An Improvement of the Nonparametric Repair-Limit Replacement Method by a Cubic Spline Approximation

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Abstract

In this article, we consider a nonparametric repair-limit replacement (RLR) problem based on the scaled total time on test (TTT) statistics. It should be noted that the estimation procedure for the optimal repair-time limit via the scaled TTT statistics is made on the graph, but is not always stable when the sample size of repair time is small. In order to overcome this problem, we estimate the optimal repair-time limit on an approximated scaled TTT curve. In other words, we apply a smoothing technique based on the natural cubic spline function to interpolate the scaled TTT curve.

1 Introduction

The concept of the scaled total time on test (TTT) statistics and the scaled TTT plot plays an important role to characterize the aging property of the lifetime distribution [1]. On the other hand, this concept is applicable to solve several kinds of optimal maintenance problems. Bergman [2] discovered this graphical solution method for the simple age replacement problem. Koshimae, Dohi, Kaio and Osaki [3] and Dohi, Matsushima, Kaio and Osaki [4] applied the similar technique to the repair-limit replacement (RLR) problems. More precisely, in the classical RLR problems to derive the optimal repair-time limit which minimizes the expected cost per unit time in the steady-state, the underlying optimization problem can be reduced to a graphical one on the scaled TTT curve, since the expected cost function can be represented by the repair-time distribution function and the corresponding scaled TTT transform.

On the other hand, when the empirical data of repair time are given, the graphical method above can be used to estimate the optimal repair-time limit. That is, replacing the scaled TTT curve into the scaled TTT plot, we can estimate the optimal policy without specifying the repair-time distribution function. In addition, note that the nonparametric method based on the scaled TTT plot can guarantee the strongly consist estimator

of the optimal repair-time limit. This property is powerful and has the advantage of practical use. In general, since the specification of the distribution from data is not easy, it will be useful for practitioners to determine the maintenance schedule without selection of the repair-time distribution.

If one has to estimate the optimal repair-time limit from a small sample of repair-time data, however, the estimate is expected to be poor. Since the scaled TTT plot is a step function of the empirical distribution and the scaled TTT statistics, it can not represent the real (but unknown) scaled TTT curve from a small sample. This results from the fact that the scaled TTT plot is not analytic and is not always a suitable approximate function for the scaled TTT transform with unknown repair-time distribution. In this article, we propose a method to approximate the scaled TTT plot by an analytic function and attempt to estimate the optimal repair-time limit precisely from a small sample data. Then the natural cubic Spline function is used to interpolate the scaled TTT plot. In the following section, we describe the RLR problem. In Section 3, the TTT concept is introduced and the optimal RLR policy is derived on the scaled TTT curve. Section 4 is devoted to explain the empirical estimation method based on the TTT plot. In Section 5, we apply a smoothing technique based on the natural cubic spline function to interpolate the scaled TTT curve and propose an algorithm to estimate the optimal policy. Numerical illustrations are presented in Section 6. Finally, the paper concludes with some remarks in Section 7.

2 The RLR Problem

Consider a single unit system, where each spare is provided only by an order after a lead time L (> 0) and each failed unit is repairable. The original unit begins operating at time 0. The mean lifetime for each unit is $1/\lambda$ (> 0). When the unit has failed, the repair is started immediately. If the repair is completed up to the time limit for repair $t_0 \in [0, \infty)$, then the unit is installed at

that time. It is assumed that the unit once repaired is presumed as good as new. However, if the repair time is greater than t_0 , *i.e.* the repair is not completed up to the time t_0 , the repair is retired and the failed unit is scrapped. Then, the spare unit is ordered immediately and delivered after the lead time L . The time required for replacement is negligible for convenience. The repair time for each unit has an arbitrary distribution $G(t)$ with density $g(t)$ and finite mean $1/\mu$ (> 0), where the function $G(\cdot)$ is assumed to have an inverse function, *i.e.* $G^{-1}(\cdot)$, and to be absolutely continuous and strictly increasing. Without any loss of generality, we assume $G(0) = 0$ and $\lim_{t \rightarrow \infty} G(t) = 1$. Under these model assumptions, we define the time interval from the start of the operation to the following start as one cycle. The configuration of the RLR model under consideration is depicted in Fig. 1.

Fig. 1: The configuration of the RLR model.

Next, we consider the cost structure. The costs considered in this paper are the following:

k_r (> 0): a cost per unit repair time

k_s (> 0): a cost per unit shortage period

c (> 0): an ordering cost for each spare unit.

We make the assumption:

(A-1) $k_r L < c$.

This assumption implies that the unit ordering cost is greater than the repair cost during the interval $[0, L]$, *i.e.* until the delivery of a new unit. For an infinite planning horizon, it will be appropriate to adopt an expected cost

per unit time in the steady-state. Since the mean time of one cycle is

$$T(t_0) = \int_0^{t_0} (1/\lambda + t) dG(t) + \int_{t_0}^{\infty} (1/\lambda + t_0 + L) dG(t) \\ = 1/\lambda + \int_0^{t_0} \bar{G}(t) dt + L\bar{G}(t_0) \quad (1)$$

and the total expected cost for one cycle is

$$V(t_0) = (k_r + k_s) \int_0^{t_0} \bar{G}(t) dt + (k_s L + c) \bar{G}(t_0), \quad (2)$$

where $\bar{G}(t) = 1 - G(t)$, then the expected cost per unit time in the steady-state is, from the well-known renewal reward argument [5],

$$C(t_0) \equiv \lim_{t \rightarrow \infty} \frac{[\text{the total cost on } (0, t]]}{t} \\ = V(t_0)/T(t_0) \quad (3)$$

and the problem is to determine the optimal repair-time limit t_0^* such as

$$C(t_0^*) = \min_{0 \leq t_0 < \infty} C(t_0). \quad (4)$$

It is straightforward to seek t_0^* by differentiating $C(t_0)$ with respect to t_0 , but we employ a different graphical method in the following section.

3 The TTT Concept

Before solving the problem in Eq.(4), we describe the total time on test (TTT) transform for the repair time distribution. Define the scaled total time on test (TTT) transform of the repair-time distribution $p \equiv G(t)$ by

$$\phi(p) \equiv \mu \int_0^{G^{-1}(p)} \bar{G}(t) dt, \quad 0 \leq p \leq 1, \quad (5)$$

where

$$G^{-1}(p) = \inf\{t \geq 0 : G(t) \geq p\}. \quad (6)$$

The curve $\mathcal{L} = (p, \phi(p)) \in [0, 1] \times [0, 1]$ is called the *scaled TTT transform* or simply the *scaled TTT curve*. We shall propose a graphical method to solve the problem in Eq.(4) on the scaled TTT curve.

The following result is due to Koshimae, Dohi, Kaio and Osaki [3].

Lemma 3.1: Suppose that the assumption (A-1) holds. The minimization problem in Eq.(4) is equivalent to obtain p^* ($0 \leq p^* \leq 1$) satisfying

$$\min_{0 \leq p \leq 1} : M(p, \phi(p)) \equiv \frac{\phi(p) + \xi}{p + \eta}, \quad (7)$$

where

$$\xi \equiv \frac{(k_s L + c)\mu}{(c - k_r L)\lambda} > 0 \quad (8)$$

and

$$\eta \equiv -\left(1 + \frac{k_r + k_s}{(k_r L - c)\lambda}\right). \quad (9)$$

From Lemma 3.1, the optimal policy is $p^* = G(t_0^*)$ which minimizes the tangent slope from $(-\eta, -\xi)$ to the curve $\mathcal{L}$. Hence, it is straightforward to calculate $t_0^* = G^{-1}(p^*)$.

More precisely, we characterize the optimal policy from the aging property of $G(t)$.

Definition 3.2: (i) The repair-time distribution $G(t)$ is IRR (DRR) if and only if the instantaneous repair rate $r(t) = g(t)/\bar{G}(t)$ is increasing (decreasing).
(ii) $G(t)$ is IRR (DRR) if and only if $\phi(p)$ is concave (convex) in $p \in [0, 1]$.

The relationship (ii) between the aging and the scaled TTT transform was proved by Barlow and Campo [1]. In the plane $(x, y) = (-\infty, +\infty) \times (-\infty, +\infty)$, define the following three points

$$B \equiv (x_B, y_B) = (-\eta, -\xi), \quad (10)$$

$$Z \equiv (x_Z, y_Z) = \left(\frac{(k_s L + c)r(0)}{(k_r L - c)\lambda}, -\xi\right) \quad (11)$$

and

$$I \equiv (x_I, y_I) = \left(-\eta, 1 + \frac{(k_r + k_s)\mu}{(k_r L - c)\lambda r(\infty)}\right). \quad (12)$$

Theorem 3.3: (1) Suppose that the scaled TTT curve $\mathcal{L}$ is strictly convex under the assumption (A-1).

(i) If $x_B > x_Z$ and $y_B > y_I$, then there exists a unique optimal solution $p^* = G(t_0^*)$ ($0 < t_0^* < \infty$) minimizing the expected cost per unit time in the steady-state given by Eq. (3), where p^* is given by the x -coordinate in the point of contact for the curve $\mathcal{L}$ from the point B, where

$$\max(0, -\eta) < p^* < 1. \quad (13)$$

(ii) If $x_B \leq x_Z$, then the optimal repair-limit policy is $p^* = G(0) = 0$.

(iii) If $y_B \leq y_I$, then the optimal repair-limit policy is $p^* = G(\infty) = 1$.

(2) Suppose that the scaled TTT curve $\mathcal{L}$ is concave under the assumption (A-1). Then, the optimal solution is $p^* = 0$ or $p^* = 1$.

Proof: Differentiating $M(p, \phi(p))$ with respect to p and setting it equal to zero implies

$$q(p) \equiv \phi'(p)(p + \eta) - (\phi(p) + \xi) = 0, \quad (14)$$

where

$$\phi'(p) = \frac{\mu}{r(G^{-1}(p))} \quad (15)$$

and the symbol ' denotes the differentiation. Further, we have

$$q'(p) = \phi''(p)(p + \eta). \quad (16)$$

When the scaled TTT curve $\mathcal{L}$ is strictly convex, then $q'(p) > 0$ and the function $M(p, \phi(p))$ is strictly convex in p .

In the plane $(x, y) \in (-\infty, +\infty) \times (-\infty, +\infty)$, we define the point B = (x_B, y_B) . Since the tangent line for the point $(p^*, \phi(p^*))$ on the curve $\mathcal{L}$ is

$$y = \frac{\mu}{r(G^{-1}(p^*))}(p - p^*) + \phi(p^*), \quad (17)$$

the condition that the point B is on the above tangent line is $q(p^*) = 0$. Define the intersection Z = (x_Z, y_Z) of the tangent line for the origin O = $(0, 0)$ on the curve $\mathcal{L}$ and $y = -\xi$. If the x -coordinate of B is strictly greater than the x -coordinate of Z, $q(0) < 0$, otherwise, $q(0) \geq 0$ under the assumption (A-1). Similarly, define the intersection I = (x_I, y_I) of the tangent line for the point U = $(1, 1)$ on the $\mathcal{L}$ and $x = -\eta$. If the y -coordinate of B is strictly greater than the y -coordinate of I, $q(1) > 0$, otherwise, $q(1) \leq 0$ under (A-1). From these, we obtain the results (1).

Secondly, consider the case where $G(t)$ is IRR. In this case, $\phi(p)$ becomes a concave function of p . If the x -coordinate of B is strictly negative and if the slope of the straight line BO is strictly smaller than that of the line BU, we have

$$(k_s L + c)/\lambda - \left\{k_r L - c + (k_r + k_s)/\lambda\right\}/\mu < 0, \quad (18)$$

which is equivalent to the condition of $M(0, \phi(0)) < M(1, \phi(1))$. Conversely, if $x_B < 0$ and if the slope of the straight line BO is not small than that of the line BU, $M(0, \phi(0)) \geq M(1, \phi(1))$ is satisfied. On the other hand, the condition $x_B \geq 0$ implies $M(0, \phi(0)) > M(1, \phi(1))$. Thus the proof is completed.

4 An Empirical Method

Based on the graphical ideas in Section 3, we propose a statistical method to estimate the optimal RLR policy. Suppose that the optimal repair-time limit has to be estimated from an ordered complete sample $0 = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n$ of repair times from an absolutely continuous repair-time distribution G , which is unknown. The estimator of $G(t) = p$ is the empirical distribution given by

$$G_n(x) = \begin{cases} i/n & \text{for } x_i \leq x < x_{i+1}, \\ 1 & \text{for } x_n \leq x \end{cases} \quad (19)$$

($i = 0, 1, 2, \dots, n-1$). Then the scaled TTT statistics based on this sample are

$$\phi_{i,n} \equiv S_i/S_n, \quad (20)$$

where $S_0 = 0$ and

$$S_i \equiv \sum_{j=1}^i (n-j+1)(x_j - x_{j-1}), \quad i = 1, 2, \dots, n. \quad (21)$$

By plotting the point $(i/n, \phi_{i,n})$, ($i = 0, 1, 2, \dots, n$), and connecting them by line segments, we obtain the so-called *scaled TTT plot*, $\mathcal{L}_n \in [0, 1] \times [0, 1]$.

As an empirical counterpart of Lemma 3.1, we propose a nonparametric estimator of the repair-time limit.

Theorem 4.1: The optimal repair-time limit can be estimated by $\hat{t}_{0n} = x_{i^*}$, where

$$\left\{ i^* \mid \min_{0 \leq i \leq n} \frac{\phi_{i,n} + \xi}{i/n + \eta} \right\}. \quad (22)$$

The proof is omitted for brevity.

For a better understanding of the theorem above, we give the following simple example.

Example 4.2: The repair-time data were made by the random number following the Weibull distribution with shape parameter $\alpha = 0.2$ and scale parameter $\beta = 2.0$. The other model parameters are $1/\lambda = 0.8000$ [day], $L = 0.2000$ [day], $c = 6.5000$ [\$], $k_r = 4.0000$ [\$] and $k_s = 6.0000$ [\$]. The scaled TTT plot based on the 200 sample data is shown in Fig. 2. Since $B = (-0.4035, -0.4771)$, the optimal point with minimum slope from B becomes $(i^*/n, \phi_{i^*,n}) = (119/200, \phi_{119,200}) = (0.5980, 0.4873)$. Thus, the estimator of the optimal repair-time limit $\hat{t}_{0,200} = x_{119} = 1.7006$.

Of our next interest is the investigation of asymptotic property of the estimator $\hat{t}_{0n}$ in Theorem 4.1. The following theorem guarantees the asymptotic optimality of the estimator above.

Theorem 4.3: (i) The expected cost $C(\hat{t}_{0n})$ of using the repair-time limit $\hat{t}_{0n}$ tends with probability one to $C(t_0^*)$ as n tends to infinity if t_0^* is positive and finite. (ii) The minimum expected cost per unit time in the steady-state $C(t_0^*)$ may be estimated by

$$\hat{C}(\hat{t}_{0n}^*) = \frac{(k_r + k_s)\phi_{i^*,n}^*/\mu_n + (k_s L + c)(1 - p_n^*)}{1/\lambda + \phi(p_n^*)/\mu_n + L(1 - p_n^*)} \quad (23)$$

where $1/\mu_n$ is the empirical mean of the repair time. Then the estimator is strongly consistent.

(iii) If a unique optimal repair-time limit exists then $\hat{t}_{0n}$ is strongly consistent.

The results above can be seen from the analogies by Bergman [2]. However, it will be clear that the estimator $\mathcal{L}_n$ for $\mathcal{L}$ is very rough for the small sample data,

since its asymptotic convergence rate is rather small experimentally. For instance, we need the number of data more than 100 to obtain a nice value closed to the real optimum. Hence, we propose to approximate $\phi_{i,n}$ by using the natural cubic spline function in the following section.

Fig. 2: Estimation of the optimal repair-time limit on the scaled TTT plot.

5 The Spline Approximation

During the last years, spline functions have found widespread application, mainly for the purpose of interpolation. The spline function is one of the piecewise continuous polynomials which is useful to interpolate arbitrary discrete points so as to satisfy any continuous condition. In the two-dimensional plane, let $x_i, y_i, i = 0, 1, \dots, n$ be given and assume $x_0 = 0 < x_1 < \dots < x_n = 1$ and $y_0 = 0 < y_1 < \dots < y_n = 1$. Then the function $z(x)$ is called the spline function with order m , where $x = (x_0, x_1, \dots, x_n)$, if $z(x)$ is the polynomial whose order is less than or equal to m in the interval $[x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) and is belonging to C^{m-1} class. Also, the spline function $z(x)$ is called the N (natural)-spline function, if $z(x)$ with order $2k-1$ ($k = 1, 2, \dots$) can be represented by a polynomial with order $k-1$ in the two intervals, $(-\infty, x_1]$ and $[x_n, \infty)$.

In general, it is well known that spline functions are better to approximate arbitrary functions than other polynomials with same number of parameters. Also, if we take account of the computational effort and precision in functional approximation, the natural cubic

spline is probably the best candidate to use for interpolation. From the reason above, we approximate the scaled TTT plot by the natural cubic spline function. For $n + 1$ points $(x_i, y_i) = (i/n, \phi_i)$, $(i = 0, 1, \dots, n)$ on the scaled TTT plot and $t \in [x_i, x_{i+1}]$, define the piecewise continuous polynomial with order three, $z_i(t)$, which is through the point x_i , as

$$z_i(t) = \frac{(x_{i+1} - t)^3}{6h_i} M_i + \frac{(t - x_i)^3}{6h_i} M_{i+1} + A_i(x_{i+1} - t) + A_{i+1}(t - x_i), \quad (24)$$

where

$$h_i = x_{i+1} - x_i \quad (25)$$

and

$$A_i = \frac{y_i}{h_i} - \frac{h_i}{6} M_i, \quad i = 0, 1, \dots, n-1. \quad (26)$$

In Eq. (24), the parameter M_i ($i = 0, 1, \dots, n$) is called the spline coefficient and satisfies the following simultaneous equations.

$$\frac{1}{4} M_{i-1} + M_i + \frac{1}{4} M_{i+1} = \frac{3(y_{i-1} - 2y_i + y_{i+1}))}{2h_i^2}, \quad i = 1, 2, \dots, n-1, \quad (27)$$

$$M_0 + \frac{1}{2} M_1 = \frac{3}{h_i} \left(\frac{y_1 - y_0}{h_i} \right), \quad (28)$$

$$M_{n-1} = \frac{y_n - 2y_{n-1} + y_{n-2}}{h_i^2}. \quad (29)$$

Finally, we can determine the cubic spline function $z_i(t)$ ($i = 0, 1, \dots, n-1$) for all $t \in [x_0, x_n] = [0, 1]$ with M_i by solving the simultaneous equations numerically.

We develop an algorithm to estimate the optimal repair-time limit by approximating the scaled TTT plot.

(Step 1) Observe n order statistics of the repair-time data $x_1, x_2, \dots, x_n$.

(Step 2) Plot $(p_{i,n}, \phi_{i,n}) = (i/n, \phi_i)$, $i = 0, 1, \dots, n$ on $\mathcal{R}^2 \in [0, 1] \times [0, 1]$.

(Step 3) Calculate the Spline coefficients from Eqs. (27)-(29) and interpolate the points $(p_{i,n}, \phi_{i,n})$ by the cubic N-Spline curve.

(Step 4) Derive the x -coordinate of the point to minimize the tangent slope from B to the N-Spline curve.

(Step 5) Interpolate the empirical distribution in Eq.(19) by the N-Spline curve and generate an approximated continuous distribution. Taking the inversion transform, seek the estimate of the optimal repair-time limit.

6 Numerical Illustration

In this section, we compare two estimation procedures in terms of predictive ability. Figure 3 shows respective estimates of the optimal repair-time limit for two methods, where we distinguish the existing method and the approximated method developed in Section 5. From this figure, it is observed that New Method can give the better estimation performance than Existing one and that both estimates tend to take the same value as the number of repair-time data increases. Roughly speaking, two methods provide the similar result for more than 26 data. In Fig. 4, we examine the behavior of the estimate for minimum expected cost value. Comparing with Fig. 3, it is concluded that New Method can estimate the corresponding cost value precisely than Existing one, even if the number of data becomes large.

Fig. 3: Behavior of the Estimate for Optimal Repair-Time Limit.

Fig. 4: Behavior of the Estimate for Minimum Expected Cost.

In order to investigate the dependence of the number of repair-time data in estimates, we calculate the

mean squared error between estimates and the real value for the optimal repair-time limit and the minimum expected cost in Tables 1 and 2, respectively, using the different data from Figs. 3 and 4. From these results, we conclude that the method based on the Spline approximation can be recommended to estimate the optimal repair-time limit, especially for the small sample problem.

Table 1: Comparison of Two Methods for the Optimal Repair-Time Limit.

No. Data	New Method	Existing Method
5	0.076178	0.224651
10	0.051879	0.227414
15	0.019075	0.194286
20	0.009594	0.194286
25	0.055516	0.211917
30	0.081512	0.246708
35	0.069092	0.259429
40	0.075008	0.255393
45	0.089604	0.246708
50	0.089478	0.246708

Table 2: Comparison of Two Methods for the Minimum Expected Cost.

No. Data	New Method	Existing Method
5	0.711362	0.711362
10	0.652048	0.652048
15	0.678579	1.472546
20	0.678579	1.472546
25	0.894121	0.894121
30	0.126263	0.126263
35	0.292141	0.349452
40	0.023012	0.023012
45	0.126263	0.126263
50	0.126263	0.126263

7 Concluding Remarks

Numerical examples show that the approximated method proposed here gives the better predictive performance than the existing TTT method. It should be noted that the improved method never guarantee the strong consistency for the estimator of the optimal repair-time limit, but is useful for the small sample problem which is an important problem in actual maintenance engineering practice.

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