Static Output Feedback Control of Flexible Structures subject to Random Disturbances

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Abstract : In this report, we propose a theoretical essence of the stabilization of flexible structures by using the static output feedback control. This shows that not only strain outputs but also other outputs are valid in this control method for flexible structures of many boundary conditions including cantilevers, free ends, and other general boundary conditions. Sufficient conditions of structures and observation mechanisms are obtained for the closed loop systems to be asymptotically stable. When the observation mechanism is nicely chosen, the closed loop system becomes exponentially stable.

Keywords : Distributed parameter systems, Flexible structures, Flexible robot arms, Flexible manipulators, Direct strain feedback

1 Introduction

The direct strain feedback control of flexible structures is the one of the static output controls [6]. The stabilization of distributed parameter systems using static output feedback control are studied by many researchers[2][3] [4][5].

In all these previous papers, observation operators which generate outputs are bounded. But in the above static strain feedback control, output operators are unbounded. In this problem we show that unbounded output operators are transformed to bounded output operators. (This point is not described in the previous papers.) This transformation is valid for not only strain output but also other unbounded outputs. The theoretical essence of the static strain output feedback control is extended to other kinds of outputs. Thus, we can introduce static GENERAL output feedback control which contains the direct strain feedback control.

2 Dynamics and Observation Mechanisms of Flexible Structures

Consider a flexible structure and let $u_1(t, x) \in R^q$ ($q = 1, 2, 3$) is an elastic displacement at time t , position $x \in \Omega$ where Ω represents a domain of structures and $\Omega \subset R^p$ ($p = 1, 2, 3$). In general, the displacement $u_1(t, x)$ of a flexible structure at time t at position x satisfies a partial differential equation

$$\frac{\partial^2 u_1}{\partial t^2} + D_x \frac{\partial u_1}{\partial t} + A_x u_1(t, x) = f_2(t, x) + w_{(sys)2}(t, x) \quad (1)$$

for suitable partial differential operators D and A . For fixed time t , the spatial function $u_1(t, \cdot)$ can be thought as an element of a suitable Hilbert space H . The operator A represents stiffness and D represents damping of flexible structures. $f_2(t, x)$ is a control input, and $w_{(sys)2}(t, x)$ is a disturbance to the system (the flexible structure).

By introducing $u_2(t, x) = \partial u_1 / \partial t$, the dynamics Eq.(1) of the flexible structure becomes

$$\frac{\partial}{\partial t} \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix} = \begin{pmatrix} u_2(t, x) \\ -A_x u_1(t, x) - D_x u_2(t, x) \end{pmatrix} + \begin{pmatrix} 0 \\ f_2(t, x) \end{pmatrix} + \begin{pmatrix} 0 \\ w_{(sys)2}(t, x) \end{pmatrix} \quad (2)$$

Let $u(t, x) = (u_1(t, x), u_2(t, x))^T$, $f(t, x) = (0, f_2(t, x))^T$, $w_{sys}(t, x) = (0, w_{(sys)2}(t, x))^T$ and

$$\begin{aligned} \mathcal{A}_x u(x) &= (u_2(x), -A_x u_1(x) - D_x u_2(x))^T \\ \text{for } u(x) &= (u_1(x), u_2(x))^T \end{aligned} \quad (3)$$

Then, Eq.(2) can be rewritten as

$$\frac{\partial u}{\partial t} = \mathcal{A}_x u(t, x) + f(t, x) + w_{sys}(t, x) \quad (4)$$

Consider the following observation mechanism, i.e. suppose we can measure the following data

$$u_{(obs)}(t) = \mathcal{C}u(t, \cdot) + w_{obs}(t) \quad (5)$$

$$= \begin{pmatrix} (c_1(\cdot), \mathcal{C}u_2(t, \cdot))_H \\ \vdots \\ (c_N(\cdot), \mathcal{C}u_2(t, \cdot))_H \end{pmatrix} + \begin{pmatrix} w_{(obs)1}(t) \\ \vdots \\ w_{(obs)N}(t) \end{pmatrix} \quad (6)$$

where $w_{obs}(t) = (w_{(obs)1}(t) \dots w_{(obs)N}(t))^T$ is an observation noise, $c_k(\cdot)$ is a spatial weighting function ($k = 1, 2, \dots, N$), $(\cdot, \cdot)_H$ is an inner product in a Hilbert space H , and the observation operator $\mathcal{C}$ is defined as follows.

$$\mathcal{C}u(\cdot) = \begin{pmatrix} (c_1(\cdot), \mathcal{C}u_2(\cdot))_H \\ \vdots \\ (c_N(\cdot), \mathcal{C}u_2(\cdot))_H \end{pmatrix} \quad (7)$$

CONDITIONS

We suppose that

- C^{-1} is a bounded operator on H ,
- $\tilde{A} = CAC^{-1}$ is a self-adjoint positive definite operator on H ,
- $\tilde{D} = CDC^{-1}$ is a positive semidefinite operator on H , and
- $\tilde{D} = CDC^{-1}$ is positive definite or $(\tilde{\mathcal{C}}, \tilde{A})$ is observable in infinite dimensional sense.

where

$$\begin{aligned} \tilde{\mathcal{C}} &: \mathcal{D}(\tilde{A}^{1/2}) \times H \rightarrow R^N \\ \tilde{\mathcal{C}}y &= [(c_k, y_2)_H; k \downarrow 1, 2, \dots, N] \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_x y(x) &= (y_2(x), -\tilde{A}_x y_1(x) - \tilde{D}_x y_2(x))^T \\ \text{for } y(x) &= (y_1(x), y_2(x))^T \in \mathcal{D}(\tilde{A}^{1/2}) \times H \end{aligned}$$

Examples of A and C which satisfies this condition are as follows.

Cantilevers When the flexible structure is a cantilever, the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0, L)$ where L is a length of the cantilever. Let $H = L^2((0, L); R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \{u \in H^4(0, L); 0 = u(0) = u'(0) = u''(L) = u'''(L)\} \quad (8)$$

$$\mathcal{D}(C) = \{u \in H^2(0, L); 0 = u(0) = u'(0)\} \quad (9)$$

which reflect the boundary conditions of the both ends ($x = 0, L$) of the structure. $H^m(0, L)$ is the Sobolev space of order m on an open interval $(0, L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ ($k = 1, 2, \dots, N$). The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \dots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied. (See Appendix.)

Free Beams When the flexible structure is a free beam (a beam whose both ends are free), the spatial domain Ω of the structure is one-dimensional and Ω becomes an open interval $(0, L)$ where L is a length of the beam. Let $H = L^2((0, L); R)$, $A_x = EId^4/dx^4$, $D = \delta A$, and $C_x = d^2/dx^2$, where δ and EI are positive constants and

$$\mathcal{D}(A) = \{u \in H^4(0, L); 0 = u''(0) = u'''(0) = u''(L) = u'''(L)\} \quad (10)$$

$$\mathcal{D}(C) = \{u \in H^2(0, L); 0 = u(0) = u'(0)\} \quad (11)$$

which reflect the boundary conditions of the both ends ($x = 0, L$) of the structure. $H^m(0, L)$ is the m -th order Sobolev space on an open interval $(0, L)$ and EI is a positive constant.

Let $c_k(x)$ be a positive continuous function which approximates $\delta(x - a_k)$ ($k = 1, 2, \dots, N$). The time derivative of the strains (the bending moments) of the structure at $x = a_1, a_2, \dots, a_N$ are measured. A and CAC^{-1} becomes positive definite self-adjoint operators on H and the above conditions are satisfied. (The proof is similar to Appendix.)

General Structures The above conditions are satisfied when A is a self-adjoint positive definite operator on the Hilbert space H , $C = A^\alpha$ ($0 \leq \alpha \leq 1$ is a constant) and $D = \delta A$ where δ is a positive constant, since $CAC^{-1} = A$. If $C = A^\alpha$ ($0 \leq \alpha \leq 1$ is a constant), the observed data correspond to time derivatives of the elastic displacements of the structure when $\alpha = 0$, to the time derivative of the strains when $\alpha = 1/2$, and to the shear forces when $\alpha = 3/4$.

3 Statistic Properties of the System Disturbance and the Observation Noises

Suppose that the disturbance $w_{sys}(t, \cdot)$ to the system and the noise $w_{obs}(t)$ of the observation mechanism are generated by a cylindrical Brownian motion (H -Brownian motion) $B_{sys}(t)$ (whose time derivative is a spatially and temporally white noise) and a standard Brownian motion $B_{obs}(t)$ respectively, i.e.,

$$dw_{sys}(t) = H_{sys}w_{sys}(t)dt + Q dB_{sys}(t) \quad (12)$$

$$dw_{obs}(t) = H_{obs}w_{obs}(t)dt + R dB_{obs}(t) \quad (13)$$

where H_{sys} is a stable bounded linear operator on a Hilbert space H . That is, the all real parts of spectra of the operators H_{sys} are smaller than some negative constant. Q is a Hilbert-Schmidt operator on a Hilbert space H , H_{obs} is a stable matrix, and R is a matrix.

Then, the extended system becomes as follows.

$$d \begin{pmatrix} u(t) \\ w_{sys}(t) \\ w_{obs}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & I & 0 \\ 0 & H_{sys} & 0 \\ 0 & 0 & H_{obs} \end{pmatrix} \begin{pmatrix} u(t) \\ w_{sys}(t) \\ w_{obs}(t) \end{pmatrix} dt + \begin{pmatrix} f(t) \\ 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ Q dB_{sys}(t) \\ R dB_{obs}(t) \end{pmatrix} \quad (14)$$

4 Static Output Feedback Control of Flexible Structures

Consider a static output feedback control

$$f(t, x) = G(x)u_{obs}(t) \quad (15)$$

where the feedback gain $G(x)$ is

$$G(x) = -\tilde{g} \begin{pmatrix} 0, \dots, 0 \\ (C^{-1}c_1)(x), \dots, (C^{-1}c_N)(x) \end{pmatrix} \quad (16)$$

where $\tilde{g}$ is a positive constant.

By this static output feedback, we obtain the following closed loop system.

$$d \begin{pmatrix} u(t) \\ w_{sys}(t) \\ w_{obs}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A} + GC & I & G \\ 0 & H_{sys} & 0 \\ 0 & 0 & H_{obs} \end{pmatrix} \begin{pmatrix} u(t) \\ w_{sys}(t) \\ w_{obs}(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ Q dB_{sys}(t) \\ R dB_{obs}(t) \end{pmatrix} \quad (17)$$

If we can choose G such that $\mathcal{A} - GC$ is sufficiently stable, then the variance of the extended state variable $(u(t), w_{sys}(t), w_{obs}(t))$ becomes small.

5 Exponential Stability of the Transformed Closed Loop Operator and the Variance of the State Variable of the Closed Loop System

When neither system disturbances nor observation noises exist, i.e. $w_{sys}(t, x) \equiv 0$ and $w_{obs}(t) \equiv 0$, the closed loop system becomes

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix} &= \\ \begin{pmatrix} u_2(t, x) \\ -A_x u_1(t, x) - D_x u_2(t, x) \end{pmatrix} &- \begin{pmatrix} 0 \\ \tilde{g} \sum_{k=1}^N (C^{-1}c_k)(x) (c_k(\cdot), Cu_2(t, \cdot))_H \end{pmatrix} \end{aligned} \quad (18)$$

Introduce a new variable $y_i(t) = Cu_i(t) (i = 1, 2)$ and $y(t) = (y_1(t), y_2(t))^T$, the closed loop system Eq.(18) is rewritten as follows.

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} y_1(t, \cdot) \\ y_2(t, \cdot) \end{pmatrix} &= \\ \begin{pmatrix} y_2(t, \cdot) \\ -\tilde{A}y_1(t, \cdot) - \tilde{D}y_2(t, \cdot) \end{pmatrix} &- \begin{pmatrix} 0 \\ \tilde{g} \sum_{k=1}^N c_k(\cdot) (c_k(\cdot), y_2(t, \cdot))_H \end{pmatrix} \end{aligned} \quad (19)$$

where $\tilde{A} = CAC^{-1}$ and $\tilde{D} = CDC^{-1}$.

By defining an operator $\tilde{C} : \mathcal{D}(\tilde{A}^{1/2}) \times H \rightarrow R^N$ as

$$\tilde{C}y = [(c_k, y_2)_H; k \downarrow 1, 2, \dots, N] \quad (20)$$

for $y = (y_1, y_2)^T \in \mathcal{D}(\tilde{A}^{1/2}) \times H$, the transformed closed loop equation Eq.(19) can be rewritten as

$$\frac{d}{dt}y(t) = (\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C})y(t) \quad (21)$$

The operator $\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}$ in the above equation becomes dissipative on the Hilbert space $\mathcal{D}(\tilde{A}^{1/2}) \times H$. In fact, let $\langle \cdot, \cdot \rangle$ be an inner product in the Hilbert space $\mathcal{D}(\tilde{A}^{1/2}) \times H$, id est,

$$\begin{aligned} \langle u, v \rangle &= (\tilde{A}^{1/2}u_1, \tilde{A}^{1/2}v_1)_H + (u_2, v_2)_H \\ \text{for } u &= (u_1, u_2)^T, v = (v_1, v_2)^T \in \mathcal{D}(\tilde{A}^{1/2}) \times H \end{aligned} \quad (22)$$

then

$$\langle (\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C})y, y \rangle = -(\tilde{D}y_2, y_2)_H - \tilde{g}(\tilde{C}y, \tilde{C}y)_{R^N} \leq 0 \quad (23)$$

We show that this system Eq.(19) is asymptotically stable in the Hilbert space $\mathcal{D}(\tilde{A}^{1/2}) \times H$. (This equation looks like the equation in [4]) For the solution $y(t) = (y_1(t), y_2(t))^T$ of the closed system Eq.(19), we have

$$\frac{d}{dt}\langle y(t), y(t) \rangle = -2(\tilde{D}y_2(t), y_2(t))_H - 2\tilde{g}(\tilde{C}y(t), \tilde{C}y(t))_{R^N} \leq 0 \quad (24)$$

Therefore, for $\frac{d}{dt}\langle y(t), y(t) \rangle = 0$, it is necessary that $y_2(t) = 0$, since $(\tilde{C}, \tilde{A})$ is observable or $\tilde{D}$ is a positive definite operator. This leads to the conclusion that $dy_2/dt = 0$ and $y_1(t) = 0$ from Eq.(19). Thus, $\langle y(t), y(t) \rangle$ tends to 0 as $t \rightarrow \infty$. Since $u_i(t) = C^{-1}y_i(t)$ ($i = 1, 2$) and C^{-1} is a bounded operator, $u(t) = (u_1(t), u_2(t))^T$ tends to 0 as $t \rightarrow \infty$ and the closed loop system Eq.(18) becomes asymptotically stable.

Moreover, we can show the exponential stability of the transformed closed loop system Eq.(19). In fact, for any eigenvalue $-\mu_i^*$ of the transformed closed loop operator $\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}$ ($i = 1, 2, \dots$),

$$-\mu_i^* \leq -\alpha(\tilde{g}) - \beta(\tilde{g}) \quad (25)$$

where

$$\alpha(\tilde{g}) = \min \{(D\psi_{i2}^*, \psi_{i2}^*)_H; i = 1, 2, \dots\}, \quad \beta(\tilde{g}) = \min \left\{ \sum_{k=1}^N (c_k, \psi_{i2}^*)_H^2; i = 1, 2, \dots \right\} \quad (26)$$

and ψ_i^* is the i -th normalized eigenfunction of the transformed closed loop operator $\tilde{A} - \tilde{g}\tilde{C}^*\tilde{C}$.

6 Conclusions

In this paper, we found the theoretical essence of static output feedback control of flexible structures. This essence shows that not only strain but also other outputs are effective

in the stabilization of flexible structures by the static output feedback control, and this control method is applicable for many types of flexible structures including cantilevers and free ends beams and other structures who has general boundary conditions. The sufficient conditions (see **CONDITIONS**) of outputs are clarified for the static output feedback control to be effective. When the observation mechanism is nicely chosen, the closed loop system becomes exponentially stable.

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Appendix

Let A and C be operators on the Hilbert space $H = L^2(0, L)$ defined as follows.

$$A_x = EI \frac{d^4}{dx^4}, \quad \mathcal{D}(A) = \left\{ u \in H^4(0, L); 0 = u(0) = u'(0) = u''(L) = u'''(L) \right\} \quad (27)$$

$$C_x = \frac{d^2}{dx^2}, \quad \mathcal{D}(C) = \left\{ u \in H^2(0, L); 0 = u(0) = u'(0) \right\} \quad (28)$$

For any $u \in \mathcal{D}(CAC^{-1})$, since

$$\mathcal{D}(C) \ni AC^{-1}u = \frac{d^2 u}{dx^2} \quad (29)$$

we have

$$0 = u''(0) = u'''(0) \text{ and } u \in H^4(0, L) \quad (30)$$

and since $u \in \mathcal{D}(AC^{-1})$, we have $C^{-1}u \in \mathcal{D}(A)$, we have

$$0 = u(L) = u'(L) \quad (31)$$

For $u, v \in \mathcal{D}(CAC^{-1})$, by integrating by parts, we have

$$(CAC^{-1}u, v) = EI \left(\frac{d^4 u}{dx^4}, v \right) = EI \left(\frac{d^2 u}{dx^2}, \frac{d^2 v}{dx^2} \right) = (u, CAC^{-1}v) \quad (32)$$