On the replicating portfolio of some exotic options

Takahiko Fujita
fujita@math.hit-u.ac.jp
and
Sachiyo Futagi
ggcd911@srv.cc.hit-u.ac.jp
Faculty of Commerce
Hitotsubashi University

Abstract

The aim of this paper is to calculate the replicating portfolio of two geometric average options in the Black-Sholes model. Seeing this calculation, we can observe that the delta hedge of these options have simple forms. So, it is easy to create the hedging scheme of these options.

1 Introduction

Kemna and Voast [4] obtain an explicit formula for pricing a geometric average fixed strike option, using that the distribution of the geometric average of lognormal distribution is also lognormal. But they do not refer to the delta hedge of the option.

Geman and Yor [3] present an analytical study of Asian options with payoff dependent on the arithmetic average of the underlying asset. They give a closed-form expression of the Asian option price when the option is in the money. The form is similar to the Black-Sholes formula. But it seems rather complicated to use in practice.

In this paper, we would like to show closed-formulas of delta hedges for a geometric average fixed strike option with pay-off max{\(M(T) - K\)} and a geometric average floating strike option with pay-off max{\(S(T) - M(T)\)} in the Black-Sholes model where \(M(T)\) is defined as a geometric average of the stock price until the expiration \(T\).

The rest of the paper is organized as follows. In the second section, we review the replication. In the third section, we review the pricing for two average options. In the fourth section, we execute evaluating the delta hedge of a geometric average fixed strike option with pay-off max{\(M(T) - K\)}. Seeing this, we seem to have the easily understandable closed-form of the delta hedge \(\phi_t\). From this point, this option can be a very tractable derivative. In the fifth section, we execute evaluating the delta hedge of a geometric average floating strike option with pay-off max{\(S(T) - M(T)\)}. The last section concludes this paper.

2 Review of the replication of derivatives

We assume a perfect market which consists of a stock and a riskless bond. In addition, we assume that the current time is zero and the expiration is \(T\). The bond price \(B(t)\) follows

\[ dB(t) = rB(t)\, dt. \]
\[ B(0) = 1 \]

where \(r\) is constant. Under the risk-neutral probability \(Q\), the stock price \(S(t)\) follows

\[ dS(t) = rS(t)\, dt + \sigma S(t)\, dw(t). \]
\[ S(0) = S \]
for all time $t$ not greater than $T$, where $w(t)$ is a standard Brownian motion.

Following Baxter and Rennie [2], replicating portfolio can be defined as follows.

Replicating portfolio of a derivative $D$ with a payoff $X$ at the time $T$ in the Black-Scholes Model: Suppose we are in a market of a riskless bond $B$ with riskless interest $r$ and a risky security $S$ with volatility $\sigma$ and given a derivative $D$ on events up to time $T$. A replicating portfolio for $D$ is defined as a self-financing portfolio $(\phi_t, \psi_t)$ such that $\phi_T S_T + \psi_T B_T = X$ and $E \left[ \int_0^T \phi_t^2 S(t)^2 dt \right] < \infty$.

Self-financing portfolio: A portfolio $(\phi_t(\text{previsible}), \psi_t)$ which describe respectively the number of units of stock $S$ and of riskless bond $B$ which we hold at time $t$ has $V_t = \phi_t S_t + \psi_t B_t$ as its value. Then the portfolio is called self-financing if $dV_t = \phi_t dS_t + \psi_t dB_t$, which means that the change in its value only depends on the change of the stock price and the riskless bond price.

After this, we specify how to make the self-financing portfolio and the price of the options. We define $E_t$ as a conditional expectation under $Q$:

$$E_t = E \left[ B_t^{-1} X \mid \mathcal{F}_t \right]$$

where the filtration $\mathcal{F}_t$ is the history of the stock up to time $t$.

We define $Z_t$ as the discounted stock price: $Z_t = B_t^{-1} S_t$.

From the martingale representation theorem, there uniquely exists a previsible process $\phi_t$, which satisfies the condition that $E \left[ \int_0^T \phi_t^2 S(t)^2 dt \right] < \infty$ and $E_t$ can be written as

$$dE_t = \phi_t dZ_t + \sigma \phi_t S_t dW(t)$$

because the discounted stock price $Z_t$ and the conditional expectation $E_t$ are both martingale under $Q$.

Then we know that $(\phi_t, \psi_t)$ becomes a replicating portfolio where $\psi_t$ is defined as $\psi_t = E_t - \phi_t Z_t$.

Thus we can prove that

$$B_t E_t = \phi_t S_t + \psi_t B_t (= V_t)$$

$$dV_t = \phi_t dS_t + \psi_t dB_t$$

hold for all time $t$ not greater than $T$ and especially

$$E_0 = \phi_0 S_0 + \psi_0 B_0 (= V_0)$$

$$B_T E_T = \phi_T S_T + \psi_T B_T (= X).$$

So, from no arbitrage point of view, we can identify a derivative $D$ as its replicating portfolio as above.

Then we remark that $E_0$ is the price of this option at time 0. And we call $\phi_t$ the delta hedge, which is a very important parameter not only theoretically but also practically.

For example, the delta hedge for any derivative with any pay-off dependig only on the final stock price in the Black-Scholes model can be easily calculated and given by $\phi_t = \frac{\partial V}{\partial S}$. This reason goes as follows. Its value at the time $t$ is defined as $V_t = V(S_t, t)$ and given by

$$V(S_t, t) = E \left[ e^{-r(T-t)} F(S_T) \mid S_t = S \right]$$

where a derivative has a pay-off $F(S_T)$ at the time $T$. Then $V_t$ has the following form:

$$dV_t = \sigma S_t \frac{\partial V}{\partial S} dw(t) + \left( r S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \right) dt$$

From the self-financing condition, we have

$$dV_t = \sigma S_t \frac{\partial V}{\partial S} dw(t) + rV_t dt$$

because $dB(t) = r B(t) dt$, $dS(t) = r S(t) dt + \sigma S(t) dw(t)$ and $V_t = \phi_t S_t + \psi_t B_t$. Comparing volatility terms and drift terms of the above two SDEs, we have $\phi_t = \frac{\partial V}{\partial S}$ and

$$r S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} = rV.$$

However it must be more difficult to obtain the delta hedge of derivatives with pay-off depending on various conditions, for example, derivatives with pay-off depending on the whole trajectory (exotic options), than with pay-off depending only on the final stock price in the Black-Scholes model as the above example.

### 3 Pricing for two geometric average options
In this section, we review the pricing for two geometric average options. 

$M(t)$ is defined as a geometric average at the time $t$ and represented as follows:

$$M(t) = e^{\frac{1}{2} \int_0^t \log S(r) \, dr}$$

where

$$S(t) = S \exp \left[ \left( r - \frac{\sigma^2}{2} \right) t + \sigma w(t) \right]$$

So, $M(t)$ can be represented as

$$M(t) = S \exp \left[ \left( r - \frac{\sigma^2}{2} \right) t + \frac{\sigma}{t} \int_0^t w(u) \, du \right]$$

The payoff of a geometric average fixed strike option is defined as $\max \{M(T) - K, 0\}$, where $K$ is the exercise price.

Let $E^0$ be the option price at $0$. Then, $E^0$ can be represented as

$$E^0 = E \left[ e^{-rT} \max \{M(T) - K, 0\} \right]$$

The distribution of $M(T)$ is lognormal:

$$\log(M(T)) \sim N \left( \log S + \frac{T}{2} \left( r - \frac{\sigma^2}{2} \right), \frac{T}{3} \sigma^2 \right)$$

where $N(\mu, \sigma^2)$ represents a normal distribution with a parameter $(\mu, \sigma^2)$.

As a result, a geometric average fixed strike option price follows an explicit formula shown in A.G.Z.Kemna and A.C.F.Voast [4]:

$$E^0 = S e^{-\frac{T}{2} \left( r - \frac{\sigma^2}{6} \right)}$$

$$\Phi \left( \frac{1}{\sigma} \sqrt{\frac{3}{T}} \left\{ \log \left( \frac{S}{K} \right) + \frac{T}{2} \left( r + \frac{\sigma^2}{6} \right) \right\} \right)$$

$$- K e^{-rT}$$

$$\Phi \left( \frac{1}{\sigma} \sqrt{\frac{3}{T}} \left\{ \log \left( \frac{S}{K} \right) + \frac{T}{2} \left( r - \frac{\sigma^2}{2} \right) \right\} \right)$$

$$\Phi \left( \frac{1}{\sigma} \sqrt{\frac{3}{T}} \left\{ \log \left( \frac{S}{K} \right) + \frac{T}{2} \left( r - \frac{\sigma^2}{2} \right) \right\} \right)$$

where $\Phi$ is the standard normal distribution function.

The question we have to solve in this section is the replicating portfolio $(\phi^1, \psi^1)$ of a geometric average fixed strike option with pay-off $\max \{M(T) - K, 0\}$.

If $E^1$ is defined as a discounted conditional expectation for a geometric average fixed strike option for $0 \leq t \leq T$, $E^1$ can be represented as follows:

$$E^1 = E \left[ e^{-rT} \max \{M(T) - K, 0\} \mid \mathcal{F}_t \right]$$

$$= E \left[ e^{-rT} \max \left\{ S \exp \left[ \left( r - \frac{\sigma^2}{2} \right) T + \frac{\sigma}{T} \int_0^t w(u) \, du \right] \right\} \right]$$

$$- K, 0 \} \mid \mathcal{F}_t \}$$

Here we define that $\tilde{w}(u) = w(u + t) - w(t)$, for the reason that $\int_0^T w(u) \, du$ can be rewritten:

$$\int_0^T w(u) \, du = \int_0^t w(u) \, du + (T - t)w(t)$$

$$+ \int_0^{T-t} \{w(u + t) - w(t)\} \, du$$

It is clear that $\int_0^{T-t} \tilde{w}(u) \, du$ is independent of $\mathcal{F}_t$ and the distribution of $\int_0^{T-t} \tilde{w}(u) \, du$ is normal:

$$\int_0^{T-t} \tilde{w}(u) \, du \sim N \left( 0, \frac{1}{3} (T - t)^3 \right)$$

4 The delta hedge of a geometric average fixed strike option

The distribution of $(\log M(T), \log S(T))$ is joint normal:

$$(\log M(T), \log S(T)) \sim N \left( \begin{pmatrix} \log S + \frac{T}{2} \left( r - \frac{\sigma^2}{2} \right) \\ \log S + T \left( r - \frac{\sigma^2}{2} \right) \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{4} T & \frac{\sigma^2}{2} T \\ \frac{\sigma^2}{2} T & \sigma^2 T \end{pmatrix} \right)$$

From this, we can derive the closed-form of the price as follows:

$$E^2 = S \left\{ \Phi \left( \frac{1}{\sqrt{T}} \left( r - \frac{\sigma^2}{2} \right) \right) \right\}$$

$$- e^{-\frac{T}{2} \left( r - \frac{\sigma^2}{3} \right)} \Phi \left( \frac{T}{2} \left( r + \frac{2\sigma^2}{3} \right) \right) \}$$

-109-
Consequently, we can calculate $E_t$ as follows.

\[
E_t^1 = E \left[ e^{-RT} \max \{ M(T) - K, 0 \} \right] w(t), \int_0^t w(u) \, du \\
= E \left[ e^{-RT} \max \left\{ S \exp \left( \left( r - \frac{\sigma^2}{2} \right) T + \frac{\sigma}{\sqrt{T}} \int_0^T w(u) \, du \right) \right\} \right] \\
+ \frac{\sigma}{T} \left\{ A + (T-t)B + \int_0^{T-t} w(u) \, du \right\} \\
-K, 0 \\
\right]_{w(t)=A,\int_0^tw(u)\,du=B} \\
= \int_{-\infty}^{+\infty} e^{-rT} \max \left\{ Se^{\left( r - \frac{\sigma^2}{2} \right) \frac{T}{2} \frac{T}{3} (T-t)^3} \right\} \\
\exp \left[ \frac{1}{2} \left\{ u - \int_0^t w(u) \, du - (T-t)w(t) \right\}^2 \right] \, du \\
= S(t) \left[ \frac{3}{2} M(t) \frac{1}{2} e^{CT} \left( D + \frac{\sigma}{T} \sqrt{\frac{1}{3} (T-t)^3} \right) \\
- Ke^{-RT} \Phi(D) \right]
\]

where

\[
C = \frac{-r}{2T} (3T^2 - 2T + t^2) + \frac{\sigma^2}{4T} (T-t)^3
\]

\[
D = \frac{1}{\sigma \sqrt{\frac{1}{3} (T-t)^3}} \left\{ -T \log K + t \log M(t) \right\} \\
+(T-t) \log S(t) + \frac{1}{2} (T-t)^2 \left( r - \frac{\sigma^2}{2} \right)
\]

Moreover Ito's formula gives:

\[
dE_t^1 = \int_{-\infty}^{+\infty} e^{-rT} \max \left\{ Se^{\left( r - \frac{\sigma^2}{2} \right) \frac{T}{2} \frac{T}{3} (T-t)^3} - K, 0 \right\} \\
\exp \left[ \frac{1}{2} \left\{ u - \int_0^t w(u) \, du - (T-t)w(t) \right\}^2 \right] \, dw(t)
\]

\[
dw(t)
\]

Consequently,

\[
\phi_t^1 = \frac{1}{\sigma S(t) \int_{-\infty}^{+\infty} e^{-r(T-t)} \max \left\{ Se^{\left( r - \frac{\sigma^2}{2} \right) \frac{T}{2} \frac{T}{3} (T-t)^3} - K, 0 \right\} \\
\exp \left[ \frac{1}{2} \left\{ u - \int_0^t w(u) \, du - (T-t)w(t) \right\}^2 \right] \, du \right]
\]

Of course, we can have the same result(1) as the previous section when $t = 0$. 

-110-
where \( \phi \) is the standard normal density function. Moreover \( \psi^1 \) can be decided as defined before, putting the obtained \( \phi^1 \) to the definition. Here, it is worth noting that \( \phi^1 \) is easily evaluated and this fact gives us managing this option easily.

5 The delta hedge of a geometric average floating strike option

The question we have to solve in this section is the replicating portfolio \( (\phi^2, \psi^2) \) of a geometric average floating strike option with pay-off \( \max\{S(T) - M(T), 0\} \).

If \( E_t^2 \) is defined as a discounted conditional expectation for a geometric average floating strike option for \( 0 \leq t \leq T \), \( E_t^2 \) can be represented as follows:

\[
E_t^2 = E \left[ e^{-rT} \max\{S(T) - M(T), 0\} \big| F_t \right] \\
= E \left[ e^{-rT} \max\left\{ S \exp\left[ \left( r - \frac{\sigma^2}{2} \right) T + \sigma w(T) \right] - S \exp\left[ \left( r - \frac{\sigma^2}{2} \right) T + \frac{\sigma^2}{T} \int_0^T w(u) \, du \right], 0 \right\} \bigg| F_t \right]
\]

If \( \bar{w}(t) \) is defined in previous section, \( \bar{w}(T-t) \) and \( \int_0^{T-t} \bar{w}(u) \, du \) are independent of \( F_t \).

Thus \( E_t \) can be rewritten as follows:

\[
E_t^2 = E \left[ e^{-rT} \max\left\{ S \exp\left[ \left( r - \frac{\sigma^2}{2} \right) T + \sigma w(T) \right] + \sigma (\bar{w}(T-t) + w(t)) \right\} \right]
\]

Provided that random variables \( U \) and \( V \) such that

\[
U = B + \bar{w}(T-t), \quad V = A + (T-t)B + \int_0^{T-t} \bar{w}(u) \, du,
\]

the distribution of \( (U, V) \) is normal:

\[
(U, V) \sim N \left( \left( \frac{B}{A + (T-t)B}, \frac{T-t}{2}(T-t)^2 \right), \frac{1}{3}(T-t)^3 \right)
\]

In the same way as the previous section, Ito's formula gives:

\[
dE_t^2 = \int_{-\infty}^{t} \int_{-\infty}^{t} e^{-rT} S \left( \max \left\{ e^{(r - \frac{\sigma^2}{2}) T + \sigma u} - e^{(r - \frac{\sigma^2}{2}) \frac{T}{T} + \frac{\sigma^2}{T} \int_0^T w(u) \, du}, 0 \right\} \right) \left( \frac{-2\sqrt{3} \left( 2(T-t)u - 3v + 3 \int_0^t w(u) \, du + (T-t)w(t) \right)}{\pi(T-t)^4} \exp \left( \frac{-2}{(T-t)^{\frac{3}{2}}} \left( \int_0^t w(u) \, du - v \right)^2 \right) \right) \right) \, du \, dv
\]

In the same way as the previous section, we can obtain as follows:

\[
\phi^2_t = \frac{1}{\sigma S_t} \int_{-\infty}^{t} \int_{-\infty}^{t} e^{-r(T-t)} S \max \left\{ e^{(r - \frac{\sigma^2}{2}) T + \sigma u} - e^{(r - \frac{\sigma^2}{2}) \frac{T}{T} + \frac{\sigma^2}{T} \int_0^T w(u) \, du}, 0 \right\} \left( \frac{-2\sqrt{3} \left( 2(T-t)u - 3v + 3 \int_0^t w(u) \, du + (T-t)w(t) \right)}{\pi(T-t)^4} \exp \left( \frac{-2}{(T-t)^{\frac{3}{2}}} \left( \int_0^t w(u) \, du - v \right)^2 \right) \right) \right) \, du \, dv
\]

-111-
Moreover \( \psi_t^2 \) can be decided as defined before, putting the obtained \( \phi_t^2 \) to the definition.

6 Conclusion

The essential points of this paper is to present not only closed-forms of the replicating portfolio of a geometric average fixed strike option and a geometric average floating strike option but also the easy calculation of the replication.

In general, average options are said to be cheaper and easier to be hedged than standard options. Indeed, "delta" and "gamma", which are considered as sensitivity parameters and components in the hedging strategy, seem to be small. Moreover, Ge- man and Yor point out an uncommon situation of "delta" and "gamma" of Asian call options. So, we expect such a phenomenon occur in our case.

Using results of our paper, we would like to go on to study sensitivity analysis further.

References


