New trends in system identification

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In this paper we discuss three topics of system identification such as multivariable system identification (Subspace identification methods), identification for control (Robust identification), and nonlinear system identification (Nonlinear black-box modeling).

1 Introduction

Recent major topics in system identification include: System identification for robust control, Identification of nonlinear systems, Subspace system identification, Continuous-time identification, and Time varying system identification etc. Among them we select and review three topics, i.e., Subspace system identification, Robust identification, and Nonlinear system identification.

2 Subspace methods

2.1 Problem formulation

We assume that a linear discrete time system which is unknown can be described by the following state-space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k + v_k \end{aligned} \quad (1)$$

where x_k is an n -dimensional state vector, u_k an m -dimensional input vector, and y_k an ℓ -dimensional output vector. We also assume that the system is observable and v_k be a zero-mean white stationary random process uncorrelated with the input u_k .

Define the stacked output vector $y_i(k)$ as

$$y_i(k) \triangleq \begin{bmatrix} y_k^T & y_{k+1}^T & \cdots & y_{k+i-1}^T \end{bmatrix}^T \text{ for } i \geq n+1.$$

The stacked vectors of the input and noise vectors are defined similarly. From (1), the following relations are easily derived:

$$y_i(k) = \Gamma_i x_k + H_i u_i(k) + v_i(k) \quad (2)$$

$$x_k = A^i x_{k-i} + \Delta_i u_i(k-i) \quad (3)$$

where

$$\Gamma_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix},$$

$$H_i = \begin{bmatrix} D & & & O \\ CB & D & & \\ \vdots & \vdots & \ddots & \\ CA^{i-2}B & CA^{i-3}B & \cdots & D \end{bmatrix}.$$

Γ_i is called an extended observability matrix. The reversed extended controllability matrix Δ_i is defined as

$$\Delta_i = [A^{i-1}B, A^{i-2}B, \dots, B].$$

From (2) and (3), we obtain

$$Y_p = \Gamma_i X_p + H_i U_p + V_p \quad (4)$$

$$Y_f = \Gamma_i X_f + H_i U_f + V_f \quad (5)$$

$$X_f = A^i X_p + \Delta_i U_p \quad (6)$$

where matrices Y_p , Y_f , X_p , and X_f are defined as follows:

$$Y_p = [y_i(0) \ y_i(1) \ \cdots \ y_i(N-1)] \quad (\ell i \times N)$$

$$Y_f = [y_i(i) \ y_i(i+1) \ \cdots \ y_i(i+N-1)] \quad (\ell i \times N)$$

$$X_p = [x_0 \ x_1 \ \cdots \ x_{N-1}] \quad (n \times N)$$

$$X_f = [x_i \ x_{i+1} \ \cdots \ x_{i+N-1}] \quad (n \times N)$$

U_p , V_p and U_f , V_f are defined in a similar manner as Y_p and Y_f , respectively.

2.2 The Algorithm

One of the typical algorithms of subspace identification[1] is as follows:

(step 1)

Form a matrix and perform QR-factorization as

$$\begin{bmatrix} U_f \\ \Phi \\ Y_f \end{bmatrix} = \begin{bmatrix} R_{11} & & \\ R_{21} & R_{22} & \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix} \quad (7)$$

where R_{11} , R_{22} , and R_{33} are some lower triangular matrices, and $[Q_1 \ Q_2 \ Q_3]$ is a matrix whose column vectors are orthonormal.

(step 2)

Compute Singular Value Decomposition (SVD) of R_{32} as follows:

$$R_{32} = [E_n \ E_n^\perp] \begin{bmatrix} S_n & \\ & S_2 \end{bmatrix} \begin{bmatrix} F_n^T \\ (F_n^\perp)^T \end{bmatrix} \quad (8)$$

(step 3) Let $\hat{A} = (E_n^{(1)})^\dagger (E_n^{(2)})$ and $\hat{C}$ be the first ℓ rows of the E_n . Here $E_n^{(1)}$ and $E_n^{(2)}$ consist of first and last $\ell \times (i-1)$ rows of E_n , respectively.

(step 4) Compute $\hat{B}$ and $\hat{D}$ solving a certain set of linear equations.

In the sequel we explore the implication of the algorithm.

2.3 Multivariate Linear Regression

Denoting by $\Gamma_i^\dagger$ the Moore-Penrose pseudoinverse of Γ_i , it follows from (4) that

$$X_p = \Gamma_i^\dagger [Y_p - H_i U_p - V_p].$$

Thus substituting the above into (6), we get

$$X_f = K\Phi - A^i \Gamma_i^\dagger V_p \quad (9)$$

where

$$K = [\Delta_i - A^i \Gamma_i^\dagger H_i \mid A^i \Gamma_i^\dagger], \quad \Phi = \begin{bmatrix} U_p \\ Y_p \end{bmatrix}.$$

Substituting the X_f into (5), we see that Y_f can be expressed as

$$Y_f = L_i \Phi + H_i U_f + V_f - A^{2i} \Gamma_i^\dagger V_p \quad (10)$$

where $L_i = \Gamma_i K$.

Consider the problem minimizing the following criterion for L_i and H_i , regarding the (10) as a multivariate linear regression :

$$\|Y_f - (L_i \Phi + H_i U_f)\|_F^2 \quad (11)$$

where $\|\cdot\|_F$ denotes the Frobenius norm. $\hat{H}_i$ and $\hat{L}_i$ minimizing the above criterion are

$$\begin{aligned} \hat{H}_i &= (Y_f - \hat{L}_i \Phi) U_f^T (U_f U_f^T)^{-1} \\ \hat{L}_i &= Y_f \Pi_{U_f}^\perp \Phi^T (\Phi \Pi_{U_f}^\perp \Phi^T)^{-1}, \end{aligned} \quad (12)$$

respectively, where $\Pi_{U_f}^\perp$ is defined as

$$\Pi_{U_f}^\perp = I - U_f^T (U_f U_f^T)^{-1} U_f.$$

Using the RQ decomposition, the criterion (11) can be rewritten as

$$\begin{aligned} &\|Y_f - (L_i \Phi + H_i U_f)\|_F^2 \\ &= \left\| [R_{31}, R_{32}] - [H_i \ L_i] \begin{bmatrix} R_{11} & \\ R_{21} & R_{22} \end{bmatrix} \right\|_F^2 \\ &\quad + \|R_{33}\|_F^2. \end{aligned}$$

Thus, the estimates $\hat{L}_i$ and $\hat{H}_i$ are the solutions of the following equation

$$[R_{31}, R_{32}] = [\hat{H}_i \ \hat{L}_i] \begin{bmatrix} R_{11} & \\ R_{21} & R_{22} \end{bmatrix}.$$

It follows from this equation that the estimate $\hat{L}_i$ satisfies a relation

$$\hat{L}_i R_{22} = R_{32}. \quad (13)$$

2.4 Estimation of extended observability matrix Γ_i

To understand a fundamental notion of what the subspace methods are, it is relevant to recognize the extended observability matrix Γ_i plays a central role in the method.

It has been proposed[1] that SVD of the quantity R_{32} shown in (8) is utilized to obtain the extended observability matrix Γ_i as in

$$\Gamma_i = E_n \quad (14)$$

This section discusses why SVD of R_{32} leads to computation of Γ_i .

$U_f = R_{11}Q_1^T$ follows from QR factorization, and then

$$\Pi_{U_f} = U_f^T(U_f U_f^T)^{-1}U_f = Q_1 Q_1^T$$

and we see also

$$\Pi_{U_f}^\perp = I - Q_1 Q_1^T$$

Furthermore since $\Phi = R_{21}Q_1^T + R_{22}Q_2^T$, we have

$$\Phi \Pi_{U_f}^\perp = R_{22}Q_2^T.$$

Postmultiplying both side of (13) by Q_2^T , we obtain the relation

$$\hat{L}_i \Phi \Pi_{U_f}^\perp = R_{32}Q_2^T. \quad (15)$$

On the other hand, from (5) and noting $U_f \Pi_{U_f}^\perp = 0$, we obtain

$$\begin{aligned} \hat{L}_i \Phi \Pi_{U_f}^\perp &= \Gamma_i X_f \Pi_{U_f}^\perp \Phi^T (\Phi \Pi_{U_f}^\perp \Phi^T)^{-1} \Phi \Pi_{U_f}^\perp \\ &+ V_f \Pi_{U_f}^\perp \Phi^T (\Phi \Pi_{U_f}^\perp \Phi^T)^{-1} \Phi \Pi_{U_f}^\perp. \end{aligned}$$

By assumption

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} V_f U_f^T &= 0 \\ \lim_{N \rightarrow \infty} \frac{1}{N} V_f \Phi^T &= 0 \end{aligned}$$

so we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} V_f \Pi_{U_f}^\perp \Phi^T = 0 \quad (16)$$

and for sufficiently large N we can verify that

$$\hat{L}_i \Phi \Pi_{U_f}^\perp \approx \Gamma_i X_f \Pi_{U_f}^\perp \Phi^T (\Phi \Pi_{U_f}^\perp \Phi^T)^{-1} \Phi \Pi_{U_f}^\perp.$$

Note that both the left singular vectors of R_{32} and $R_{32}Q_2^T$ are identical since $Q_2^T Q_2 = I$. Therefore we can obtain Γ_i via SVD of R_{32} for sufficiently large N .

Above discussions lead to a conjecture that SVD of not only $\hat{L}_i \Phi \Pi_{U_f}^\perp$, but also of $\hat{L}_i \Phi$, again of $Y_f \Pi_{U_f}^\perp \Phi^T$ can lead to Γ_i according to the property of (16). In fact, we can see [2] that

N4SID method is a version of subspace method based on SVD of

$$\hat{L}_i \Phi = Y_f \Pi_{U_f}^\perp \Phi^T (\Phi \Pi_{U_f}^\perp \Phi^T)^{-1} \Phi$$

Furthermore in PI-MOESP methods [3], we obtain Γ_i via SVD of

$$Y_f \Pi_{U_f}^\perp U_p^T$$

while, replacing U_p with Φ in the above equation, we can see [6] it is no other than a subspace method based on SVD of

$$Y_f \Pi_{U_f}^\perp \Phi^T = R_{32} R_{22}^T.$$

To concluding this section, we point out that different versions of subspace methods correspond to various estimates of Γ_i 's; but all the estimates are essentially represented as

$$\Gamma_i = E_n T_s \quad (17)$$

where T_s is an arbitrary real nonsingular matrix. We should stress that one of the most principal point of subspace methods is extracting the extended observability matrix from input-output data perturbed by noise.

3 Robust identification [4]

Assume that a linear discrete time SISO system can be described by the following transfer function $G^0(z)$

$$G^0(z) = G(z) + \Delta(z) \quad (18)$$

where $\Delta(z)$ denotes an additive uncertainty, and $G(z)$ a nominal model represented by

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} \quad (19)$$

where z is a forward shift operator.

Consider a minimization problem of $\mathcal{H}_\infty$ norm of the additive uncertainty $\Delta(z)$:

$$J = \|\Delta(z)\|_\infty.$$

subject to the condition that the model can explain some given input and output

Fig. 1 The System

data. We can approach the problem with the *Carathéodory-Fejér* theorem:

« Theorem 1 »

$$\begin{aligned} \min_{p(z) \in \mathcal{H}_\infty} \|f_0 + \dots + f_{N-1}z^{-(N-1)} + z^{-N}p(z)\|_\infty \\ = \|f_0 + f_1z^{-1} + \dots + f_{N-1}z^{-(N-1)}\|_H \\ = \bar{\sigma}(F_N) \end{aligned} \quad (20)$$

where $\bar{\sigma}(\cdot)$ represents the maximum singular value of $(\cdot)$, and $N \times N$ Toeplitz matrix F_N is defined as

$$\begin{bmatrix} f_0 & & & O \\ f_1 & f_0 & & \\ \vdots & \ddots & \ddots & \\ f_{N-1} & \dots & f_1 & f_0 \end{bmatrix} \quad (21)$$

for a function "F(z)" ($f_0 + f_1z^{-1} + f_2z^{-2} + \dots$) or signal "f" ($f_0, f_1, f_2, \dots$). For the signals y, u , and transfer function $G^0(z)$, we denote the corresponding Toeplitz matrices by Y_N, U_N and G_N^0 , respectively. Then we have $Y_N = G_N^0 U_N$, and thus

$$G_N^0 = Y_N U_N^{-1} \quad (22)$$

provided $u_0 \neq 0$. Since $G^0(z) = G(z) + \Delta(z)$ from (18), denoting the Toeplitz matrices of $G(z)$ and $\Delta(z)$ by G_N and Δ_N , respectively, we see

$$\Delta_N = G_N^0 - G_N$$

and $G_N = \mathcal{B}_N \mathcal{A}_N^{-1}$ from (19), where $\mathcal{B}_N$ and $\mathcal{A}_N$ are the Toeplitz matrices for $b(z)$ and $a(z)$, respectively. Using above equation, we have

$$\Delta_N = Y_N U_N^{-1} - \mathcal{B}_N \mathcal{A}_N^{-1}.$$

Thus, the polynomial coefficients of Δ_N , $\delta_0, \delta_1, \dots, \delta_{N-1}$, are obtained up to $N-1$, and we can see from Theorem 1 that

$$\begin{aligned} \min_{p(z) \in \mathcal{H}_\infty} \|\delta_0 + \dots + \delta_{N-1}z^{-(N-1)} + z^{-N}p(z)\|_\infty \\ = \bar{\sigma}(\Delta_N) = \bar{\sigma}(Y_N U_N^{-1} - \mathcal{B}_N \mathcal{A}_N^{-1}) \end{aligned}$$

« Theorem 2 »

Assume that the system has no noise and be with zero initial condition. Given the input-output time series data ($u_i, y_i, u_0 \neq 0$), an error function $\Delta(z)$ exists such that $\|\Delta(z)\|_\infty \leq \varepsilon$ if and only if the following condition holds:

$$\bar{\sigma}(\Delta_N) = \bar{\sigma}(Y_N U_N^{-1} - \mathcal{B}_N \mathcal{A}_N^{-1}) \leq \varepsilon$$

Note that if the Toeplitz matrix $\mathcal{A}_N$ corresponding to the denominator of the model, can be obtained via some other schemes, then Δ_N is a linear function of $\mathcal{B}_N$ which is of the numerator. A method to obtain some kind of linear relation of the denominator has been proposed via formulating a series of estimates $\bar{\sigma}(\Delta_N^{(i)})$ for $\bar{\sigma}(\Delta_N)$ where

$$\Delta_N^{(i)} = (\mathcal{A}^{(i-1)} U_N)^{-1} (\mathcal{A}^{(i)} Y_N - \mathcal{B}^{(i)} U_N).$$

$\Delta_N^{(i)}$ is linear with respect to $\mathcal{B}^{(i)}$ as well as $\mathcal{A}^{(i)}$ and we may solve the problem numerically by iteration.

4 Nonlinear black-box model [5]

In this section, for simplicity, we consider the SISO case. Collect a finite number of past inputs u_k and outputs y_k into the vector φ_k

$$\varphi_k = [y_{k-1}, \dots, y_{k-n}, u_{k-1}, \dots, u_{k-n}]^T$$

The problem then is to understand the relationship between the next output y_k and φ_k :

$$y_k \leftrightarrow \varphi_k$$

Non-linear regression set-up

For a particular value φ_k the corresponding y_k is assumed to be

$$y_k = g_0(\varphi_k) + e_k \quad (23)$$

where $\{e_k\}$ is supposed to be a sequence of independent random variables, with zero mean and variance σ_e^2 . To find the function g_0 in (23) we have the following information available:

1. A parametrized family of functions
 $\mathcal{G}_m = \{g(\varphi_k, \theta) | \mathcal{D}_M\}$
2. A collection of observed y_k, φ -pairs:
 $Z^N = \{y_k, \varphi_k | k = 1, \dots, N\}$

The typical way to estimate g_0 is then to form the scalar valued function

$$J_N(\theta) = \frac{1}{N} \sum_{k=1}^N (y_k - g(\varphi_k, \theta))^2$$

and determine the parameter estimate $\hat{\theta}_N$ as its minimizing argument:

$$\hat{\theta}_N = \arg \min J_N(\theta)$$

The estimate of g_0 will then be

$$\hat{g}_N(\varphi) = g(\varphi, \hat{\theta}_N).$$

Usually we use function expansions of the type

$$g(\varphi, \theta) = \sum_{i=1}^q \theta_i g_i(\varphi)$$

where $\{g_i(\varphi)\}$ is a basis for the function $g_0(\varphi)$. Out of the many possible choice of basis functions, a large family of special ones have received most of the current interest. They are all based on just one fundamental function $\mu(\varphi)$, which is scaled in various ways, and centered at different points, *i.e.*

$$g_i(\varphi) = \mu(\beta_i^T \varphi + \gamma_i) = \mu(\varphi, \eta_i).$$

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