Chaotic Analysis for Stochastic Signal based on Local Tangent Space

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Abstract
Statistical distribution of local expansion ratios and local q-order Lyapunov exponents are employed as indices for diagnostic techniques. These indices characterize change in statistical properties of dynamical systems that have internal disturbance for the systems’ parameters; they give a foothold for detailed analysis of engineering systems.

Introduction
For large and complex engineering systems, especially nuclear power plants, research and development of diagnostic techniques are indispensable to ensure the safety. Chaotic analysis in a sense of inverse problem is one of the trials to be done when nonlinear effects may arise. In engineering system, nonlinear dynamics are generally accompanied by noise. So far some studies have focussed on this problem 1). However, works have mainly dealt with external noise using conventional estimation techniques of Lyapunov exponents or fractal dimensions. Concerning this, our previous studies have demonstrated effects of additive parametric and observation noise on a chaotic system 2). In deterministic chaotic systems, the tangent space on any point of a trajectory is uniquely determined by the conditions. However, in noisy systems, it should be statistically determined in the neighborhood of the point concerned. Conventional Lyapunov estimation techniques do not consider this fact.

Furthermore, conventional estimation methods of the Lyapunov exponent need convergency of numerical computations for the ensemble average or the number of iteration that corresponds to long term average on the trajectory (see Ref.3). This causes poor sensitivity in estimating Lyapunov exponents for noisy chaotic systems. Numerical computations should sometimes show a positive Lyapunov exponent in chaos-like systems, which are defines as ones such that a positive Lyapunov exponent may arise even if their deterministic parts do not have a chaotic property. In order to analyse the non-uniformity of trajectories in deterministic chaotic systems, Fujisaka introduces a concept of the q-order Lyapunov exponents 4), which is an expansion of the conventional methods.

In this paper we propose a new scheme of chaotic analysis which is based on the local statistical properties over a set of tangent spaces on trajectories. Our method is readily applied to the q-order Lyapunov exponents; the index obtained seems very useful in identifying the local structure of trajectories. Our proposed method is particularly hopeful in the view of surveillance and diagnosis for engineering systems.

Some Remarks on Numerical Simulation
Numerical simulation is based on two kinds of systems:
\[
\frac{dX}{dt} = f(X,t,p+w), \quad x \in \mathbb{R}^n \quad (1-a)
\]
\[
\frac{dX}{dt} = f(X,t,p+w,u), \quad x \in \mathbb{R}^n \quad (1-b)
\]
where \(p,w\) and \(u\) mean the constant system parameter, withe noise and the external disturbance, deterministic or stochastic, respectively. We assume:
- The systems approximately hold statistical stationary state in a sufficiently long period in terms of the solution process.
- The deterministic parts of stochastic systems need not be chaotic. Practically our method is discussed based on noisy Lorenz systems, noisy van der Pol systems, and noisy Duffing systems.
Local Property Estimation for Non-uniform Trajectory

In chaotic analysis, the Lyapunov exponent is estimated on a reconstructed trajectory $x(t)$ from one dimensional time series data $s(t)$ by using delay time coordinate 5)

$$x(t) = \{s(t), s(t+T), \ldots, s(t+(m-1)T)\},$$

(2)

where $m$ is an embedding dimension and $T$ a delay time. On the reconstructed trajectory, the evolution of a tangent vector $\xi$ in a tangent space can be described as

$$\dot{\xi} = DF(x(t)) \cdot \xi$$

(3)

where $DF(x(t))$ is the Jacobian matrix of a dynamical system $F(\cdot)$ at $x(t)$. In a chaotic plus stochastic system or chaos-like system, Eq. (3) is understood as stochastic differential equation. The solution can be given by

$$\xi(t) = A^t \xi(0)$$

(4)

In Eq. (3) and (4) the mathematical structure of the dynamical system is unknown from observed data. To estimate the property of tangent space, the evolution of the displacement vector $\delta x$ for relatively short evolution time $\tau$ in the local space described by

$$\delta x'(\tau) = A_j \delta x'(0)$$

(5)

is employed as an approximation of Eq. (4) and the linear operator $A_j$ computed by the least square method. The initial condition of the displacement vector $\delta x(0)$ is defined as

$$\delta x = \{x_k - x_i \mid \|x_k - x_i\| \leq \epsilon\}$$

(6)

In practical computations a small percentage of the trajectory size is employed as the value of $\epsilon$. From the set of linear operator $A_j$, the series of Lyapunov exponents, namely, Lyapunov spectra can be computed by

$$\lambda_i = \lim_{n \rightarrow \infty} \frac{1}{n \tau} \sum_{j=1}^{n} \ln \|A_j e_j\|$$

(7)

for $i=1,2,\ldots,m$, where $\{e_j\}$ form an orthogonal basis in the tangent space. This method to estimate the Lyapunov spectrum is described in the literature 6 and 7. If the system is deterministic, the maximum element of the Lyapunov spectrum $\lambda_1$ corresponds to the largest Lyapunov exponent which is estimated by the ensemble average 3) of expansion ratio for short time evolution of distance between nearby points on the trajectory.

To estimate local property of the reconstructed trajectory, some small domains $\mathcal{D}_0$ are defined on the trajectory. In the numerical experimentations described below, size of the local domain $\mathcal{D}_0$ is equal to $\epsilon$ in this paper. Searching points in one of the domains, a set of the standard point $\{x_i\}$ is decided for calculating the displacement vectors $\delta x_i$ (see eq.(6)). The linear operator $A_j$ is estimated for each standard point $x_i$. In m dimensional phase space, the linear operator $A$ in eq. (5) is estimated as a $nm \times m$ matrix.

From the real eigenvalues of the linear operators $\{A\}$ that correspond to local expansion ratio $A(\tau)$, q-order Lyapunov exponents 3) 4) are calculated by:

$$\lambda_q = \frac{1}{\tau} \left\langle \ln \left\{ A(\tau)^q \right\} \rightangle$$

(8)

where $\langle \ldots \rangle$ means ensemble average. The q-order Lyapunov exponent is a measure that describes fluctuations of the Lyapunov exponents in the ensemble: $[\lambda_{q=0} \ldots \lambda_{q=\infty}]$ corresponds to $[\lambda_{min} \ldots \lambda_{max}]$. When the ensemble average of the local expansion ratio $A(\tau)$ is taken for all over the trajectory $\lambda_{q=0}$ agrees with the largest Lyapunov exponent $\lambda$ of the system. In this process to estimate the stochastic distribution of local expansion ratio $A(\tau)$, it does not require convergency for the estimated indices. Therefore, there is no need for the system to be chaotic or chaos-like. The statistical distribution of the local expansion ratios $\{A(\tau)\}$ and the local q-order Lyapunov exponents are employed as the indices for diagnosis in this paper.
Local Property of Non-uniform Chaotic Attractor

To verify our assertion, the expanded method is applied to the Lorenz model:

\[
\begin{align*}
\frac{dx}{dt} &= a(y - x) \\
\frac{dy}{dt} &= -xz + y - x \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\] (9)

where the parameters \( \{a,b,\gamma\}=\{10.0,8/3,50.0\} \) are chosen for the Lorenz model to have chaotic property. Following the process described in the previous paragraph, the trajectory is reconstructed from the time series obtained from time evolution of the state variable \( x \) in eq. (9). In several local domains defined on the reconstructed trajectory, the statistical distribution of the local expansion ratios is estimated as shown in fig. 1. The results say that non-uniformity of the Lorenz attractor can be shown as the diversity of local expansion ratios. Furthermore, the statistical distribution of the local expansion ratios are estimated for a noisy chaotic system where one of the parameters, \( a \), in eq. (9) is disturbed by white noise with zero mean value. Figure 2 shows the results. In the figure, variance \( \sigma^2 \) corresponds to the intensity of the system noise. The change in the property of the dynamical system, i.e., the increasing of internal disturbance can be characterized by deformation of the distribution of the local expansion ratios for the fixed local domain on the trajectory. In practical application of this method the distribution of local expansion ratios should be estimated for several fixed local domains to improve the accuracy and sensitivity for change in the statistical properties. This is true since the influence of change in the statistical properties that appear on the distribution of local expansion ratios depend on the local structure of tangent space. The following is the condition for this numerical experimentation: sampling time=0.1[sec], \( T=0.1[sec] \), embedding dimension \( m=3, \varepsilon=0.02 \).

Local Property of Periodic Attractor

In this paragraph, the expanded method is applied to periodic nonlinear systems. One of them is the van der Pol model and the other is the Duffing model. In the same way as in the previous paragraph, internal disturbance is considered for the systems' parameters.

I. Noisy van der Pol model

The van der Pol model, that is well known as a model of a triode, is given by the next equation:

\[
\ddot{x} - e(1 - x^2)\dot{x} + x = 0
\] (10)

For the parameter \( e=10.0 \), the trajectory reconstructed from time evolution of the state variable \( x \) has slight non-uniform structure as shown in fig. 3(a), however the motion is periodic. In eq. (10), white noise with zero mean value is added to the system parameter \( e \) as internal disturbance. In two local domains, \( E \) and \( F \), defined on the reconstructed trajectory, the local q-order Lyapunov exponents are estimated. Figure 4 shows the results. Non-uniformity of the trajectory can be seen as difference between the local q-order Lyapunov exponents obtained from the time series with intensity of internal disturbance \( e^2=0 \) in two local domains \( E \) and \( F \). Moreover, the non-uniformity, i.e., diversity of property in tangent space appears as change in the local q-order Lyapunov exponents for intensity \( e^2 \). Especially, in the case of \( e^2=1.0 \), the fluctuation of the local Lyapunov exponents increases in local domain \( E \). On the contrary, the fluctuation decreases in local domain \( F \). The following is the condition for this numerical experimentation: sampling time=0.2[sec], \( T=2.0[sec] \), embedding dimension \( m=3, \varepsilon=0.02 \).

II. Noisy Duffing model

The Duffing model, that has an external force and friction term be given by

\[
\ddot{x} + k\dot{x} + x^3 = B\cos t.
\] (11)

The dynamical system does not have chaotic property for the parameters \( \{k,B\}=[0.08,0.2] \). The time series obtained from time evolution of state variable \( x \) also is periodic, the shape of the reconstructed trajectory is simple, as shown in fig. 3(b). On the other hand, this system has a kind of dependency for initial condition in the shape of the attractor. Therefore, the simulations have been done to obtain the same type of attractor for different intensity of internal disturbances. Figure 5 shows the local q-order Lyapunov exponents estimated in two local domains, \( G \) and \( H \), defined on the reconstructed trajectory (see fig. 3(b)). Although, the degree of non-uniformity is very low in the shape of the reconstructed trajectory, we can see a certain difference between the local property in
the case of $e^2=0$. Moreover, the change in the local q-order Lyapunov exponents for intensity of the internal disturbance also is distinctive of the local tangent space. In the domain $G$, a shift in the distribution of the local Lyapunov exponents is caused by the internal disturbance. The following is the condition for this numerical experimentation: sampling time=0.1[sec], $T=0.7[sec]$, embedding dimension $m=3$, $e=0.02$.

The effect of internal disturbance also appears in numerical characteristics of the linear operator $A_j$, that approximately describes time evolution in the tangent space. The fractal dimensions of these models, the van der Pol and the Duffing model, are lower than 2.0. However, in the case where the embedding dimension $m=2$, the eigenvalues of the linear operator $A_j$ tend to be imaginary number with the increasing of internal disturbance. To get meaningful estimation of the local properties, a higher embedding dimension should be employed. Therefore, the estimation for the noisy models was executed with embedding dimension $m=3$.

Conclusion

In order to clarity the non-uniformity of trajectory, we have proposed the method based on the statistical distribution of the local expansion ratios estimated as eigenvalues of linear transition operators in local space. The method is applied to the noisy chaotic system and the noisy periodic systems, showing to be effective for surveillance and diagnosis of engineering systems. Applications to stability monitoring in power plants and anomaly detection in rotary machines are expected. To do this more refinement from both numerical and analytical point of view are necessary. Furthermore, routes to chaos caused by noise should be studied.

References

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Fig. 1 Lorenz attractor and the distribution of local expansion ratios

Fig. 2 Change in the distribution of local expansion ratios estimated in domain A for internal disturbance
Fig. 3 Reconstructed trajectory and local domains to estimate the local property

Fig. 4 Local q-order Lyapunov exponents of noisy van der Pol model

Fig. 5 Local q-order Lyapunov exponents of noisy Duffing model