatrix} c_{i0}(\bar{x}_j) \\ c_{i1}(\bar{x}_j) \\ c_{i2}(\bar{x}_j) \\ c_{i3}(\bar{x}_j) \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_{ij}/\delta_{ij} & \alpha_{ij}^2/\delta_{ij}^2 & -\alpha_{ij}^3/\delta_{ij}^3 \\ 0 & 1/\delta_{ij} & -2\alpha_{ij}/\delta_{ij}^2 & 3\alpha_{ij}^2/\delta_{ij}^3 \\ 0 & 0 & 1/\delta_{ij}^2 & -3\alpha_{ij}/\delta_{ij}^3 \\ 0 & 0 & 0 & 1/\delta_{ij}^3 \end{bmatrix} \begin{bmatrix} g(y_1) \\ g(y_2) \\ \delta_{ij} g_{x_j}(y_1) \\ \delta_{ij} g_{x_j}(y_2) \end{bmatrix},$$

$$\bar{x}_j = [x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n]^T,$$

$$y_1 = [x_1, \dots, x_{j-1}, \alpha_{ij}, x_{j+1}, \dots, x_n]^T,$$

$$y_2 = [x_1, \dots, x_{j-1}, \alpha_{ij} + \delta_{ij}, x_{j+1}, \dots, x_n]^T,$$

$$g_{x_j} = \frac{\partial}{\partial x_j} g.$$

Moreover, a product of operator, $\Psi_i^{[j]}$ is defined as

$$\Psi_i^{[j]} \equiv \Psi_i^j \circ \dots \circ \Psi_i^2 \circ \Psi_i^1, \quad \Psi_i^{[0]} g(x) = g(x). \quad (26)$$

Let $\bigcap_{i=1}^n C^4(x_i)$ be a set of four times continuously differentiable functions with respect to $x_i (i = 1, \dots, n)$. If $f(x) \in \bigcap_{i=1}^n C^4(x_i)$, $g(x)$ also belongs to $\bigcap_{i=1}^n C^4(x_i)$, and an error bound of approximation by the cubic Hermite interpolation with respect to x_j is⁽⁴⁾

$$\sup_{x \in D_i} |g(x) - \Psi_i^j g(x)| \leq \frac{1}{384} \delta_{ij}^4 \sup_{x \in D_i} \left| \frac{\partial^4}{\partial x_j^4} g(x) \right|. \quad (27)$$

First, we have the following lemma about an error bound between $g(x)$ and $\Psi_i^{[n]} g(x)$.

Lemma 1: If $f(x) \in \bigcap_{i=1}^n C^4(x_i)$, the error bound of the cubic Hermite interpolation on subdomain D_i is

$$\sup_{x \in D_i} |g(x) - \Psi_i^{[n]} g(x)| \leq \frac{1}{384} \bar{\delta}_i^4 \sum_{j=1}^n M_{ij}, \quad (28)$$

where $\bar{\delta}_i = \max_{1 \leq j \leq n} \delta_{ij}$, $M_{ij} = \sup_{x \in D_i} \left| \frac{\partial^4}{\partial x_j^4} \Psi_i^{[j-1]} g(x) \right|$.

Proof: The error bound of function approximation on D_i is

$$\begin{aligned} & \sup_{x \in D_i} |g(x) - \Psi_i^{[n]} g(x)| \\ &= \sup_{x \in D_i} |\Psi_i^{[0]} g(x) - \Psi_i^{[n]} g(x)| \\ &\leq \sum_{j=1}^n \sup_{x \in D_i} |\Psi_i^{[j-1]} g(x) - \Psi_i^{[j]} g(x)| \\ &= \sum_{j=1}^n \sup_{x \in D_i} |\Psi_i^{[j-1]} g(x) - \Psi_i^j(\Psi_i^{[j-1]} g(x))|. \end{aligned} \quad (29)$$

From Eq.(27),

$$\begin{aligned} & \sup_{x \in D_i} |g(x) - \Psi_i^{[n]} g(x)| \\ &\leq \sum_{j=1}^n \frac{1}{384} \delta_{ij}^4 \sup_{x \in D_i} \left| \frac{\partial^4}{\partial x_j^4} \Psi_i^{[j-1]} g(x) \right| \\ &\leq \frac{1}{384} \bar{\delta}_i^4 \sum_{j=1}^n M_{ij}. \end{aligned} \quad (30)$$

Q.E.D

Furthermore, an approximation of a vector function $G(x)$ of Eq.(15) by the cubic Hermite interpolation on D_i is expressed by

$$\Psi_i^{[n]} G(x) = [\Psi_i^{[n]} g_1(x), \dots, \Psi_i^{[n]} g_{4^n-1}(x)]^T. \quad (31)$$

Let $\varepsilon_i(x)$ be an approximation error on D_i

$$\varepsilon_i(x) \equiv G(x) - \Psi_i^{[n]} G(x). \quad (32)$$

We have

$$\begin{aligned} \sup_{x \in D_i} \|\varepsilon_i(x)\| &= \sup_{x \in D_i} \max_{1 \leq k \leq 4^n-1} |g_k(x) - \Psi_i^{[n]} g_k(x)| \\ &= \max_{1 \leq k \leq 4^n-1} \sup_{x \in D_i} |g_k(x) - \Psi_i^{[n]} g_k(x)|. \end{aligned} \quad (33)$$

From lemma 1,

$$\sup_{x \in D_i} \|\varepsilon_i(x)\| \leq \frac{1}{384} \bar{\delta}_i^4 \max_{1 \leq k \leq 4^n-1} \sum_{j=1}^n M_{ijk}, \quad (34)$$

where $M_{ijk} = \sup_{x \in D_i} \left| \frac{\partial^4}{\partial x_j^4} \Psi_i^{[j-1]} g_k(x) \right|$.

Therefore, we have the following lemma about an error bound of the vector function approximation.

Lemma 2: If $f(x) \in \bigcap_{i=1}^n C^4(x_i)$,

$$\varepsilon \equiv \max_{1 \leq i \leq m} \sup_{x \in D_i} \|\varepsilon_i(x)\| \leq \frac{M}{384} \bar{\delta}^4, \quad (35)$$

where $\bar{\delta} = \max_{1 \leq i \leq m} \bar{\delta}_i$, $M = \max_{1 \leq i \leq m} \max_{1 \leq k \leq 4^n-1} \sum_{j=1}^n M_{ijk}$.

Finally, we have the following theorem about an error bound between $x(t)$ and $\hat{x}(t)$.

Theorem: An error bound of the formal linearization is

$$\begin{aligned} & \|x(t) - \hat{x}(t)\| \\ &\leq e^{at} \|\phi(x(0)) - z(0)\| + \varepsilon(e^{at} - 1)/a, \end{aligned} \quad (36)$$

where

$$\begin{aligned} a &= \max_{1 \leq i \leq m} \|A_i\|, \\ \varepsilon &= \max_{1 \leq i \leq m} \sup_{x \in D_i} \|\varepsilon_i(x)\| \\ &\leq \frac{M}{384} \bar{\delta}^4 \quad (\text{if } f(x) \in \bigcap_{i=1}^n C^4(x_i)). \end{aligned} \quad (37)$$

Proof: If $x(t) \in D_i$ for any i , a differential equation of difference between $\phi(x)$ of Eq.(13) and z of Eq.(21) is given by

$$\begin{aligned} \frac{d}{dt} (\phi(x) - z) &= \dot{\phi}(x) - \dot{z} = G(x) - A_i z - b_i \\ &= A_i (\phi(x) - z) + (G(x) - A_i \phi(x) - b_i). \end{aligned} \quad (38)$$

The solution of this equation is

$$\begin{aligned} \phi(x(t)) - z(t) &= e^{A_i t} [\phi(x(0)) - z(0)] \\ &\quad + \int_0^t e^{A_i(t-\tau)} \varepsilon_i(x(\tau)) d\tau. \end{aligned} \quad (39)$$

The norm of Eq.(39) is

$$\begin{aligned}
& \| \phi(\mathbf{x}(t)) - \mathbf{z}(t) \| \\
& \leq e^{\| \mathbf{A}_i \| t} \| \phi(\mathbf{x}(0)) - \mathbf{z}(0) \| \\
& \quad + \sup_{\mathbf{x} \in D_i} \| \varepsilon_i(\mathbf{x}) \| \int_0^t e^{\| \mathbf{A}_i \| (t-\tau)} d\tau \\
& \leq e^{at} \| \phi(\mathbf{x}(0)) - \mathbf{z}(0) \| + \varepsilon \int_0^t e^{a(t-\tau)} d\tau \\
& = e^{at} \| \phi(\mathbf{x}(0)) - \mathbf{z}(0) \| + \varepsilon(e^{at} - 1)/a. \tag{40}
\end{aligned}$$

Eq.(40) is valid on all subdomains. From Eqs.(22) and (24),

$$\begin{aligned}
& \| \mathbf{x}(t) - \hat{\mathbf{x}}(t) \| = \| \mathbf{K} \phi(\mathbf{x}(t)) - \mathbf{K} \mathbf{z}(t) \| \\
& \leq \| \mathbf{K} \| \| \phi(\mathbf{x}(t)) - \mathbf{z}(t) \| \\
& = \| \phi(\mathbf{x}(t)) - \mathbf{z}(t) \|. \tag{41}
\end{aligned}$$

Eqs. (40) and (41) yield Eq.(36). Q.E.D

Theorem indicates that an initial error causes the first term, and the approximation error by the Hermite interpolation dose the second term which can be reduced when $\bar{\delta}$ is reduced. Therefore, if $\mathbf{z}(0) = \phi(0)$ and $\bar{\delta} \rightarrow 0$, then $\hat{\mathbf{x}}(t) \rightarrow \mathbf{x}(t)$ at any time t .

4. A Nonlinear Filter

We here synthesize a nonlinear filter as an application of the linearization. Assume that a nonlinear dynamic and a nonlinear measurement equations are given as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \mathbf{x}(0) = \mathbf{x}_0, \tag{42}$$

$$\zeta(t) = \mathbf{h}(\mathbf{x}(t)) + \mathbf{v}(t), \tag{43}$$

where $\zeta = [\zeta_1, \dots, \zeta_r]^T$: r measurement vector, $\mathbf{h} = [h_1, \dots, h_r]^T$: r nonlinear measurement vector function, $\mathbf{v}(t)$: white Gaussian noise of $E(\mathbf{v}(t)) = 0$ and $E(\mathbf{v}(t)\mathbf{v}^T(\tau)) = \mathbf{V}\delta(t - \tau)$, δ : Dirac delta function.

Eq.(42) is transformed into the linear system of Eq.(21) by the formal linearization. For Eq.(43), approximate it by the cubic Hermite interpolation

$$\mathbf{h}(\mathbf{x}) \approx \mathbf{C}_i \phi(\mathbf{x}) + \mathbf{d}_i, \mathbf{x} \in D_i. \tag{44}$$

Thus we have linearized dynamic and measurement equations:

$$\dot{\mathbf{z}} = \mathbf{A}_i \mathbf{z}(t) + \mathbf{b}_i, \tag{45}$$

$$\zeta(t) = \mathbf{C}_i \mathbf{z}(t) + \mathbf{d}_i + \mathbf{v}(t). \tag{46}$$

We apply the linear estimation theory, Kalman filter⁽⁵⁾, to Eqs.(45) and (46). and have a nonlinear filter algorithm:

$$\dot{\hat{\mathbf{z}}} = (\mathbf{A}_i - \mathbf{L}\mathbf{C}_i)\hat{\mathbf{z}} + \mathbf{b}_i + \mathbf{L}(\zeta - \mathbf{d}_i), \tag{47}$$

$$\mathbf{L} = \mathbf{P}\mathbf{C}_i^T \mathbf{V}^{-1}, \tag{48}$$

$$\dot{\mathbf{P}} = \mathbf{A}_i \mathbf{P} + \mathbf{P} \mathbf{A}_i^T - \mathbf{P} \mathbf{C}_i^T \mathbf{V}^{-1} \mathbf{C}_i \mathbf{P}, \tag{49}$$

$$\hat{\mathbf{x}} = \mathbf{K} \hat{\mathbf{z}} \in D_i, \tag{50}$$

$\hat{\mathbf{x}}(t_0), \mathbf{P}(t_0)$: given.

Matrices $\mathbf{A}_i, \mathbf{b}_i, \mathbf{C}_i$ and $\mathbf{d}_i$ must be changed into $\mathbf{A}_j, \mathbf{b}_j, \mathbf{C}_j$ and $\mathbf{d}_j$ respectively, when $\hat{\mathbf{x}}(t)$ goes onto different subdomain D_j from D_i during state estimation.

5. Numerical Examples

We have numerical experiments of the formal linearization and the nonlinear filter. Differential equations are solved by Runge-Kutta method.

Example 1: Consider the following two dimensional nonlinear system:

$$\begin{cases} \dot{x}_1 = -1000(x_1 - x_2^4) + 4x_2^3, \\ \dot{x}_2 = 1, \end{cases} \tag{51}$$

where the initial value is $\mathbf{x}(0) = [0, 0]^T$. The analytic solution of this system is $x_1(t) = t^4, x_2(t) = t$. To apply our linearization method, set the domain as $\mathbf{x} \in [-1, 1] \times [-1, 1]$, and let N be a parameter such as $\delta_{ik} = 2^{-N}$ ($m = 4^{2N}, 1 \leq i \leq m, k = 1, 2$) of Eq.(16). For the purpose of comparison, we apply Runge-Kutta method to the original nonlinear system of Eq.(51) with time step $\Delta t = 0.001$. Fig. 1 shows the trajectories of $x_1(t)$ and $\hat{x}_1(t)$, and Fig. 2 shows the error defined as

$$J(t) = \int_0^t \| \mathbf{x}(\tau) - \hat{\mathbf{x}}(\tau) \| d\tau. \tag{52}$$

Those results of Figs.1 and 2 show that the accuracy of the formal linearization is improved for smaller size of subdomains.

Example 2: Consider the following nonlinear system:

$$\dot{x}(t) = \cos(x(t)), x(0) = 0.0,$$

$$\zeta(t) = \sin \frac{x(t)}{2} + v(t), V = 0.2.$$

To apply our nonlinear filter algorithm to the system, set the domain of x as $[0, 2)$, and $\delta_{ik} = 0.5$ ($m = 4$, $1 \leq i \leq m$, $k = 1$). Give the initial values $\hat{x}(0) = 1.5$, $P(0) = 5I$. For the purpose of comparison, we apply the extended Kalman filter⁽⁵⁾ to the above system with initial values $\hat{x}(0) = 1.5$, $P(0) = 5$. Fig. 3 shows the results of state estimation. These results indicate that the estimation by the proposed filter is better than the result by the extended Kalman filter.

6. Conclusions

A numerical computational method of the formal linearization by the cubic Hermite interpolation is considered. This linearization method is applicable to practical nonlinear systems and easily computable with the aid of computers. Based on this method, a new nonlinear filter algorithm is derived. Finally the linearization and the nonlinear filter algorithm are validated through numerical experiments.

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Fig. 1: Trajectories of $x_1(t)$ and $\hat{x}_1(t)$.

Fig. 2: Approximation errors $J(t)$.

Fig. 3: Estimation results.