Robust Filtering of Systems Evolving on Tree with Energy Constraints

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Abstract

In this paper we derive robust filtering algorithm for systems evolving on dyadic tree. It has been reported that such systems can capture quite rich phenomena. Our results include merge step at each scale which has no equivalent form in time domain state-space systems. We will derive our robust filtering with uncertain quantities that satisfy energy constraints. The derived algorithm will take the form of ML-filter on tree.

1 Introduction

In this paper we derive robust filtering. This work is motivated by the recently flurry activities on wavelet transform. In particular we expand the works of some authors e.g. [1] [2] [3], in robust environment. Different from ordinary treatment on robust filtering, in this paper we deal with the systems defined on dyadic tree. It has been reported that equivalent Kalman filtering works well on such tree [1].

The systems defined on dyadic tree is able to capture variety phenomena, for example fractal, optical flow, to name a few [4]. All of the works in this related field, in our observation, so far based on the stochastic information of the systems. Here we will derive our robust filtering without knowing the stochastic behaviour of the initial state of the systems, except that the uncertain quantities satisfy energy constraints.

2 State-space Models on Tree

In this section we will explain our basic terms and notations related to our works. We need to introduce a state-space models on tree for our purpose. This models follow the definitions in [1]. Let T denotes the set of all nodes as in Figure 1, t denotes abstract index of each node of the tree, m(t) is the scale of m-component of t. In addition there are some basic shift operators on T, i.e. backward shift γ and two forward shifts α and β (with increasing m denoting forward direction).

Now it is possible to define dynamic models on T evolving from coarse to fine scales:

\[ x(t) = A(t)x(t\gamma) + B(t)u(t) \]
\[ y(t) = C(t)x(t) + v(t) \]

(1)
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We consider the dynamic systems as in equation (1) with noise-corrupted measurements
\[ z(t) = y(t) + v(t) = C(t)x(t) + v(t) \] (2)

The matrices \( A(t) \), \( B(t) \) and \( C(t) \) have the appropriate dimensions. Assume the initial state \( x(t_0) \) and the disturbances \( u(t), v(t) \) are unknown, except they satisfy the energy constraints
\[
\begin{align*}
[z(t_0) - x_0]^T \Psi^{-1} [z(t_0) - x_0] \\
+ \sum_{i=0}^{N-1} [u(i)^T Q^{-1}(i) u(i)] \\
+ v^T (i + 1) R^{-1}(i + 1) v(i + 1) \leq 1
\end{align*}
\] (3)

where \( x_0 \) is a given vector, \( \Psi, Q(t) \) and \( R(t) \) are given positive-definite symmetric matrices. Let \( t \) be an arbitrary scale on the tree, \( t_0 \) fixed initial scale at the bottom of the tree, and let \( Z(t) \) denotes the measured system output function evolving up to scale \( t \)
\[ Z(t) = \{ (z(s), s) : s \in [t_0, t] \} \] (4)

Our task is to find the set \( X(t|t) \) of possible system states \( x(t) \) that are consistent with the constraint in equation (3) and the output function \( Z(t) \) observed at scale \( t \). This is equal to find the minimization of the following equation, see [5],
\[ J^*[\xi, t] \triangleq \min_u J[\xi, t; u] \leq 1 \] (5)

Then the problem becomes the standard tracking problem, although in this case we work on the dyadic tree, so that it is natural that we have some new phenomena, notably the merge step at each scale.

**Proposition 1** The solution to our filtering problem is the ellipsoid \( X(t|t) \) given for all \( t \in [t_0, t] \) by
\[ X(t|t) = \{ \xi : [\xi - \hat{x}(t)] P(t|t)[\xi - \hat{x}(t)] \leq 1 - \delta^2(t) \} \] (6)

\[ P(t|t) \] is the solution to the Riccati equation
\[
P(t|t) = P^{-1}(\alpha t|t) + P^{-1}(\beta t|t) + C^T(t) R^{-1}(t) C(t)
\] (7a)
\[
P(\gamma^{-1} t|t) = A(t) P(t|t) A^T(t) + B(t) Q(t) B^T(t)
\] (7b)
\[
P(0|0) = \Psi
\] (7c)

The \( n \) vector \( \hat{x}(\gamma^{-1} t|t) \) is the solution to the linear differential equation
\[
\hat{x}(\gamma^{-1} t|t) = [S(\alpha t) \hat{x}(t|\alpha t) + S(\beta t) \hat{x}(t|\beta t)] + C^T(t) R^{-1}(t) y(t)
\] (8a)
\[
S(t) = P(\gamma^{-1} t|t) A^{-1}(t) P(t|t)
\] (8b)
\[
\hat{x}(t_0) = x_0
\] (8c)

and the positive real number \( \delta^2(t) \) is given by
\[
\delta^2(t) = \sum_{i=1}^{t} [z(i) - C(i) A(i-1) \hat{x}(i-1)]^T [C(i) P(\gamma^{-1} i|i) C^T(i) + R(i)]^{-1} [z(i) - C(i) A(i-1) \hat{x}(i-1)]
\] (9)

Further explanations and direction of proof of the above proposition can be seen in the Appendix A.

4 Summary

In this paper we have described and proposed a new phenomenon in robust filtering. We begin with a definition of a new kind of systems, viz systems evolving on the dyadic tree. It is said that this kind of systems can capture quite broad phenomena, notably those with fractal and self-similar features. Our derived method is different from the ordinary filtering, that is we introduce a merging step at each scale, due to the tree structure. The solution is lying on the ellipsoid. We employ the method of triangularization Hamiltonian matrix for smoothing problem to guarantee that our solution exists.

**Acknowledgement**

The author, J.S., would like to thank The Hitachi Scholarship Foundation for their generous support on his study in Japan.

**Appendix A**

The condition in Proposition 1 is satisfied following the standard solution to the linear quadratic tracking
The existence of the Riccati equation is almost the same with the one derived in [4]. The major difference is that in this paper we consider that the input \( u(\cdot) \) and the measurement noise \( v(\cdot) \) satisfy the energy constraint, in other words we don’t know anything about the stochastic behaviour of our systems. Consider the following scale-invariant model defined with downward dynamics and its associated measurement equation for an \( M \)-level tree.

\[
x(t) = Ax(\gamma^{-1}) + B(t)u(t) \\
y(t) = Cx(t) + v(t)
\]

The Hamiltonian function for the smoothing problem is as follows.

\[
H(x, \lambda) = \sum_{t=1}^{T} \left[ \frac{1}{2} [y(t) - Cx(t)]^T R^{-1} [y(t) - Cx(t)] \\
+ \sum_{t \neq t_0} \frac{1}{2} u^T(t) Q^{-1}(t) u(t) \\
+ \frac{1}{2} [x(t_0) - \bar{x}(t_0)]^T P_0^{-1} [x(t_0) - \bar{x}(t_0)] \\
+ \sum_{t \neq t_0} \lambda^T(t) [x(t) - Ax(\gamma^{-1}t) - Bu(\cdot)]
\]

Finding the optimal estimates of the state \( x \), the noise \( u \), and the Lagrange multiplier \( \lambda \), then followed by rewriting the results in matrix form, we will have the following Hamiltonian dynamics in terms of the points \( t, \alpha t, \beta t \) for \( t \neq t_0 \)

\[
A [\begin{bmatrix} \hat{x} \\ \lambda \end{bmatrix} _t] + \Theta_\alpha [\begin{bmatrix} \hat{x} \\ \lambda \end{bmatrix} _\alpha t] + \Theta_\beta [\begin{bmatrix} \hat{x} \\ \lambda \end{bmatrix} _\beta t] = [\begin{bmatrix} -A & 0 \\ -A & 0 \end{bmatrix} C^T R^{-1} y(t)]
\]

To tri-angularize the dynamics defined by equation (A - 5) into the form in which there is an upward recursion for the \( x^u \) decoupled from \( \hat{x} \), we transform the dynamics for \( x^u \) and \( \hat{x} \) into the following structure.

\[
S_t A T^{-1}_t [\begin{bmatrix} x^u \\ \hat{x} \end{bmatrix} _t] + S_t \Theta_\alpha T^{-1}_t [\begin{bmatrix} x^u \\ \hat{x} \end{bmatrix} _\alpha t] \\
+ S_t \Theta_\beta T^{-1}_t [\begin{bmatrix} x^u \\ \hat{x} \end{bmatrix} _\beta t] = [\begin{bmatrix} C^T R^{-1} y(t) \\ 0 \\ 0 \end{bmatrix}]
\]

Thus we can write the up-down algorithm as an upward recursion for \( x^u \) followed by a downward correction sweep involving \( \hat{x} \) and \( x^u \). For all \( t \) such that \( t \neq t_0 \) and \( m(t) \neq M \) the upward recursion is as follows.

\[
x^u(t) = F_{t+} [x^u(\alpha t)] + C^T R^{-1} y(t)
\]

\[
F_{t+} = -F_{t+} = P_t^{-1} A_t^{-1} y(t)
\]

For \( t = t_0 \) we have

\[
x^u(t_0) = F_{t+} [x^u(\alpha t_0)] + x^u(\beta t_0)] \\
+ C^T R^{-1} y(t_0) + P_0^{-1} \hat{x}(t_0)
\]
where

\[ F_{t_0^+} = P_{t_0^-}^{-1} A^{-1} \Gamma_{t_0^+}^{-1} \quad (A - 11a) \]
\[ \Gamma_{t_0^+} = 2P_{t_0^-}^{-1} + C^T R^{-1} C + P_0^{-1} \quad (A - 11b) \]

The initial condition for the upward recursion of the equation \( (A - 9) \) is determined by considering our constraints for all \( t \) such that \( m(t) = M \). We must now define \( x^u \) along the bottom level to provide our desired initial conditions. Consider the following definition for all \( s \) such that \( m(s) = M \).

\[ x^u(s) = \Gamma_M \hat{x}(s) + \hat{\lambda}(s) \quad (A - 12a) \]
\[ \Gamma_M = C^T R^{-1} y(s) \quad (A - 12b) \]

Finally, for all \( t \) such that \( t \neq t_0 \) and \( m(t) \neq M \)

\[ \hat{x}(\gamma^{-1}t|t) = [S(\alpha|t)\hat{x}(t|\alpha t) + S(\beta|t)\hat{x}(t|\beta t)] \]
\[ + C^T(t)R^{-1}(t)y(t) \quad (A - 13a) \]
\[ S(t) = P(\gamma^{-1}|t)A^{-1}(t)P(t|t) \]
\[ P(t|t) = P^{-1}(\alpha|t) + P^{-1}(\beta|t) \]
\[ + C^T(t)R^{-1}(t)C(t) \quad (A - 13b) \]
\[ P(\gamma^{-1}t|t) = A^{-1}(t)P(t|t)A^{-T}(t) \]
\[ + A^{-1}(t)B(t)Q(t)B^T(t)A^{-T}(t) \quad (A - 13c) \]

Initial conditions

\[ \hat{x}(\gamma^{-1}t|t) = 0 \quad (A - 14a) \]
\[ P^{-1}(\gamma^{-1}|t) = 0 \quad (A - 14b) \]

for all \( t \) such that \( m(\gamma^{-1}t) = M \).

References