On Stochastic Parabolic Model for Term Structure Dynamics and Mean-variance Efficient Portfolio

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Abstract We consider the term structure modeling by using a appropriate stochastic parabolic system with boundary noises. After finding a sufficient condition for the no arbitrage opportunity, we solve the mean-variance optimal control problem in the incomplete market.

1 Introduction

Bonds are tradable assetes in a financial market. The market price \( P(t, T) \) is a function of the time \( t \) and the maturity \( T \) and changes its value randomly as shown in Fig.1 [1]. For the mathematical modeling of this process, we rewrite \( P(t, T) = \exp\{- \int_0^T f(t, x)dx\} \) and the so-called forward rate dynamics for \( f \) is constructed in [1].

It is well known that there are two different motivations for term structure of interest rates modeling. The first is concerned with the pricing of interest rate derivative securities. In the sense of arbitrage-free, the dynamics of the interest rates is developed in [2]. Starting from the classical modeling of the short rate to the forward rate including the Musiela equation, there are many excellent books for example [3].

The second motivation is the actual "econometric" modeling of the real-world term structure dynamics which gives the statistical description of the movements of interest rate in [4]. The actual econometric phenomena can be also found in [5].

In this paper, we consider the second viewpoint. First we present the motivating discussions of the term structure model. Then the parabolic stochastic equation with boundary noises is proposed for the term structure. After studying the existence of a unique solution to this stochastic partial differential equation, we derive the sufficient condition to support the no arbitrage opportunity. The final section is devoted to consider the mean variance optimal control problem for the wealth process in the incomplete market situation.

The simple infinite-dimensional term structure model is given by

\[
df(t, x) = \frac{\partial f(t, x)}{\partial x} dt + \nu(t, x) dt + dw(t, x), \quad (1.1)
\]

where \( \nu(t, x) \) is identified by using the argument of absence of arbitrage. Recently from empirical observations, Bouch et.al. and Cont [4] proposed the parabolic type systems to support the smoothness property of \( f(t, x) \) with

![Figure 1: The Market value \( P(t, T) \) of a bond](image-url)
The natural extension of (1.1) to the parabolic system is
\[
df(t, x) = \frac{k}{2} \frac{\partial^2 f(t, x)}{\partial x^2} dt + \frac{\partial f(t, x)}{\partial x} dt + \nu(t, x) dt + dw(t, x).
\]
(1.2)

In Cont [4], in order to formulate (1.2) first the two factors, i.e., the short rate and the spread are taken out of the model. The short rate and the spread are modeled by a bivariate diffusion process. In such a modeling procedure the short rate and the spread are independent of the new variable of the constructed model explicitly.

Here we formulate (1.2) without taking out these two terms. By constructing the boundary conditions, we get the short rate \( r(t) \) and the long rate \( \ell(t) \) as the boundary values \( f(t, 0) \) and \( f(t, 1) \).

First we set the simple boundary condition to (1.2),
\[
\frac{\partial f(t, 0)}{\partial x} = 0.
\]
(1.3)
The discrete version of (1.3) becomes
\[
\frac{f(t, \Delta x) - f(t, 0)}{\Delta x} = 0
\]
i.e.,
\[
r(t) = f(t, 0) = f(t, \Delta x).
\]
This implies that the short rate \( r(t) \) is the same value as the nearest term structure value. It is well known that if the considered situation has no randomness, then \( f(t, x) \) becomes a constant for all \( x \). So we need to adjust (1.3) to fit the stochastic situation, i.e.,
\[
\frac{k}{2} \frac{\partial f(t, 0)}{\partial x} = \sigma_0(t) \frac{dw_0(t)}{dt}
\]
where \( w_0(t) \) is a standard Brownian motion process. The discrete version of (1.4) with respect to \( t \) and \( x \) is given by
\[
\frac{k}{2} \frac{f(t, \Delta x) - f(t, 0)}{\Delta x} = \sigma_0(t) \frac{w_0(t) - w_0(t-1)}{\Delta t}
\]
i.e.,
\[
r(t_i) = f(t_i, \Delta t) - \sigma_0(t_i) \frac{w_0(t_i) - w_0(t_i-1)}{\Delta t}.
\]
Hence the spot rate \( r(t) \) is fluctuated by \( w_0 \) and also depends on \( f(t, x), x > 0 \). It is also possible to generalize the boundary condition (1.4) to
\[
\frac{k}{2} \frac{\partial f(t, 0)}{\partial x} = \mu_1(f(t, 0), f(t, 1))
\]
\[
+ \sigma_{00}(f(t, 0), f(t, 1)) \frac{dw_0(t)}{dt}
\]
\[
+ \sigma_{11}(f(t, 0), f(t, 1)) \frac{dw_1(t)}{dt}
\]
This formulation is exactly the generalization of the bivariate diffusion formulation given by Brennan & Schwarz [4] to the stochastic boundary conditions.

2 Mathematical Formulation

In order to make our idea clear, we consider the following simple situation:
\[
df(t, x) = \frac{k}{2} \frac{\partial^2 f(t, x)}{\partial x^2} dt + \frac{\partial f(t, x)}{\partial x} dt + \nu(t, x) dt + dw(t, x),
\]
\[
f(0, x) = f_s(x), x \in G
\]
\[
\frac{k}{2} \frac{\partial f(t, 0)}{\partial x} = \sigma_0(t) \frac{dw_0(t)}{dt}, t \in ]0, t_f[ (2.3)
\]
\[
- \frac{k}{2} \frac{\partial f(t, 1)}{\partial x} = \sigma_1(t) \frac{dw_1(t)}{dt}, t \in ]0, t_f[ (2.4)
\]
We work in the following Hilbert spaces:
\[
V = H^1(G) \subset H = L^2(G) \subset V' = \text{dual of } V.
\]
Define \( \forall \phi_1, \phi_2 \in V \)
\[
< A\phi_1, \phi_2 > = \int_G \left\{ \frac{k}{2} \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial x} \phi_2 \right\} dx.
\]
The weak form of the proposed system is
\[
(f(t), \phi) + \int_0^t < Af(s), \phi > ds
\]
\[
+ (\sigma w_h(t); \phi)_G = (f_0, \phi)
\]
\[
- \int_0^t (\nu(s), \phi) ds + (w(t), \phi) \forall \phi \in V
\]
(2.5)
where \( \Gamma = L^2(\partial G) \) and
\[
(\sigma w_h(t), \phi)_G = \sigma_0 w_0(t)\phi(0) + \sigma_1 w_1(t)\phi(1)
\]
and \( w, w_1 \) and \( w_0 \) are mutually independent Brownian motion processes; \( \forall \phi_1, \phi_2 \in H \)
\[
E\{(w(t), \phi_1)(w(t), \phi_2)\} = t(\phi_1, Q\phi_2)
\]
\[
E\{|w_0(t)|^2\} = E\{|w_1(t)|^2\} = t
\]
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Theorem 2.1 Under (2.6),
\[ k > 0, \quad f_0 \in L^2(\Omega; H) \]
and
\[ \nu \in L^2(\Omega \times [0, t_f]; V') \]
(2.5) has a unique solution in
\[ L^2(\Omega; C([0, t_f]; H) \cap L^2([0, t_f]; H)). \]

Proof: The parabolic type stochastic evolution equation with boundary noise has been studied by many authors. For example the method used can be found in the book by Rozovskii [6] and [7].

Proposition 2.1 Under
\[ f_0 \in L^2(\Omega; V), \quad \nu \in L^2(\Omega \times [0, t_f]; H) \]
and
\[ Tr\left\{ \frac{\partial}{\partial x} \left( \frac{\partial Q}{\partial x} \right)^* \right\} < \infty, \]
(2.8)
we have
\[ f \in L^2(\Omega; C([0, t_f]; V) \cap L^2([0, t_f]; H^2) \]
(2.9)
and the spot rate \( r(t) = f(t, 0) \) and the long rate \( \ell(t) = f(t, 1) \) respectively satisfy
\[ r, \ell \in L^2(\Omega; C([0, t_f]; R^1)). \]
(2.10)

Proof: By using the technique proposed by Bardos [8], it is easy to derive the above regularity property.

3 Market Model

In this section \( G \) is replaced as \( G = [0, T_d] \). Our market consists of a bank account \( B(\cdot) \) and bonds for the maturity \( T, t \leq T \leq t + T_d \) where \( t \) is a present time, i.e.,
\[ P(t, T) = \exp\{- \int_0^{T-t} f(t, x) dx\} \]
(3.1)
and the bank account \( B \) is set as
\[ \frac{dB(t)}{dt} = r(t)B(t), \quad B(0) = B_0, \]
(3.2)
where \( r(t) \) is a spot rate and is given by
\[ r(t) = f(t, 0). \]

Proposition 3.1 The bond price \( P(t, T) \) is a solution of
\[
\begin{align*}
    dP(t, T) &= \left( r(t) - \frac{k}{2} f_x(t, T - t) \\
    &+ g(T - t) \right) P(t, T) dt \\
    &+ \sigma_0 P(t, T) d\omega_0(t) - P(t, T) d\tilde{w}(t, T)
\end{align*}
\]
(3.3)
where
\[
\begin{align*}
    g(T - t) &= \frac{1}{2} \int_0^{T-t} \int_0^{T-t} q(x, y) dx dy \\
    &- \int_0^{T-t} \nu(t, x) dx \\
    f_x(t, T - t) &= \frac{\partial f(t, x)}{\partial x} \bigg|_{x = T-t} \\
    \tilde{w}(t, T) &= \int_0^t \int_0^{T-s} \sqrt{\lambda_i e_i(\tau)} d\tau d\beta_i(s)
\end{align*}
\]
and
\[ Q = \int_G q(x, y)(\cdot) dy \]
(3.4)

Proof: By using Ito's formulat to (3.1), (3.3) can be obtained.

4 No Arbitrage Opportunity

In the field of mathematical finance, it is important that the proposed model is arbitrage free. In order to prevent the free-lunch opportunity, we must show that the proposed system can be transformed to the local martingale by using the Girsanov theorem.

Mathematically speaking, the discounted bond price \( \tilde{P}(t, T) = \frac{P(t, T)}{B(t)} \) process should be a local martingale. This means that the original parabolic system should be transformed to the hyperbolic one such that
\[
(\tilde{w}(t, x, \phi) = \int_0^t \int_0^x \left( \lambda_i e_i(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i e_i(x)} \int_0^{x_i} e_i(y) dy \right) ds
\]
(3.3)
where
\[
\begin{align*}
    f(t, x, \phi) &= \int_0^t \int_0^x \left( \lambda_i e_i(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i e_i(x)} \int_0^{x_i} e_i(y) dy \right) ds \\
    &+ \int_0^t \int_0^x \left[ \nu(s, x) - \sum_{i=1}^{\infty} \sqrt{\lambda_i e_i(x)} \int_0^{x_i} e_i(y) dy \right] ds
\end{align*}
\]
(3.3)
If we can show that the process $\tilde{w}(t, x)$ is a Brownian motion process under the suitable measure, $\hat{P}$ becomes a local martingale.

**Theorem 4.1** Under

$$Q = \sum_{i=1}^{m} \sqrt{\lambda_i} e_i \otimes e_i, \quad e_i \in H^2,$$  \hspace{1cm} (4.5)

we can define a Martingale measure $\hat{P}$

$$\frac{d\hat{P}}{dP} = \exp \left\{ \int_{t}^{T} k \frac{\partial^2 f(s, x)}{\partial x^2} - \nu(s, x) \right. \\ + \tilde{q}(x, m), d\tilde{w}(s)) \\
- \frac{1}{2} \int_{t}^{T} k \frac{\partial^2 f(s, x)}{\partial x^2} - \nu(s, x) + \tilde{q}(x, m) \right\}_{|t}^{T} ds \right\}$$

and $\tilde{w}(t, x)$ is a Brownian motion process with respect to $\hat{P}$ where

$$\tilde{q}(x, m) = \sum_{i=1}^{m} \sqrt{\lambda_i} e_i(x) \int_{0}^{x} e_i(y) dy.$$

It should be noted that the operator $Q$ is an $m$-dimensional operator but the state process $\tilde{f}(t, x)$ is still an infinite-dimensional one, because the initial condition $f_0$ is still an infinite-dimensional state.

5 Mean-variance Optimal Control

We consider a portfolio comprised of $\beta$ shares of the money market fund, and $\gamma(\cdot, T)$ shares of bond maturing at dates $T$: $c(t) = \beta(t)B(t) + \int_{t}^{T} \gamma(t, T)P(t, T)dT dt$  \hspace{1cm} (5.1)

where we assume that $T < T_d.$  \hspace{1cm} (5.2)

For the self-financing portfolio, the derivatives of $\beta(t)$ and $\gamma(t, \cdot)$ with respect to $t$ become zero. So in our case, the instantaneous change in portfolio value is

$$dc(t) = \beta(t)dB(t) + \int_{t}^{T} \gamma(t, T)dP(t, T)dT.$$  \hspace{1cm} (5.3)

It is easy to show that

$$dc(t) = r(t)c(t)dt - \int_{t}^{T} \frac{k}{2} f_x(t, T - t) \gamma(t, T)P(t, T)dT dt$$

$$+ \sigma_0 \int_{t}^{T} \gamma(t, T)P(t, T)dT dw_0(t)$$

$$- \int_{t}^{T} \gamma(t, T)P(t, T)d\tilde{w}(t, T)dT.$$  \hspace{1cm} (5.4)

Setting

$$u(t, T) = \begin{cases} \gamma(t, T)P(t, T) & \text{for } t \leq T \\ 0 & \text{for otherwise} \end{cases}$$  \hspace{1cm} (5.5)

and

$$d\tilde{w}(t, T) = \sigma_0 dw_0(t) - d\tilde{w}(t, T),$$  \hspace{1cm} (5.6)

we have

$$dc(t) = r(t)c(t)dt - \frac{k}{2} f_x(t, \cdot - t)$$

$$- g(\cdot - t, u(t)) \phi_{\beta}(t) dt - (u(t), d\tilde{w}(t, \cdot))_{\tilde{T}}$$  \hspace{1cm} (5.7)

where

$$(\phi, \psi)_{\tilde{T}} = \int_{0}^{T} \phi(T) \psi(T) dT.$$  \hspace{1cm} (5.8)

From (5.2) we can not treat the bond $P(t, T)$ for all maturity $T \leq T_d$ and the market becomes incomplete [1]. In such a situation, we need to consider the mean-variance control problem instead of the usual option pricing. Hence instead of finding a portfolio $(\beta, \gamma)$, we want to construct a control $u(t, T)$ to achieve $c(t_f) = \xi$ for $a$-priori given $\xi$. It is almost impossible to achieve $c(t_f) = \xi$ a.s. for the stochastic process $c(t)$. So our control problem is to find the control $u$ to minimize

$$J(u) = \frac{1}{2} E \{ \xi^2 \}.$$  \hspace{1cm} (5.9)

For (5.8) an admissible control is set as

$$u(\cdot) \in L^2([0, t_f]; \hat{H}), u(t) \in U_{ad} a.e., a.s.$$  \hspace{1cm} (5.10)

where $U_{ad}$ is a convex closed non empty subset of $\hat{H} = L^2(0, T)$. It should be noted that our control problem is not a usual linear one, because the system (5.7) contains the random coefficients $r(t)$ and $f_x$. We introduce the adjoint system:

$$-dp(t) = r(t)p(t)dt - (h(t), d\tilde{w}(t, \cdot))_{\tilde{T}}$$  \hspace{1cm} (5.10)

$$p(t_f) = c(t_f) - \xi$$  \hspace{1cm} (5.11)

where

$$h \in L^2(\Omega \times [0, t_f]; \hat{H})$$  \hspace{1cm} (5.12)
For the details about the above stochastic adjoint equation, we refer to Bennoussan [9], Kohlmann [10].

Define $Z$ to be the solution of

\[
\begin{cases}
    dZ(t) = r(t)Z(t)dt - \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t), v(t)\right)_T dt - (v(t), d\tilde{\omega}(t, T))_T \\
    Z(0) = 0, \quad v \in U_{ad}
\end{cases}
\]

Noting that $p(t)$ is $\mathcal{F}_t$-measurable, we get

\[
p(t)Z(t) - p(0)Z(0) = -\int_0^t \left(\frac{k}{2}f_x(s, \cdot - s) - g(\cdot - s), v(s)\right)_T ds - \int_0^t p(s)(v(s), d\tilde{\omega}(s))_T \\
- \int_0^t Z(s)(h(s), d\tilde{\omega}(s))_T + \int_0^t (v(s), \tilde{Q}h(s))_T ds, \quad (5.13)
\]

where $\tilde{Q}$ is an incremental covariance operator of $\tilde{w}$. Taking the mathematical expectation to $(5.13)$ and using the initial condition $Z(0) = 0$ and the terminal condition $p(t_f) = c(t_f) - \xi$, we get

\[
E\{p(t_f)Z(t_f)\} = E\{(c(t_f) - \xi)Z(t_f)\} \\
= E\{\int_0^{t_f} [(v(s), -(\frac{k}{2}f_x(s, \cdot - s) - g(\cdot - s))p(s) + \tilde{Q}(s)h(s))_T ds) \}
\]

Hence from the stochastic maximum principle, the optimal process $h$ satisfies

\[
\tilde{Q}h(s) = \left\langle \frac{k}{2}f_x(s, \cdot - s) - g(\cdot - s) \right\rangle p(s)
\]

Now we set the following assumption:

\[
\tilde{Q} \left\{ \frac{k}{2}f_x - g \right\} p \in L^2(\Omega \times]0, t_f[; \tilde{H}). \quad (5.15)
\]

Hence the optimal system becomes

\[
dc(t) = r(t)c(t)dt - \left( u^\circ(t), \tilde{\omega}(t)\right)_T \\
- \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t), u^\circ(t)\right)_T dt, \quad (5.16)
\]

\[
dp(t) = r(t)p(t)dt - (Q \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right) \right)_T dt \\
- g(\cdot - t)p(t), d\tilde{\omega}(t)\right)_T, \quad (5.17)
\]

Here we set

\[
p(t) = P(t)(c(t) - q(t)). \quad (5.18)
\]

Hence from $(5.16)$ we get

\[
dp(t) = \tilde{P}(t)(c(t) - q(t))dt
\]

\[
+ P(t)(dc(t) - \dot{q}(t)dt)
\]

\[
= (\tilde{P}(t)c(t) - \tilde{P}(t)q(t) - P(t)\dot{q}(t))dt
\]

\[
- \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t), u^\circ(t)\right)_T P(t)dt \\
+ P(t)r(t)c(t)dt \\
- P(t)(u^\circ(t), d\tilde{\omega}(t))_T, \quad (5.19)
\]

and substituting $(5.18)$ into $(5.17)$, we obtain

\[
dp(t) = -r(t)P(t)(c(t) - q(t))dt \\
+ (Q \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right) \right)_T \times P(t)(c(t) - q(t), d\tilde{\omega}(t))_T. \quad (5.20)
\]

Comparing $(5.19)$ with $(5.20)$ and assuming

\[
P(t) > 0 \quad (5.21)
\]

we obtain

\[
u^\circ(t) = -Q \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right) \times (c(t) - q(t)) \quad (5.22)
\]

and

\[
\tilde{P}(t)c(t) - \tilde{P}(t)q(t) - P(t)\dot{q}(t) \\
+ P(t)r(t)c(t) \\
+ (\tilde{Q} \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right) \right)_T \times (c(t)P(t) - q(t)P(t)) \\
= -r(t)P(t)c(t) + r(t)P(t)q(t). \quad (5.23)
\]

Hence we have the following two equations

\[
\tilde{P}(t) + 2r(t)P(t) + (\tilde{Q} \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right)^2 \right)_T \\
- g(\cdot - t))^2 P(t) = 0, P(t_f) = 1 \quad (5.24)
\]

and

\[
\tilde{P}(t)q(t) + P(t)\cdot q(t) + (\tilde{Q} \left(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)\right) \right)_T \\
- g(\cdot - t))^2 q(t)P(t) = -r(t)P(t)q(t) \quad (5.25)
\]

Substituting $(5.24)$ into $(5.25)$, we have

\[
\left\{ \begin{array}{l}
\dot{q}(t) = r(t)q(t) \\
q(t) = \xi \end{array} \right. \quad (5.26)
\]

It is obvious that $P(t) > 0$ is satisfied from $(5.24)$. At this point we should notice that the solution $q(t)$ of $(5.26)$ must be a $\mathcal{F}_t$-measurable
solution. So we shall introduce the following process:

\[ q(\bar{t}, t) = E\{\exp(-\int_{\bar{t}}^{t} r(s) ds) | \mathcal{F}_{\bar{t}}) \xi, \text{ for } \bar{t} \geq t, \]

i.e.,

\[ \frac{dq(\bar{t}, t)}{dt} = E\{r(\bar{t}) \exp(\int_{\bar{t}}^{t} r(s) ds) | \mathcal{F}_{\bar{t}}\} \]

and

\[ \lim_{\bar{t} \to t} q(\bar{t}, t) = q(t, t). \]

Hence we get

\[ \lim_{\bar{t} \to t} \frac{dq(\bar{t}, t)}{dt} = r(t)q(t, t). \]

So the derivative of the left hand side of (5.26) should be as stated above and

\[ q(t) = q(t, t). \quad (5.28) \]

In order to realize the optimal control (5.22) we only need to obtain the explicit form of (5.28). The Riccati equation (5.24) is not used anymore as stated in Kohlmann et. al. [11]

**Proposition 5.1** Define the Dirichlet map \( D \) such that

\[ D\phi(s) = \phi(s, 0), \quad \text{for } \phi(s) \in H^{1}[0,T]. \]

The \( \mathcal{F}_{t} \)-measurable solution of (5.26) is given by

\[ q(t) = \exp\{\int_{t}^{T} -D(\Phi(s,t))f(t) + \int_{t}^{T} \Phi(s,\tau) \nu(\tau) d\tau \}
\]

\[ \times \exp\{\frac{1}{2} \int_{t}^{T} [D \int_{\tau}^{T} \Phi(s,\tau) d\tau]^{T} \Phi(s,\tau) d\tau \}
\]

\[ \times (D \int_{\tau}^{T} \Phi(s,\tau) d\tau \}
\]

\[ + \sigma_{s}^{2} [D \int_{t}^{T} (D \Phi^{*}(s,\tau) d\tau \}^{T} \sigma_{s}^{2} d\tau \}
\]

\[ q(t) = \exp\{\int_{t}^{T} -D(\Phi(s,t))f(t) + \int_{t}^{T} \Phi(s,\tau) \nu(\tau) d\tau \}
\]

\[ \times \exp\{\frac{1}{2} \int_{t}^{T} [D \int_{\tau}^{T} \Phi(s,\tau) d\tau]^{T} \Phi(s,\tau) d\tau \}
\]

\[ \times (D \int_{\tau}^{T} \Phi(s,\tau) d\tau \}
\]

\[ + \sigma_{s}^{2} [D \int_{t}^{T} (D \Phi^{*}(s,\tau) d\tau \}^{T} \sigma_{s}^{2} d\tau \}
\]

\[ q(t) = \exp\{\int_{t}^{T} -D(\Phi(s,t))f(t) + \int_{t}^{T} \Phi(s,\tau) \nu(\tau) d\tau \}
\]

\[ \times \exp\{\frac{1}{2} \int_{t}^{T} [D \int_{\tau}^{T} \Phi(s,\tau) d\tau]^{T} \Phi(s,\tau) d\tau \}
\]

\[ \times (D \int_{\tau}^{T} \Phi(s,\tau) d\tau \}
\]

\[ + \sigma_{s}^{2} [D \int_{t}^{T} (D \Phi^{*}(s,\tau) d\tau \}^{T} \sigma_{s}^{2} d\tau \}
\]

where \( \Phi(s,t) \) is a semigroup generated by \( \Lambda \).

**Remark 5.1** From above proposition, to realize the optimal control \( u(t) \), we need the information for \( e(t) \) and \( f(t) \) processes.

**6 Conclusions**

From the empirical consideration of [4], the term structure is modeled by the stochastic parabolic systems with boundary noises. The arbitrage-free opportunity can be found under the condition that the number of the random source is finite (4.13). In the mean-variance optimal control problem, we set the assumption (5.15). This assumption is also recovered under the same arbitrary-free condition.

**References**


