Stochastic Process on CL Multiwavelet

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Abstract

This paper describes an early attempt to build a new multi-scale stochastic model on multiwavelet. Previously such stochastic model are defined either on a single or multiple binary tree as well as on an ordinary wavelet structure. Several interesting results are obtained due to the specific properties of multiwavelet which are different from its wavelet counterpart. Smoothing of a certain fractal signal on the proposed model will be given.

1 Introduction

Recently research topics focusing on the descriptions and applications of the multi-scale stochastic process has gained momentum thanks to the nice properties of such model on dealing with fractal signal [1, 2, 3, 4, 5, 6]. Earlier model is defined on a single binary tree, see e.g. [1, 2], then [4] moves further with model on wavelet. The authors gave another model from different point of view in [6]. On the other development, the wavelet theory gets another impetus driven from the introduction of what the so called multiwavelet that can be seen e.g. in [7, 8]. This multiwavelet has been known to have several advantages over ordinary wavelet such as short support, symmetry and orthogonality. In this paper those advantages will be further investigated in relation to the stochastic modeling.

The basic construction of a multi-scale stochastic model is a state space evolving on a dyadic tree $\mathcal{T}$, given in Fig. 1. Upon going down towards the finest scale N , the resolution doubles from scale to scale, and vice versa. In wavelet theory it has been known that the Haar wavelet decomposition or a pure decimator behave just like the upward evolution, and the synthesis process is equivalent to the downward path. This paper expands the model further from ordinary wavelet to multiwavelet structure.

Figure 1: Dyadic tree $\mathcal{T}$

2 Multiwavelet

2.1 Multi-resolution Analysis

Multiwavelet theory as an addition to the body of wavelet theory, falls on the realm of multi-resolution analysis (MRA). But instead of assuming single scaling function $\phi(t)$, the multiwavelet, extends the MRA by N scaling function $\Phi(t) = [\phi_1(t), \dots, \phi_N(t)] \in \mathbb{L}^2(\mathbb{R})^N$, such that subspaces

$$V_j \equiv \text{span}\left\{2^{j/2}\phi_i\left(\frac{t}{2} - k\right) : 1 \leq i \leq N, k \in \mathbb{Z}\right\} \quad (1)$$

are nested, i.e

$$\dots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \dots \quad (2)$$

and the conditions

$$\overline{\bigcap_{j \in \mathbb{Z}} V_j} = \mathbb{L}^2(\mathbb{R}) \text{ and } \bigcup_{j \in \mathbb{Z}} V_j = \{0\} \quad (3)$$

hold. It is understood that all scaling functions $\phi_1(t), \dots, \phi_N(t)$ lies in V_0 where $V_0 \subset V_1$ whose basis are $\{\sqrt{2}\phi_i(2t - k) : 1 \leq i \leq N, k \in \mathbb{Z}\}$ that the following two-scale matrix dilation equation is satisfied

$$\Phi(t) = \sqrt{2} \sum_k G_k \Phi(2t - k) \quad (4)$$

where $G_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})^{N \times N}$ are $N \times N$ scaling coefficients. For orthonormal scaling function, it can be shown that

$$\sum_k G_k G_{2l+k}^T = I_N \delta_{0,l}, \text{ for } l \in \mathbb{Z} \quad (5)$$

Let W_j be the complement spaces of V_j in V_{j-1} such that

$$V_{j-1} = V_j \oplus W_j \text{ and } V_j \cap W_j = 0 \quad (6)$$

Let $\Psi(t) = [\psi_1(t), \dots, \psi_N(t)] \in \mathbb{L}^2(\mathbb{R})^N$, then define

$$W_j \equiv \overline{\text{span}\{2^{j/2}\psi_i(\frac{t}{2} - k) : 1 \leq i \leq N, k \in \mathbb{Z}\}} \quad (7)$$

$\psi(t)$ is called a multiwavelet if $\{\psi_i(t - k) : 1 \leq i \leq N, k \in \mathbb{Z}\}$ is Riesz basis of W_0 . The multiscaling and multiwavelet function satisfy two-scale matrix wavelet equation

$$\Psi(t) = \sqrt{2} \sum_k H_k \Phi(2t - k) \quad (8)$$

where $H_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})^{N \times N}$ are $N \times N$ wavelet coefficients. In orthogonal multi-resolution analysis the W_j are orthogonal complement to V_j in V_{j-1} . This fact leads to the following relation

$$\sum_k G_k H_{2l+k}^T = 0_N, \text{ for } l \in \mathbb{Z} \quad (9)$$

In particular, the matrix H_k in orthonormal wavelet system satisfies the condition below

$$\sum_k H_k H_{2l+k}^T = I_N \delta_{0,l}, \text{ for } l \in \mathbb{Z} \quad (10)$$

The pair of Eqs. (4) and (8) can be realized with matrix filter banks operating on N input data stream and producing $2N$ decimated by 2 output streams. If $x(t) \in V_0$ denotes input signal then multiwavelet decomposition of this signal can be obtained by recursively calculate the following

$$v_{j,k} = \sum_m G_{m-2k} v_{j-1,m} \quad (11a)$$

$$w_{j,k} = \sum_m H_{m-2k} v_{j-1,m} \quad (11b)$$

The original signal $x(t)$ can be found through synthesis equation

$$v_{j-1,k} = \sum_m \tilde{G}_{k-2m}^T v_{j,m} + \sum_m \tilde{H}_{k-2m}^T w_{j,m} \quad (12)$$

If $\phi_1(t - k), \dots, \phi_N(t - k), k \in \mathbb{Z}$ form an orthogonal basis of V_0 then the multiwavelet system will be orthogonal.

This paper will focus on a certain type of orthogonal multiwavelet known as *Chui-Lian* (CL) pair multiwavelet whose coefficients G and H are determined

as

$$G_0 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & -1 \\ \frac{\sqrt{7}}{2} & -\frac{\sqrt{7}}{2} \end{bmatrix} \quad G_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad (13a)$$

$$G_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{7}}{2} & -\frac{\sqrt{7}}{2} \end{bmatrix}$$

$$H_0 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \quad H_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} -4 & 0 \\ 0 & 2\sqrt{7} \end{bmatrix}$$

$$H_2 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad (13b)$$

This choice is motivated by orthogonality of the pre-filtering needed in the process.

2.2 Pre-filtering

As described in the previous subsection, the multiwavelet system needs N row of input data stream, and in particular choosing CL wavelet results in $N = 2$ input data streams. The choice of such multiwavelet is related to the properties of the pre-filtering involved in multiwavelet system. For simplicity this paper will use critical sampling method described in [7] as a pre-filtering method. This is due to the fact that the covariance matrix of pre-filtering result retain diagonal matrix form needed to maintain orthogonality in the subsequent derivation.

The pre-filtering procedure involves a matrix $P_0 = \begin{bmatrix} 1/4 & 1/4 \\ -1/6 & 1/6 \end{bmatrix}$ to obtained data stream at level 0 written as

$$v_{0,k} = P_0 \begin{bmatrix} v_{2k} \\ v_{2k+1} \end{bmatrix} \quad (14)$$

so that high pass filtering $\sum_m H_{m-2k} v_{0,m}$ is zero. For input comprises a signal corrupted by zero-mean noise $X_k = D_k + v_k$, it can be obtained that the covariance matrix $\text{cov}[X_k X_k]$ at level 0 for $[v_{0,0}^0, v_{0,0}^1, v_{0,1}^0, v_{0,1}^1]$ remain diagonal

$$\text{cov}[X_k X_k] = \sigma_\epsilon^2 \begin{bmatrix} 0.125 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix} \quad (15)$$

where σ_ϵ^2 is the variance of the noise. With this property and the orthogonality of the multiwavelet, one can retain the orthogonality throughout the system.

3 Stochastic Model

Stochastic model based on wavelet decomposition has been covered in [4]. The extension to multiwavelet structure, given apparent advantages of multiwavelet, is the main topics in this paper. As described in the previous section, the multiwavelet transformation consists of filtering $G_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})^{N \times N}$ and $H_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})^{N \times N}$

followed by a decimation by two. In this paper, the process filtering G_k with such decimation will be referred as α , and β for H_k filter respectively. In other words the operators α and β works from $l^2(\mathbb{Z})$ to $l^2(2\mathbb{Z})$. For given signal or vector X_n at level n then αX_n is the coarse approximation at level $n+1$, whereas βX yields the detail required upon going from one level to the next. From Eq. (12), the *reverse* process, i.e. going from coarse to fine involves adjoint operators $\bar{\alpha}$ and $\bar{\beta}$ for filter $\tilde{G}_k$ and $\tilde{H}_k$ respectively. From Eqs. (5), (9), (10), and (12), it is clear that the following conditions are satisfied

$$\bar{\alpha}\alpha + \bar{\beta}\beta = 1 \quad (16a)$$

$$\bar{\alpha}\alpha = I = \bar{\beta}\beta \quad (16b)$$

$$\alpha\bar{\beta} = 0 = \beta\bar{\alpha} \quad (16c)$$

Now the relation of the transformation above with the stochastic modeling can be explained as follows. Let us define a multi-scale stochastic model written in the following forms

$$X_{n+1} = (\bar{\alpha}_n A_n + \bar{\beta}_n B_n) X_n + W_{n+1} \quad (17a)$$

$$Y_{n+1} = C X_n + V_n \quad (17b)$$

where n denotes scale index increasing towards finest scale N , $\bar{\alpha}, \bar{\beta}$ are multiwavelet transformation adjoint operators. X_n is the state vector with stationary white noise W_n , and Y_n is the observation vector also with stationary white noise V_n . These noises are independent each other. The variables A_n, B_n and C_n are matrices with appropriate dimensions. It can be seen that the model written in Eq. (17) can be associated with a synthesis process of multiwavelet transformation, see Eq. (12), such that the evolution of Eq. (17) is towards finer scale through adjoint operators $\bar{\alpha}$ and $\bar{\beta}$. The initial process at X_0 is called the root node, and the finest level at X_N is referred to leaf nodes.

Once transformation or decomposition through multiwavelet is carried out, the coefficients becomes the data set. Then there exist several facts as follows

- (1) Let w_n consists of $\mathcal{W}_n = \{\alpha, \beta\}$ for $0 \leq n < N$.
- (2) Then $w_n X_n$ is linked to the coarser scale through $w_{n-1} X_{n-1}$, and it is conditionally independent given X_{n-1} .
- (3) $w_n Y_n$ is linked to X_n through $w_n X_n$, and it is conditionally independent given X_n .

Following the term used in [4], one can get a smoother on each board of coefficient decomposition where the smoothing process in each board is independent. Different from wavelet structure where there are N board depending on the filter taps, in multi-wavelet decomposition there are at least twice board as much. This can be seen in Eq. (1) where $\Phi(t) =$

$[\phi_1(t), \dots, \phi_N(t)] \in \mathbb{L}^2(\mathbb{R})^N$, $N = 2$ for CL multi-wavelet.

With *Chui-Lian* (CL) multiwavelet, the two rows of input signal will have diagonal matrix correlation. This fact in addition to the orthogonality of the multiwavelet transformation will lead us to the implementation of a smoother on model Eq. (17) by taking advantages of the fast multi-scale smoothing algorithm [1]. It is shown through simulation that in case of smoothing of a typical fractal signal, significant improvement over previously announced model can be achieved.

4 Simulation

The subject of this section is the smoothing procedure of a certain fractal signal using the method described in the previous section. Assume that there exists an observation signal corrupted by white Gaussian noise ($\text{SNR} = \sqrt{2}$) depicted as dashed line in Figs. 2, 3, and 4. The true signal of this noisy observation is generated through multi-scale system elaborated in [1, 2] and given by the dotted line in those figures. Then the smoothing results on a variety of wavelets are given by solid line. Upon examining the graphs closely, a smoothing based on Haar wavelet reveals some blocky nature mentioned in [1]. A better result can be achieved through Daubechies4 wavelet, see Fig. 3, refer also [4]. Fig. 4 depicts the smoothing of the signal on the CL multiwavelet structure which shows further improvement over the two previously announced methods.

The quality of the proposed smoothing result based on CL multiwavelet are given numerically in Table 1, as well as the result using Haar and Daubechies4 wavelet for comparison. The *MSE* value of each process shows that the proposed method gives better performance. Other important quality measurement is the Bhattacharyya distance which is given to show the closeness of two random vector. It is based on the following formula

$$B_d(R_1, R_2) = \frac{1}{2} \ln \left[\left| \frac{1}{2}(R_1 + R_2) \right| \right] - \frac{1}{4} \ln \left[\left| (R_1 R_2) \right| \right] \quad (18)$$

where R_1, R_2 denote the covariance matrices of both random vectors, and $|R|$ is the determinant of the matrix R . Now given two random vectors with probability of obtaining a sample from either vectors is $1/2$, then the probability of error of deciding from which process the sample was is bounded by the following

$$P(\text{error}) \leq \frac{1}{2} \exp^{-B_d(R_1, R_2)} \quad (19)$$

Bhattacharyya distance and probability of error also confirm that the smoothing based on the CL multi-wavelet outperforms ordinary wavelet base.

Wavelet	MSE	B_d	$P(\text{error})$
Haar	1.035391	6.695519	0.000618
Daubechies4	1.032211	3.371325	0.017172
Chui-Lian	1.026031	1.849849	0.078630

Table 1: Quality of smoothing

Figure 2: Smoothing with Haar wavelet

Figure 3: Smoothing with Daubechies4 wavelet

Figure 4: Smoothing with Chui-Lian multiwavelet

5 Conclusion

This paper described an early attempts on modeling of stochastic system evolving on the multiwavelet structure. Simulation results hinted that several improvements over previously published results can be achieved through the proposed method.

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