Generalized Kalman Filter When Exogenous Variables Exist

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Abstract

A generalized Kalman filter is proposed, which can be applied to any state space model with exogenous variables or inputs.

1 State space model with exogenous variables

We begin with a state space model,

$$x(t+1) = Fx(t) + Lz(t) + \xi(t), \quad (1)$$

$$y(t) = Hx(t) + \eta(t). \quad (2)$$

Here $x(t)$ is an n -dimensional state vector and $z(t)$ is an m -dimensional vector of exogenous variables. The state $x(t)$ is updated to $x(t+1)$ with noise $\xi(t)$ as in (1). The output $y(t)$ is observed with noise $\eta(t)$ as in (2). The $F \in \mathbf{R}^{n \times n}$, $L \in \mathbf{R}^{m \times n}$ and $H \in \mathbf{R}^{n \times s}$ are matrices of the coefficients. We derive a generalized Kalman filter under the following assumptions.

$$E[\xi(s)\xi(t)'] = \delta_{st}Q, \quad E[\eta(s)\eta(t)'] = \delta_{st}R,$$

$$E[\xi(s)\eta(t)'] = \delta_{st}S,$$

$$E[\xi(t)x(0)'] = E[\eta(t)x(0)'] = 0,$$

$$E[z(s)\xi(t)'] = E[z(s)\eta(t)'] = 0,$$

$$E[x(0)z(s)'] = 0, \quad E[z(s)z(t)'] = U(s, t),$$

$$Q \in \mathbf{R}^{n \times n}, \quad R \in \mathbf{R}^{s \times s},$$

$$S \in \mathbf{R}^{n \times s}, \quad U(s, t) \in \mathbf{R}^{m \times m}.$$

For simplicity we further assume that $x(0)$ is a random vector with $E[x(0)] = 0$ and $E[x(0)x(0)'] = \Sigma_0$ and that $z(t)$ is a random vector with $E[z(t)] = 0$. We use the following notations,

$$Y(t) = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{pmatrix}, \quad Z(t) = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(t) \end{pmatrix}.$$

2 Estimate of state vector

We consider the case when $y(t)$ and $z(t)$ are both observable at time t . Let $\hat{y}(t | t-1)$ be the conditional expectation of $y(t)$ and $\hat{z}(t | t-1)$ be that of $z(t)$, given a σ -field $\mathcal{W}(t-1)$ generated by $w(0), w(1), \dots, w(t-1)$, where

$$w(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.$$

By defining

$$\hat{y}(t | t-1) = E[y(t) | \mathcal{W}(t-1)]$$

and

$$\hat{z}(t | t-1) = E[z(t) | \mathcal{W}(t-1)],$$

we can write the conditional expectation of $w(t)$

$$\hat{w}(t | t-1) = \begin{pmatrix} \hat{y}(t | t-1) \\ \hat{z}(t | t-1) \end{pmatrix}.$$

Denote the residual of $w(t)$ as

$$e(t) = w(t) - \hat{w}(t | t-1),$$

and the variance-covariance matrix as

$$\Sigma(t) = E[e(t)e(t)'] = \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{pmatrix}.$$

We also use the notation,

$$e(t) = \begin{pmatrix} e_y(t) \\ e_z(t) \end{pmatrix} = \begin{pmatrix} y(t) - \hat{y}(t | t-1) \\ z(t) - \hat{z}(t | t-1) \end{pmatrix}.$$

Fig. 1 : Linear system with exogenous variables $z(t)$

Theorem 1 Under the assumption that both $y(t)$ and $z(t)$ are Gaussian, a Kalman filter is given by the following formulas.

- (i) $\hat{x}(t+n|t) = F\hat{x}(t+n-1|t) + L\hat{z}(t+n-1|t) \quad (n > 1)$
- (ii) $\hat{x}(t|t) = \hat{x}(t|t-1) + xx(t,t)H'\Sigma(t)^{-1}e(t)$
- (iii) $\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Lz(t) + K(t)e(t)$
- (iv) $e(t) = y(t) - H\hat{x}(t|t-1)$
- (v) $\Sigma(t) = Hxx(t,t)H' + R$
- (vi) $K(t) = (Fxx(t,t)H' + S)\Sigma(t)^{-1}$
- (vii) $xx(t+1,t+1) = Fxx(t,t)F' + Q - K(t)\Sigma(t)K(t)'$
- (viii) $\hat{z}(t+n-1|t) = a(t,t+n-1)'U(t+1)^{-1}Z(t)$
- (ix) $zz(t+1,t+1) = U(t+1,t+1) - a(t,t+1)'U(t+1)^{-1}a(t,t+1)$
- (x) $U(t+1)^{-1} = \begin{pmatrix} A(t) & B(t) \\ B(t)' & zz(t,t)^{-1} \end{pmatrix},$

where

$$\begin{aligned}
 xx(t,t) &= E[(x(t) - \hat{x}(t|t-1))(x(t) - \hat{x}(t|t-1))'] \\
 xz(t,t) &= E[(x(t) - \hat{x}(t|t-1))(z(t) - \hat{z}(t|t-1))'] \\
 zz(t,t) &= E[(z(t) - \hat{z}(t|t-1))(z(t) - \hat{z}(t|t-1))'] \\
 A(t) &= U(t)^{-1} \\
 &\quad + U(t)^{-1}a(t-1,t)zz(t,t)^{-1}a(t-1,t)'U(t)^{-1} \\
 B(t) &= -U(t)^{-1}a(t-1,t)zz(t,t)^{-1}.
 \end{aligned}$$

Note that $\Sigma(t)$ and $e(t)$ are only a part of $\Sigma(t)$ or $e(t)$ defined previously. This is because we assume that $z(t)$ is observable together with $y(t)$ at time t so that only the part related to $y(t)$ is necessary for updating the state $x(t)$.

Proof.

To prove the formulae in Theorem 1, we need the following Lemma 1. Although the proof of the Lemma 1 is placed in Appendix, we introduce the following notations.

$$\begin{aligned}
 U(t) &= \begin{pmatrix} U(0,0) & \cdots & U(0,t-1) \\ \vdots & \ddots & \vdots \\ U(t-1,0) & \cdots & U(t-1,t-1) \end{pmatrix}, \\
 U_0(t) &= \begin{pmatrix} U(0,0) & \cdots & U(0,t-1) \\ \vdots & \ddots & \vdots \\ U(t-2,0) & \cdots & U(t-2,t-1) \\ 0 & \cdots & 0 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 a(s,t) &= \begin{pmatrix} U(0,t) \\ \vdots \\ U(s,t) \end{pmatrix}, \\
 H(l) &= \begin{pmatrix} H & & 0 \\ & \ddots & \\ 0 & & H \end{pmatrix}, H(l) \in \mathbf{R}^{sl \times nl}, \\
 F(l) &= \begin{pmatrix} 0 & & 0 \\ I & \ddots & \\ \vdots & \ddots & \ddots \\ F^{l-2} & \cdots & I & 0 \end{pmatrix}, \\
 F(l) &\in \mathbf{R}^{nl \times nl}, \\
 F(p,q) &= \begin{pmatrix} I & & 0 \\ & \ddots & \\ & & I & \\ & & & F \\ 0 & & & & F^q \end{pmatrix}, \\
 F(p,q) &\in \mathbf{R}^{np \times np}, \\
 L(l) &= \begin{pmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{pmatrix}, L(l) \in \mathbf{R}^{ml \times nl}, \\
 \Sigma_0(l) &= \begin{pmatrix} \Sigma_0 & \cdots & \Sigma_0 \\ \vdots & \ddots & \vdots \\ \Sigma_0 & \cdots & \Sigma_0 \end{pmatrix}, \Sigma_0(l) \in \mathbf{R}^{nl \times nl}, \\
 Q(q) &= \begin{pmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{pmatrix}, Q(q) \in \mathbf{R}^{nq \times nq}, \\
 Q(p,q) &= \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & & Q(q) \\ 0 & \cdots & 0 & & & \end{pmatrix}, \\
 Q(p,q) &\in \mathbf{R}^{np \times np}, \\
 R(l) &= \begin{pmatrix} R & S'H' & \cdots & S'F^{l-2}H' \\ HS & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & S'H' \\ HF^{l-2} & \cdots & HS & R \end{pmatrix}, \\
 R(l) &\in \mathbf{R}^{sl \times sl},
 \end{aligned}$$

Lemma 1 Assume that both $y(t)$ and $z(t)$ are Gaussian and $E[Y(t)Y(t)']$ is nonsingular when $L = 0$, then we obtain the following formulae,

- (i) $z(t) - \hat{z}(t|t-1)$

$$\begin{aligned}
&= z(t) - a(t-1, t)'U(t)^{-1}Z(t) \\
\text{(ii)} \quad &x(t) - \hat{x}(t | t-1) = F^t(x(0) - \hat{x}(0 | t-1)) \\
&\quad + \sum_{j=1}^t F^{j-1}(\xi(t-j) - \hat{\xi}(t-j | t-1)) \\
\text{(iii)} \quad &x(0) - \hat{x}(0 | t-1) \\
&= x(0) - (\Sigma_0 F' H', \Sigma_0 F'^2 H', \dots, \Sigma_0 F'^{t-1} H') \\
&\quad \tilde{C}(t)^{-1} Y(t) \\
&\quad + (\Sigma_0 F' H', \Sigma_0 F'^2 H', \dots, \Sigma_0 F'^{t-1} H') \\
&\quad \tilde{C}(t)^{-1} H(t) F(t) L(t) U_0(t) U(t)^{-1} Z(t) \\
\text{(iv)} \quad &\xi(s) - \hat{\xi}(s | t-1) \\
&= \xi(s) - (0, \dots, 0, S, QH', \dots, QF'^{t-1-s} H') \\
&\quad \tilde{C}(t)^{-1} Y(t) \\
&\quad + (0, \dots, 0, S, QH', \dots, QF'^{t-1-s} H') \tilde{C}(t)^{-1} \\
&\quad H(t) F(t) L(t) U_0(t) U(t)^{-1} Z(t)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{C}(t) &= H(t) \{F(t, t-1) \Sigma_0(t) F(t, t-1)'\} \\
&\quad + \sum_{s=0}^{t-2} F(t, s) Q(t, s+1) F(t, s)'\} H(t) + R(t)
\end{aligned}$$

and $s = 0, 1, \dots, t-1$.

We now evaluate $xz(t, t)$. Since $x(t) - \hat{x}(t | t-1) \perp \mathcal{W}(t-1)$ and $\hat{z}(t | t-1) \in \mathcal{W}(t-1)$,

$$xz(t, t) = E[(x(t) - \hat{x}(t | t-1))z(t)'].$$

Moreover since $x(t) - \hat{x}(t | t-1)$ is expressed by (ii), (iii) and (iv) in Lemma 1 and

$$E[Y(t)z(t)'] = H(t)F(t)L(t)U_0(t)U(t)^{-1}E[Z(t)z(t)'],$$

we obtain

$$xz(t, t) = 0.$$

For $e(t)$ since $\eta(t) \perp \mathcal{W}(t-1)$,

$$e(t) = \begin{pmatrix} y(t) - H\hat{x}(t | t-1) \\ z(t) - \hat{z}(t | t-1) \end{pmatrix}.$$

About $\Sigma(t)$, we evaluate each block of the partitioned matrix,

$$\Sigma(t) = \begin{pmatrix} E[e_y(t)e_y(t)'] & E[e_y(t)e_z(t)'] \\ E[e_z(t)e_y(t)'] & E[e_z(t)e_z(t)'] \end{pmatrix},$$

Since $e_y(t) = H(x(t) - \hat{x}(t | t-1)) + \eta(t)$ and $\eta(t) \perp \{x(t), \mathcal{W}(t-1)\}$, we obtain

$$E[e_y(t)e_y(t)'] = Hxx(t, t)H' + R.$$

From the fact that $\eta(t) \perp \{z(t), \mathcal{W}(t-1)\}$, we have

$$E[e_y(t)e_z(t)']$$

Other blocks can be evaluated in the same manner. and we have

$$\Sigma(t) = \begin{pmatrix} Hxx(t, t)H' + R & 0 \\ 0 & zz(t, t) \end{pmatrix}.$$

The formula (i) follows from

$$\begin{aligned}
\hat{x}(t+n | t) &= F\hat{x}(t+n-1 | t) \\
&\quad + L\hat{z}(t+n-1 | t) + \hat{\xi}(t+n-1 | t),
\end{aligned}$$

since $\xi(t+n-1) \perp \mathcal{W}(t)$. To prove the equation (ii), we divide the space $\mathcal{W}(t)$ into the space $\mathcal{W}(t-1)$ and its tangent space $\mathcal{T}_w(t)$, which is spanned by the elements of $e(t)$. Then we can write

$$\hat{x}(t | t) = \hat{x}(t | t-1) + K_1(t)e(t),$$

for a $K_1(t) \in \mathbb{R}^{n \times (s+m)}$.

Then by solving

$$0 = E[(x(t) - \hat{x}(t | t))e(t)'] = (xx(t, t)H', xz(t, t))\Sigma(t).$$

We have

$$K_1(t) = (xx(t, t)H', 0)\Sigma(t)^{-1}.$$

Moreover by partitioning $K_1(t)$ into $(K_1^{(1)}(t), K_1^{(2)}(t))$, we have

$$K_1^{(1)}(t) = xx(t, t)H'\Sigma_{11}(t)^{-1}, \quad K_1^{(2)}(t) = 0.$$

To show the equation (iii), we start from

$$\hat{x}(t+1 | t) = F\hat{x}(t | t) + Lz(t) + \hat{\xi}(t | t)$$

and noting that

$$\hat{\xi}(t | t) = E[\xi(t) | \mathcal{W}(t-1)] + E[\xi(t) | \mathcal{T}_w(t)],$$

we see that $\hat{\xi}(t | t)$ can be written as

$$\hat{\xi}(t | t) = K_2(t)e(t),$$

since $\xi(t) \perp \mathcal{W}(t-1)$. Therefore,

$$\begin{aligned}
\hat{x}(t+1 | t) &= F(\hat{x}(t | t-1) + (xx(t, t)H', xz(t, t))\Sigma(t)^{-1}e(t)) \\
&\quad + Lz(t) + K_2(t)e(t).
\end{aligned}$$

Then we can rewrite the right hand side of the equation above and have

$$\hat{x}(t+1 | t) = F\hat{x}(t | t-1) + Lz(t) + K(t)e(t),$$

by introducing a new $K(t)$. Moreover, since $x(t+1) - \hat{x}(t+1 | t) \perp \mathcal{W}(t)$ and $e(t) \in \mathcal{W}(t)$,

$$E[(x(t+1) - \hat{x}(t+1 | t))e(t)'] = 0.$$

Thus

$$K(t) = (Fxx(t, t)H' + S, 0)\Sigma(t)^{-1}.$$

In a similar way, partitioning $K(t)$ into $(K^{(1)}(t), K^{(2)}(t))$, we have the desired result. We now evaluate $xx(t, t)$. Since

$$\begin{aligned} x(t+1) - \hat{x}(t+1 | t) \\ = F(x(t) - \hat{x}(t | t-1)) + \xi(t) - K(t)e(t), \end{aligned}$$

we obtain

$$\begin{aligned} xx(t+1, t+1) &= Fxx(t, t)F' \\ &\quad - Fxx(t, t)H'K(t)' + Q \\ &\quad - SK(t)' - K(t)Hxx(t)F' \\ &\quad - K(t)S' + K(t)\Sigma(t)K(t)'. \end{aligned}$$

We further employ the formula (vi), then

$$\begin{aligned} Fxx(t, t)H' &= K(t)\Sigma(t) - S \\ Hxx(t, t)F' &= \Sigma(t)K(t)' - S', \end{aligned}$$

and

$$\begin{aligned} xx(t+1, t+1) &= Fxx(t, t)F' \\ &\quad + Q - K(t)\Sigma(t)K(t)'. \end{aligned}$$

The formula (viii) easily follows from the equation (i) in Lemma 1. For $zz(t+1, t+1)$, since $z(t+1) - \hat{z}(t+1 | t) \perp \mathcal{W}(t)$ and $\hat{z}(t+1 | t) \in \mathcal{W}(t)$ we have

$$zz(t+1, t+1) = E[(z(t+1) - \hat{z}(t+1 | t))z(t+1)'],$$

and

$$\begin{aligned} zz(t+1, t+1) &= U(t+1, t+1) \\ &\quad - a(t, t+1)'U(t+1)^{-1}a(t, t+1) \\ U(t+1) &= \begin{pmatrix} U(t) & a(t-1, t) \\ a(t-1, t)' & U(t, t) \end{pmatrix}. \end{aligned}$$

The formula (x) follows by a direct calculation the inverse matrix of $U(t+1)$. The proof of the theorem is now complete. ■

As an example of Theorem 1, consider the the case when no exogenous variable exists. We can rewrite the formulae in Theorem 1 to the following formulae,

- (i) $\hat{x}(t+n | t) = F\hat{x}(t+n-1 | t) \quad (n > 1)$
- (ii) $\hat{x}(t | t) = \hat{x}(t | t-1) + xx(t, t)H'\Sigma(t)^{-1}(t)$
- (iii) $\hat{x}(t+1 | t) = F\hat{x}(t | t-1) + K(t)e(t)$
- (iv) $e(t) = y(t) - H\hat{x}(t | t-1)$
- (v) $\Sigma(t) = Hxx(t, t)H' + R$
- (vi) $K(t) = Fxx(t, t)H' + S\Sigma(t)^{-1}$,
- (vii) $xx(t+1, t+1) = Fxx(t, t)F' + Q - K(t)\Sigma(t)K(t)'$.

This is exactly the same as the Kalman filter which is well known.

We next consider the case when observation of $z(t)$ is lagged behind that of $y(t)$. That is, we redefine

$$w(t) = \begin{pmatrix} y(t) \\ z(t-1) \end{pmatrix}.$$

We then obtain following theorem.

Theorem 2 Under the assumption that both $y(t)$ and $z(t)$ are Gaussian, we obtain following formulas,

- (i) $\hat{x}(t+n | t) = F\hat{x}(t+n-1 | t) + L\hat{z}(t+n-1 | t) \quad (n > 1)$
- (ii) $\hat{x}(t | t) = \hat{x}(t | t-1) + (xx(t, t)H', xz(t, t-1))\Sigma(t)^{-1}e(t)$
- (iii) $\hat{x}(t+1 | t) = F\hat{x}(t | t-1) + L\hat{z}(t | t) + K(t)e(t)$
- (iv) $e(t) = \begin{pmatrix} y(t) - H\hat{x}(t | t-1) \\ z(t-1) - \hat{z}(t-1 | t-1) \end{pmatrix}$
- (v) $\Sigma(t) = \begin{pmatrix} Hxx(t, t)H' + R & Hxz(t, t-1) \\ xz(t, t-1)'H' & zz(t-1, t-1) \end{pmatrix}$
- (vi) $K(t) = (Fxx(t, t)H', xz(t, t-1)) + L(xz(t, t)'H', 0) + (S, 0)\Sigma(t)^{-1}$
- (vii) $xx(t+1, t+1) = Fxx(t, t)F' + Fxz(t, t)L' + Lxz(t, t)'F' + Lzz(t, t)L' + Q - K(t)\Sigma(t)K(t)'$
- (viii) $\hat{z}(t+n-1 | t) = a(t, t+n-1)'U(t+1)^{-1}Z(t)$
- (ix) $zz(t+1, t+1) = U(t+1, t+1) - a(t, t+1)'U(t+1)^{-1}a(t, t+1)$
- (x) $U(t+1)^{-1} = \begin{pmatrix} A(t) & B(t) \\ B(t)' & zz(t, t)^{-1} \end{pmatrix}$
- (xi) $xz(t+1, t) = Fxz(t, t) + Lzz(t, t)$
- (xii) $xz(t, t) = Lzz'(t-1, t) - xz(t, t-1)B_1^{(t)'}(t)$
- (xiii) $zz'(t, t+1) = U(t, t+1) - a(t-1, t)'U(t)^{-1}a(t-1, t+1),$

where

$$\begin{aligned} xx(t, t) &= E[(x(t) - \hat{x}(t | t-1))(z(t) - \hat{z}(t | t))'] \\ zz(t, t) &= E[(z(t) - \hat{z}(t | t))(z(t) - \hat{z}(t | t))'] \\ zz'(t, t+1) &= E[(z(t) - \hat{z}(t | t))z(t+1)'] \\ (B_t^{(t)}, \dots, B_1^{(t)}) &= a(t-1, t)'U(t)^{-1} \end{aligned}$$

The proof of Theorem 2 is omitted to save space, but it is similar to that of Theorem 1. Update scheme given in the Theorem 2 is illustrated in Fig. 2.

Fig. 2 : Update scheme given in Theorem 2

3 Appendix

3.1 Proof of Lemma 1

Since $y(t)$ and $z(t)$ are Gaussian processes, we can write

$$z(t) - \hat{z}(t | t-1) = z(t) - A^{(t)}Y(t) - B^{(t)}Z(t).$$

From

$$E[(z(t) - \hat{z}(t | t-1))Z(t)'] = 0$$

and

$$E[(z(t) - \hat{z}(t | t-1))Y(t)'] = 0,$$

we have the equations,

$$a(t-1, t)' - A^{(t)}H(t)F(t)L(t)U_0(t) - B^{(t)}U(t) = 0.$$

and

$$-A^{(t)}\tilde{C}(t) = 0.$$

Here $\tilde{C}(t) = E[Y(t)Y(t)']$. The matrix $\tilde{C}(t)$ is of full-rank from the assumption, so that $A^{(t)} = 0$ and,

$$B^{(t)} = a(t-1, t)'U(t)^{-1}.$$

For $x(t) - \hat{x}(t | t-1)$, we have

$$x(t) - \hat{x}(t | t-1) = F^t(x(0) - \hat{x}(0 | t-1)) + \sum_{j=1}^t F^{j-1}(\xi(t-j) - \hat{\xi}(t-j | t-1)).$$

Since $y(t)$ and $z(t)$ are Gaussian processes, we can write

$$x(0) - \hat{x}(0 | t-1) = x(0) - C^{(t)}Y(t) - D^{(t)}Z(t) \\ \xi(s) - \hat{\xi}(s | t-1) = \xi(s) - M^{(t)}Y(t) - N^{(t)}Z(t),$$

for $s = 0, 1, \dots, t-1$. From

$$E[(x(0) - \hat{x}(0 | t-1))Z(t)'] = 0, \\ E[(x(0) - \hat{x}(0 | t-1))Y(t)'] = 0, \\ E[\xi(s) - \hat{\xi}(s | t-1)Z(t)'] = 0, \\ E[\xi(s) - \hat{\xi}(s | t-1)Y(t)'] = 0.$$

we obtain

$$C^{(t)} = (\Sigma_0 H', \Sigma_0 F' H', \dots, \Sigma_0 F'^{t-1} H') \tilde{C}(t)^{-1}, \\ D^{(t)} = -(\Sigma_0 H', \Sigma_0 F' H', \dots, \Sigma_0 F'^{t-1} H') \tilde{C}(t)^{-1} \\ H(t)F(t)L(t)U_0(t)U(t)^{-1}, \\ M^{(t)} = (0, \dots, S, QH', \dots, QF'^{t-1-s}H') \tilde{C}(t)^{-1}, \\ N^{(t)} = (0, \dots, S, QH', \dots, QF'^{t-1-s}H') \tilde{C}(t)^{-1} \\ H(t)F(t)L(t)U_0(t)U(t)^{-1}. \blacksquare$$

4 Concluding Remarks

The Kalman filter we have derived is one of possible filters. For example, we can derive a different Kalman filter in which $Y(t)$ are used in preference to $Z(t)$. Wonham [2] derived a Kalman filter under the assumption that $Z(t)$ is measurable with respect to $Y(t)$. His filter is exactly the same as the filter given by (ii) to (ii) in Theorem 1. It is an interesting problem to find a Kalman filter when neither of $Z(t)$ nor $Y(t)$ is unobservable. We leave such interesting problems for future research.

Reference

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