A Subspace Identification of Closed-Loop Systems based on Stochastic Realization

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Abstract
In this paper, we consider a subspace model identification for closed-loop systems based on stochastic realization approach. At first, by the joint input-output method the closed-loop identification problem is reformulated to an open-loop one, which can be solved by the recently proposed realization based subspace method ([3]). Then an effective model reduction step is adopted to obtain a plant model of the original system order from the estimated higher-order model, which is possibly unstable. We include two numerical examples to illustrate the effectiveness of the proposed method.

Keywords: stochastic realization, closed-loop system, subspace identification.

1 Introduction
Identification of systems operating in closed-loop has attracted much interest for a long period, since, in many industrial or economic processes, the feedback loop is either indispensable for the safety reason or intrinsic in the system. In [2], the closed-loop identification methods are classified into three categories based on the traditional prediction error method. Furthermore, a comprehensive study of closed-loop identification in the prediction error framework is presented recently by Forssell and Ljung [1], where it is shown that different methods correspond to different parametrizations of the process and noise models. However, these classical methods cannot avoid the inherent problems such as difficult parametrizations and non-linear optimizations, in particular for MIMO systems. On the contrary, the recently developed subspace identification methods provide a favorable alternative for the classical methods, see e.g. [7, 8, 10, 12].

Within the framework of subspace-based identification, there exists a growing literature on the subject of closed-loop identification. In [11], Verhaegen proposed a closed-loop identification method by reformulating the identification of systems operating in closed-loop to an open-loop identification problem, to which the MOESP method is applied. In [9] a method using geometric projection is developed; however, the underlying mechanism is difficult to understand.

In this paper, we consider the problem of identifying the closed-loop system from the viewpoint of stochastic realization theory. The earlier stochastic realization theory based on canonical correlation analysis (CCA) is developed by Akaike in his pioneer work. Recently Katayama and Picci [3] proposed a realization method of stochastic systems with exogenous inputs and developed some subspace identification algorithms for open-loop systems. This allows us to extend the stochastic realization to the closed-loop system identification with methodology similar to that of [11]. The basic procedure in this paper is reformulating the closed-loop identification to the identification of the transfer function from the exogenous input to joint plant input and output signals. We then obtain the state-space model of the plant by using the realization-based subspace algorithms.

The outline of this paper is as follows. In Section 2, we briefly introduce the identification problem. Then in Section 3, the stochastic realization for the closed-loop system is presented. Section 4 introduces the adopted model reduction procedure and finally in Section 5 simulation results are presented to verify the effectiveness of the proposed method.

2 Problem description
We consider the linear, time-invariant, closed-loop system shown in Fig. 1, where $G(z)$ is the plant
and \( C(z) \) is the controller, \( u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, r(t) \in \mathbb{R}^m \) denote the input, output and exogenous input measurable signals respectively. It is assumed that the exogenous reference signal \( r(t) \) is uncorrelated with the disturbance \( v(t) \) and all signals are discrete-time wide-sense stationary vector processes. From the configuration of Fig. 1, we state the closed-loop identification problem as follows.

**Given:** finite measured data of the exogenous signal \( r(t) \) and input and output signals \( u(t) \) and \( y(t) \).

**Find:** the state-space model of the plant.

We assume that the disturbance \( v(t) \) is modelled by a zero mean white noise signal \( e(t) \) and an inversely stable transfer matrix \( H(z) \). The plant input \( u(t) \) and output \( y(t) \) can be described by

\[
    u(t) = r(t) - C(z)y(t) \\
    y(t) = G(z)u(t) + H(z)e(t)
\]

(1) (2)

As is pointed in [9], the subspace identification methods do not work when the data are measured under the closed-loop experiment. A natural idea to solve it is to model the joint input-output signal by applying the open-loop subspace identification methods and then extract the plant model from the joint estimated model.

### 3 Stochastic realization for closed-loop system

In [6], the stochastic realization with exogenous input is proposed based on the orthogonal decomposition of the output signal. A more recent approach is proposed in [3], where the geometric projections are employed. In this paper, we apply the approach due to [3] to the closed-loop identification problem. Some necessary preliminaries are introduced in the following.

#### 3.1 Preliminaries

We define the joint vector signal \( s(t) \) as

\[
    s(t) \triangleq \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}
\]

(3)

where \( s(t) \) is also a stationary process with dimension \( l = p + m \).

Denote \( t \) be the present time and \( k \) be a positive integer. We define the stacked vectors of past and future exogenous inputs and stacked vectors of past and future joint vector signals as

\[
    r_-(t) \triangleq \begin{bmatrix} r(t-1) \\ r(t-2) \\ \vdots \\ r(t+k-1) \end{bmatrix}, \quad r_+(t) \triangleq \begin{bmatrix} r(t) \\ r(t+1) \\ \vdots \\ r(t+k-1) \end{bmatrix}
\]

and

\[
    s_-(t) \triangleq \begin{bmatrix} s(t-1) \\ s(t-2) \\ \vdots \\ s(t+k-1) \end{bmatrix}, \quad s_+(t) \triangleq \begin{bmatrix} s(t) \\ s(t+1) \\ \vdots \\ s(t+k-1) \end{bmatrix}
\]

Furthermore, we define the past signal and future signal as

\[
    p(t) \triangleq \begin{bmatrix} r_-(t) \\ s_-(t) \end{bmatrix}, \quad f(t) \triangleq s_+(t)
\]

Let \( \mathcal{R}_{t+}, \mathcal{S}_{t-}, \mathcal{P}_t \) be the linear spaces of second order random variables spanned by the exogenous input \( r(t) \), by joint signals \( s(t) \) and by the past \( p(t) \), respectively. By closing the spaces with respect to the norm induced by the inner product \( ||\xi||^2 = [E\{\xi^2\}]^{1/2} \), where \( E \) denotes mathematical expectation, we can consider these spaces as Hilbert subspaces of an ambient Hilbert space \( \mathcal{H} \) spanned by the joint process \( (r, s) \).

Suppose that \( \mathcal{A} \) is a subspace of \( \mathcal{H} \). We know the orthogonal projection of any \( b \in \mathcal{H} \) onto \( \mathcal{A} \) can be expressed by the following formula

\[
    b|\mathcal{A} = E\{ba'\}E\{aa'\}^{-1}a \quad (a \in \mathcal{A})
\]

Furthermore, we introduce an important Lemma on geometric projection used in [3].

**Lemma 1** Let \( s, a (a \in \mathcal{A}) \) and \( b (b \in \mathcal{B}) \) be random vectors with components in \( \mathcal{H} \), assume \( \mathcal{A} \cap \mathcal{B} \).
$B = 0$, Then

\[
\hat{E}\{s|\mathcal{A} \cup \mathcal{B}\} = \left[ \begin{array}{cc} E\{sa'|} & E\{sb'\} \\ E\{ab'|} & E\{bb'\} \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right] = \Pi(s)a + \Phi(s)b
\]

where the vectors $\Pi(s)a, \Phi(s)b$ are the oblique projections of $s$ onto $\mathcal{A}$ along $\mathcal{B}$ and of $s$ onto $\mathcal{B}$ along $\mathcal{A}$, respectively. Furthermore, $\Pi(s)$ and $\Phi(s)$ satisfy the discrete Wiener-Hopf type equations

\[
\Pi(s)\Sigma_{aa}|_{\gamma^\perp} = \Sigma_{ea}|_{\gamma^\perp}, \Phi(s)\Sigma_{bb}|_{\alpha^\perp} = \Sigma_{eb}|_{\alpha^\perp}
\]

(4)

3.2 Stochastic realization based on geometric projection

The key step of stochastic realization is to find the linear predictor of the future output vector based on the past output and future input signals. By using the signals defined above, we have reformulated the closed-loop identification problem to an open-loop one which consists of the identification of transfer function from exogenous input $r(t)$ to joint vector signals $s(t)$. Under the feedback-free condition from $s$ to $r$, it will be shown that the minimal state space of joint process $s(t)$ is the predictor space. We assume the exogenous signal is sufficient richness such that the following condition is satisfied.

\[
\mathcal{R}_{t-} \cap \mathcal{R}_{t+} = 0
\]

(5)

It should be noted that this condition is too restrictive ([3]). However, it simplifies the induction of some crucial representation results.

Under this condition and Lemma 1, we can obtain the optimal predictor $\hat{f}(t)$ described by Theorem 1 in [3] as

\[
\hat{f}(t) = \hat{E}\{f(t)|\mathcal{P}_{t-} \cup \mathcal{R}_{t+}\} = \Pi p(t) + \Phi r_+(t)
\]

where $\Pi p(t)$ is the oblique projection of the future $f(t)$ onto the past $\mathcal{P}_{t-}$ along the future exogenous input subspace $\mathcal{R}_{t+}$ and $\Phi r_+(t)$ is the oblique projection of the future $f(t)$ onto $\mathcal{R}_{t+}$ along $\mathcal{P}_{t-}$.

Furthermore, under the assumption of feedback-free condition between $s(t)$ and exogenous signal $r(t)$, according to Lemma 2 in [3], the oblique projection of $\Pi p(t)$ spans the state space of a minimal state-space model of $s$ which is causal with respect to $r$.

By employing a subsequent procedure of extended canonical correlation analysis introduced in [3], we can obtain the optimal predictor $\hat{f}(t)$ as

\[
\hat{f}(t) = Ox(t) + Dr(t) + e(t)
\]

(6)

where $x(t)$ is the state vector, $O$ is the extended observability matrix; see [3] for details.

Thus, the forward innovation state-space model can be obtained by using the state vector and the optimal predictor as

\[
x(t + 1) = Ax(t) + Br(t) + Ke(t)
\]

(7)

\[
s(t) = Cx(t) + Dr(t) + e(t)
\]

(8)

where $x(t) \in \mathbb{R}^n$ represents a basis of the stationary oblique predictor space and $e(t)$ is an innovation process with the dimension $n \times n, n \times m, l \times n, l \times m$ respectively.

By defining

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}
\]

\[
e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \quad e_1 \in \mathbb{R}^p, \quad e_2 \in \mathbb{R}^m
\]

it is easy to rewrite (7-8) as

\[
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} r(t) + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}
\]

\[
y(t) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} r(t) + \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}
\]

(9)

Thus the plant transfer function $G(z)$ is given by

\[
G(z) = G_{ry}(z)G_{ru}^{-1}(z)
\]

where the transfer functions $G_{ry}(z)$ from $r(t)$ to $y(t)$ and $G_{ru}(z)$ from $r(t)$ to $u(t)$ respectively are estimated from (9-10):

\[
G_{ry}(z) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

(11)

\[
G_{ru}(z) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

(12)
It should be noted that \( \det(D_2) \neq 0 \) from (1). Thus we can summarize the estimation of \( G(z) \) as the following proposition.

**Proposition 1** The estimated system transfer function \( \hat{G} \) is given by

\[
\hat{G} = \left[ \begin{array}{c|c}
A - BD_2^{-1}C_2 & BD_2^{-1}D_1D_2^{-1}C_2 \\
\hline
C_1 - D_1D_2^{-1}C_2 & D_1D_2^{-1}
\end{array} \right]
\]  

(13)

where

\[
C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}
\]

For the limited space, we do not introduce the stochastic realization algorithms here; see [3] for further details.

4 Model reduction

Since the estimated system model becomes higher-order, an additional model reduction procedure is necessary.

In the case of a stable transfer function \( G(z) \), we often use balance and truncate (B&T) to obtain a lower-order approximation model. Unfortunately, B&T is only limited to the estimated higher-order stable plant model \( G(z) \). An alternative approach named fractional balanced reduction (FBR) in [5], however, can handle unstable plants based on normalized fractional representations. The nice property of FBR is that it has a graph-metric error bound that allows a priori robustness analysis using graph-metric theory. The basic procedures of FBR algorithm are listed as follows; see also [4, 5] for details.

**FBR Algorithm:**

1. **Step 1**: Find a minimal state space triple \( \{A_g, B_g, C_g\} \) of \( \hat{G}(z) \).

2. **Step 2**: Solve the algebraic Riccati equation

\[
A_g^T P A_g - P - A_g^T P B_g [I + B_g^T P B_g]^{-1} B_g^T P A_g + C_g^T C_g = 0
\]

and let \( \bar{A} = A_g + B_g K_g \) where \( K_g = -[I + B_g^T P B_g]^{-1} B_g^T P A_g \).

3. **Step 3**: Obtain the balanced realization

\[
\{\bar{A}, \bar{B}, \begin{bmatrix} C_g \\ K_g \end{bmatrix}\}
\]

\( \bar{A} \), \( \bar{B} \), of the stable minimal state space triple \( \{\bar{A}, \bar{B}, \begin{bmatrix} C_g \\ K_g \end{bmatrix}\} \).

4. **Step 4**: Apply the B&T to the above realization, and partition as

\[
\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}
\]

\[
\begin{bmatrix} \bar{C} \\ \bar{K}_g \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{K}_{g1} \\ \bar{K}_{g2} \end{bmatrix}
\]

where \( \bar{A}_{11} \in \mathbb{R}^{n \times r} \).

5. **Step 5**: The \( r \)-th order reduced model is given by

\( \hat{G}_r(z) = \{\bar{A}_{11}, \bar{B}_1, \bar{C}_1\} \) with \( \bar{A}_{11} = A_{11} - B_1 K_{g1} \), \( B_1 = \bar{B}_1 \), \( C_1 = \bar{C}_1 \).

5 Numerical examples

In this section, we illustrate the proposed method with two numerical examples.

**Example 1.** This example is a slightly modified version of that in [11], which has also been employed in [9]. The plant corresponds to a discrete-time model of a laboratory plant setup of two circular plates rotated by an electrical servo motor with flexible shafts. The transfer functions of the plant and the controller and the noise model are given by

\[
G(z) = \frac{0.001(0.98z^4 + 12z^3 + 18z^2 + 3z - 0.02)}{z^5 - 4.2z^4 + 7.45z^3 - 7.0z^2 + 3.48z - 0.7}
\]

\[
C(z) = \frac{0.63z^4 - 2.08z^3 + 2.82z^2 - 1.86z + 0.49}{z^4 - 2.65z^3 + 3.11z^2 - 1.75z + 0.39}
\]

\[
H(z) = \frac{0.001(2.89z^2 + 11.15z + 2.74)}{z^3 - 2.7z^2 + 2.61z - 0.9}
\]

where \( e(t) \) is a white noise with variance 1/9. The exogenous input \( r(t) \) is chosen to be zero-mean white noise sequence with variance 1. The length of the data is taken equal to 1200. We set the time index \( k \) as 10 and 10 Monte Carlo experiments are performed. We have applied COV-b algorithm introduced in [3] to the simulation data. Firstly a ninth order state space model from \( r(t) \) to \( s(t) \) is estimated. Then we have adopted the FBR model reduction procedure to obtain a fifth order state space model. The Bode plots of the estimated model are shown in Fig. 2. The poles of the estimated reduced-order model are plotted in Fig. 3, in which (+) denotes the true poles of the plant. From the figures, we can see that the proposed method performs quite well.
The discrete-time plant is unstable with the poles of $1.0101$ and $0.9802$. We chose the reference signal $r(t)$ as a white noise with mean zero and variance 1 so that the condition (5) is satisfied. The disturbance $v(t)$ is assumed to be a white noise with mean zero and variance $(0.01)^2$. The data are collected from the original continuous-time system with the predetermined sampling period $0.01s$. For applying realization algorithm, we set the index $k = 8$. A total number of 20 Monte Carlo experiments are performed with data length of 1000. Similarly to the first example, we chose COV-b algorithm to the simulation data. First a 3rd order state space model from $r(t)$ to $s(t)$ is estimated. Then we applied the FBR model reduction procedure discussed in the previous section to obtain a 2nd order approximation model. The estimation results are shown in Figs. 4 and 5. Fig. 4 shows Bode plots of the estimated models while Fig. 5 describes the poles of the estimated reduced-order model with the same identification algorithm where (*) and (+) denote Bode plot and the poles of the true plant respectively. We see from these figures that a satisfactory estimation result is obtained by the proposed method.

**Example 2.** In the second example, we consider a discretized system of original continuous-time system discussed in [13], which consists of a second order plant $G(s)$ and a first order PI regulator $C(s)$. It should be noted that the configuration of the closed-loop continuous-time system is the same as that of Fig. 1. The transfer functions of plant and regulator are given by

$$G(s) = \frac{s + 1}{s^2 + s - 2}, \quad C(s) = \frac{10s + 15}{s}$$

where the plant is unstable. We set the sampling period as $0.01s$. Then with a zero-order hold method the corresponding discrete-time system is obtained as

$$G(z) = \frac{0.01z - 0.0099}{z^2 - 1.99z + 0.99}$$

The discrete-time plant is unstable with the poles of $1.0101$ and $0.9802$. We chose the reference signal $r(t)$ as a white noise with mean zero and variance 1 so that the condition (5) is satisfied. The disturbance $v(t)$ is assumed to be a white noise with mean zero and variance $(0.01)^2$. The data are collected from the original continuous-time system with the predetermined sampling period $0.01s$. For applying realization algorithm, we set the index $k = 8$. A total number of 20 Monte Carlo experiments are performed with data length of 1000. Similarly to the first example, we chose COV-b algorithm to the simulation data. First a 3rd order state space model from $r(t)$ to $s(t)$ is estimated. Then we applied the FBR model reduction procedure discussed in the previous section to obtain a 2nd order approximation model. The estimation results are shown in Figs. 4 and 5. Fig. 4 shows Bode plots of the estimated models, while Fig. 5 describes the poles of the estimated reduced-order model with the same identification algorithm where (*) and (+) denote Bode plot and the poles of the true plant respectively. We see from these figures that a satisfactory estimation result is obtained by the proposed method.

**6 Conclusions**

In this paper, we have developed a closed-loop subspace identification method based on the stochastic
realization theory. The main procedures in the proposed method consist of firstly making a reformulation of the identification problem to an open-loop identification of the joint plant input and output signals and then applying the stochastic realization based identification method to obtain the estimation of the plant model. Finally we adopt an effective model reduction method to obtain a lower-order approximation to the estimated higher-order model. The numerical examples have verified the effectiveness of the method.

References


