Recursive Subspace Identification Based on Rank- k Modification

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Abstract

In this paper, we present a recursive algorithm for a subspace identification. Using the results of the subspace extraction via Schur complement (SES) approach, we derive a recursive formula of an error covariance matrix in subspace methods. This recursive procedure requires the eigenvalue decomposition (EVD) of the covariance matrix at each step. Therefore we introduce the rank- k modification algorithm with the spectrum slicing theory and achieve to update the EVD recursively. The algorithm improves a computational complexity from $O(N^3)$ to $O(N^2k)$ class where $N \times N$ is the size of the covariance matrix.

1 Introduction

Subspace model identification methods have lately attracted much attention because of being essentially suitable for the identification of MIMO systems. Subspace-based methods directly realize system matrices of the state space model from the input-output data without intermediate expressions, they are characterized by the determination of the extended observability matrix from input-output data. To estimate the extended observability matrix, the QR factorization and the singular value decomposition (SVD) are used in a basic subspace method, such as the MOESP algorithm [1], [2] [3]. We considered the QR factorization in the MOESP and shown an alternative derivation of the estimate of the extended observability matrix by using the subspace extraction via Schur complement (SES) [5]. A recursive subspace identification algorithm was also proposed and we paved another way to the recursive version of subspace-based methods. The proposed recursive algorithm can be applicable to the IV-based algorithm of the subspace method, but the algorithm requires an exact EVD (or SVD) to update the each step and it is necessary to reduce the computational cost. Here we will use the rank- k modification algorithm involving the spectrum slicing theory [6] for updating the EVD of the Schur complement matrix recur-

sively and achieve the reduction of the computational complexity.

The paper is organized as follows. In the next section, we review the subspace extraction via the Schur complement based on the MOESP algorithms. In section 3, we consider a modified eigenvalue problem and present a recursive EVD update procedure for an error covariance matrix. Finally section 4 concludes this paper.

2 Review of the subspace extraction via the Schur complement

In this section, we review the subspace extraction via Schur complement (SES) [5] based on the MOESP algorithms [1], [2], [3]. Consider the following n -th order discrete time linear time-invariant state-space model:

$$x_{k+1} = Ax_k + Bu_k \quad (1)$$

$$y_k = Cx_k + Du_k + v_k \quad (2)$$

with m inputs and l outputs collected in u_k and y_k , respectively, and v_k is an unmeasurable stochastic disturbance. The noise term is assumed to be a zero-mean stationary random process uncorrelated with u_k . The unknown system matrices A , B , C and D have appropriate dimensions. It is assumed that the model is minimal, that is, the system is completely reachable and observable. We introduce a stacked input vector:

$$u_i(k+N) \triangleq [u_{k+N}^T, u_{k+N+1}^T, \dots, u_{k+N+i-1}^T]^T \quad (3)$$

where $u_i(k+N) \in \mathbb{R}^{mi \times 1}$. Stacked vectors of the output and the noise are defined similarly. We define the Hankel matrix:

$$U_{k,i,N} \triangleq [u_i(k), u_i(k+1), \dots, u_i(k+N-1)]. \quad (4)$$

Corresponding definitions for the output and the noise are in a similar way. In relation to the order of the system n , the pair i and N satisfy $i > n$ and $N \gg n$. We define a state vector sequence as follows:

$$X_{k,N} \triangleq [x_k \ x_{k+1} \ \dots \ x_{k+N-1}]. \quad (5)$$

Then we obtain a relation

$$Y_{k,i,N} = \Gamma_i X_{k,i,N} + H_i U_{k,i,N} + V_{k,i,N} \quad (6)$$

where

$$\Gamma_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, H_i = \begin{bmatrix} h_0 & & & O \\ h_1 & h_0 & & \\ \vdots & \vdots & \ddots & \\ h_{i-1} & h_{i-1} & \cdots & h_0 \end{bmatrix} \quad (7)$$

and

$$h_k = \begin{cases} D & k = 0 \\ CA^{k-1}B & k \geq 1 \end{cases} \quad (8)$$

The main problem is to estimate the column space of the extended observability matrix Γ_i , and the H_i is a block lower triangular and the Toeplitz matrix of impulse responses from u_k to y_k . To obtain the estimate of the Γ_i , if the noise term $v_k = 0$ we consider the data product moment as follows:

$$\begin{bmatrix} U_{k,i,N} \\ Y_{k,i,N} \end{bmatrix} \begin{bmatrix} U_{k,i,N}^T & Y_{k,i,N}^T \end{bmatrix} = \begin{bmatrix} U_{k,i,N} U_{k,i,N}^T & U_{k,i,N} Y_{k,i,N}^T \\ Y_{k,i,N} U_{k,i,N}^T & Y_{k,i,N} Y_{k,i,N}^T \end{bmatrix} \quad (9)$$

The Schur complement [4] S of the submatrix $U_{k,i,N} U_{k,i,N}^T$ in (9) is represented by

$$\begin{aligned} S &= Y_{k,i,N} \Pi_{U_{k,i,N}}^\perp Y_{k,i,N}^T \\ &= \hat{E}_N \hat{E}_N^T \end{aligned} \quad (10)$$

The $\Pi_{U_{k,i,N}}^\perp$ is defined as

$$\Pi_{U_{k,i,N}}^\perp = I - \Pi_{U_{k,i,N}} \quad (11)$$

where the $\Pi_{U_{k,i,N}} = U_{k,i,N} (U_{k,i,N} U_{k,i,N}^T)^{-1} U_{k,i,N}^T$ and the error $\hat{E}_N$ is denoted by

$$\hat{E}_N = Y_{k,i,N} - \hat{G}_N U_{k,i,N} \quad (12)$$

where $\hat{G}_N$ is defined as

$$\hat{G}_N \triangleq Y_{k,i,N} U_{k,i,N}^T (U_{k,i,N} U_{k,i,N}^T)^{-1}. \quad (13)$$

A recursive formula of the matrix $\hat{E}_N \hat{E}_N^T$ is derived as follows [5]:

$$\alpha = 1 + u_i^T P_N u_i \quad (14a)$$

$$e_i = y_i - \hat{G}_N u_i \quad (14b)$$

$$\hat{G}_{N+1} = \hat{G}_N + e_i u_i^T P_N / \alpha \quad (14c)$$

$$\mathcal{R}_{N+1} = \mathcal{R}_N + e_i e_i^T / \alpha \quad (14d)$$

$$P_{N+1} = P_N - P_N u_i u_i^T P_N / \alpha \quad (14e)$$

where the error covariance $\mathcal{R}_N \triangleq \hat{E}_N \hat{E}_N^T$ and $P_N \triangleq (U_{k,i,N} U_{k,i,N}^T)^{-1}$. and the input u_i and the output y_i

omitted the subscript $(k + N)$ are the new date set. Suppose the following matrix as in (9) for a stochastic case.

$$\begin{bmatrix} U_{k,i,N} \\ Y_{k,i,N} \end{bmatrix} \Pi_{\Omega_{k,i,N}} \begin{bmatrix} U_{k,i,N}^T & Y_{k,i,N}^T \end{bmatrix} = \begin{bmatrix} U_{k,i,N} U_{k,i,N}^T & U_{k,i,N} Y_{k,i,N}^T \\ Y_{k,i,N} U_{k,i,N}^T & Y_{k,i,N} \Pi_{\Omega_{k,i,N}} Y_{k,i,N}^T \end{bmatrix} \quad (15)$$

where $\Pi_{\Omega_{k,i,N}} = \Omega_{k,i,N}^T (\Omega_{k,i,N} \Omega_{k,i,N}^T)^{-1} \Omega_{k,i,N}$. The matrix $\Omega_{k,i,N}$ is defined as

$$\Omega_{k,i,N} \triangleq [\omega_i(k) \quad \omega_i(k+1) \quad \cdots \quad \omega_i(k+N-1)] \quad (16)$$

where

$$\omega_i(k) \triangleq [u_i(k)^T \quad \phi(k)^T]^T \quad (17)$$

and $\phi(k)$ is the stacked vector of an instrumental variable such as the $u_i(k)$. Then we have the Schur complement of $U_{k,i,N} U_{k,i,N}^T$ in (15)

$$\begin{aligned} \hat{S} &= Y (\Pi_{\Omega_{k,i,N}} - \Pi_{U_{k,i,N}}) Y^T \\ &= \hat{E}_N \hat{E}_N^T - \hat{E}_N^* \hat{E}_N^{*T} \\ &= \tilde{E}_N \tilde{E}_N^T \end{aligned} \quad (18)$$

where

$$\hat{E}_N^* \triangleq Y_{k,i,N} - \hat{G}_N^* \Omega_{k,i,N} \quad (19)$$

$$\hat{G}_N^* \triangleq Y_{k,i,N} \Omega_{k,i,N}^T (\Omega_{k,i,N} \Omega_{k,i,N}^T)^{-1} \quad (20)$$

Then a recursive formula of the error covariance matrix $\hat{E}_N^* \hat{E}_N^{*T}$ is summarized as follows:

$$\beta = 1 + \omega_i^T Q_N \omega_i \quad (21a)$$

$$e_i^* = y_i - \hat{G}_N^* \omega_i \quad (21b)$$

$$\hat{G}_{N+1}^* = \hat{G}_N^* + e_i^* \omega_i^T Q_N / \beta \quad (21c)$$

$$\mathcal{R}_{N+1}^* = \mathcal{R}_N^* + e_i^* e_i^{*T} / \beta \quad (21d)$$

$$Q_{N+1} = Q_N - Q_N \omega_i \omega_i^T Q_N / \beta \quad (21e)$$

where $\mathcal{R}_N^* \triangleq \hat{E}_N^* \hat{E}_N^{*T}$, $Q_N \triangleq (\Omega_{k,i,N} \Omega_{k,i,N}^T)^{-1}$. Therefore a recursive formula of $\tilde{E}_N \tilde{E}_N^T$ is obtained

$$\tilde{\mathcal{R}}_{N+1} = \mathcal{R}_{N+1} - \mathcal{R}_{N+1}^* \quad (22)$$

where $\tilde{\mathcal{R}}_N \triangleq \tilde{E}_N \tilde{E}_N^T$.

We show that the Schur complement matrix (10) and (18) are computed recursively and an estimate of the Γ_i is obtained by computing the EVD of the Schur complement matrix. Once an estimate of the Γ_i has been obtained, we can derive an estimate of the quadruple of the system matrices.

3 Rank- k modification algorithm

In the previous section, we derived a recursive formula of an error covariance matrix obtained by using the SES

procedure. But it requires the EVD update for each step. Therefore, to make effective use of the recursive form of the covariance matrix, we introduce the rank- k modification algorithm to solve the recursive update of the EVD. We apply the algorithm to the computation of the EVD and improve the computational complexity.

3.1 Modified eigenvalue problem

First we consider a general modification of a symmetric eigenproblem as follows:

$$\hat{R} = R + E \quad (23)$$

where $\hat{R}$ and $R \in \mathbb{R}^{N \times N}$ are the modified and original covariance matrices and $E \in \mathbb{R}^{N \times N}$ is the additive modification matrix. We assume that $\text{rank}(E) = k$, where k is usually much smaller than N and it has the following weighted outer product expansion:

$$E = USU^T \quad (24)$$

where $U \in \mathbb{R}^{N \times k}$ and $U \in \mathbb{R}^{k \times k}$ is a nonsingular matrix. An example of an expansion in (24) is the EVD of the E . But it can also choose a simple expansion, for example, S has diagonal elements equal to 1 or -1 , corresponding to updating or downdating, respectively. Suppose that we also have *a priori* information of the EVD of R as follows:

$$R = QDQ^T \quad (25)$$

where $D \in \mathbb{R}^{N \times N}$, $Q \in \mathbb{R}^{N \times N}$, $D = \text{diag}[d_1, d_2, \dots, d_N]$ is the diagonal eigenvalue matrix and $Q = [q_1, q_2, \dots, q_N]$ is the corresponding eigenvector matrix. Suppose that λ and x are the eigenpair of $\hat{R}$, the eigenvalue can be determined by solving for the zeros of

$$\det[\hat{R} - \lambda I] = 0. \quad (26)$$

Substituting $\hat{R}$ for (25) gives the following expression:

$$(R - \lambda I)x + USU^T x = 0. \quad (27)$$

To split away the rank- k modification aspect from the rest of the problem, we induce the following system of equations

$$(R - \lambda I)x + Uy = 0 \quad (28)$$

$$U^T x - S^{-1}y = 0 \quad (29)$$

where $y \triangleq SU^T x$ and $y \in \mathbb{R}^k$. Solving x in terms of y in (28) and substituting it into (29) gives the following:

$$W(\lambda)y = 0 \quad (30)$$

where

$$\begin{aligned} W(\lambda) &= S^{-1} + U^T (R - \lambda I)^{-1} U \\ &= S^{-1} + U^T Q (D - \lambda I)^{-1} Q^T U \end{aligned} \quad (31)$$

$W(\lambda)$ is called the Weinstein-Aronszjan (W-A) matrix. The modified eigenvalues can be obtained from solving $\det[W(\lambda)] = 0$ rather than from (26). To obtain the modified eigenvalues, a nonlinear equation $w(\lambda) = W(\lambda)$ is evaluated. A nonlinear search algorithm is used with the spectrum slicing theorem and the search can be carried out in parallel. Several situations exist for deflating the problem, which an original eigenpair remains the one of the modified matrix $\hat{R}$ and a modified eigenvalue has multiplicity [9]. For briefly we assume that all deflation procedures have been carried out, and an efficient algorithm can be used to localize the eigenvalue. The spectrum-slicing formula is described as follows:

$$N_{\hat{R}}(\lambda) = N_R(\lambda) + D^+[W(\lambda)] - D^+[S] \quad (32)$$

where $N_{\hat{R}}(\lambda)$ and $N_R(\lambda)$ are the number of eigenvalues of $\hat{R}$ and R less than λ , respectively, $D^+[W(\lambda)]$ is the positive inertia of $W(\lambda)$ and, likewise, $D^+[S]$ is the positive inertia of S . The above formula (32) provides the information of great computational value. $W(\lambda)$ is easy compute for each value of λ . The dominant cost is on the order of Nk^2 flops per evaluation. As an algorithm to evaluate the positive inertia of the W-A matrix $W(\lambda)$, we can use a LDL^T decomposition [4]. This requires k^3 number of operations. The $N_R(\lambda)$ is available readily, if we know the EVD of R . $D^+[S]$ can be determined easily. The spectrum-slicing formula provides an easy method for counting the eigenvalues of $\hat{R}$ in a user-defined interval. This information enables us to isolate the eigenvalues in disjoint intervals. The eigenvalue search algorithm can be summarized as follows:

1. Use the knowledge of the original eigensystem and the formula (32) to locate the eigenvalues to disjoint intervals.
2. For each iteration step of each eigenvalue search in the interval (l, u) set $\lambda = (l + u)/2$, and evaluate $N_{\hat{R}}(\lambda)$. Repeat until convergence to user-defined accuracy.

To use the interlacing property of the modified eigenvalues [9] with the spectrum-slicing formula, we can obtain disjoint intervals containing a single eigenvalue λ . Once we obtain the intervals, a nonlinear search algorithm is used to compute the modified eigenvalues. The following numerical example illustrates this algorithm.

Consider an rank-2 modification problem as follows:

$$\hat{R} = R + \rho_1 u_1 u_1^T - \rho_2 u_2 u_2^T \quad (33)$$

where

$$R = QDQ^T$$

and

$$\begin{aligned}
Q &= \begin{bmatrix} \text{Columns 1 through 3} & & \\ -0.84016269269677 & 0.52486135387007 & -0.06042557281944 \\ -0.41846130016525 & -0.47060033103245 & 0.37420176105139 \\ -0.25883669817876 & -0.49707709753316 & -0.41765666154376 \\ -0.17422365789405 & -0.38207361843070 & 0.19356666381097 \\ -0.12066436778059 & -0.27431210414092 & -0.61827272291544 \\ -0.08430054044055 & -0.18639438396821 & 0.51201122253660 \\ \text{Columns 4 through 6} & & \\ 0.0206778867824 & 0.06277016861903 & 0.10309359832206 \\ -0.23458922317983 & -0.32052079902856 & -0.55284068634659 \\ 0.70137976916333 & -0.10062051758428 & 0.09707093644093 \\ -0.11337958425962 & 0.86940788006208 & 0.13218480150704 \\ -0.66303361247891 & -0.13388252168866 & 0.26531814690681 \\ -0.01175752067275 & -0.33076765748022 & 0.76579995950366 \end{bmatrix} \\
D &= \text{diag}(d) \\
d &= \begin{bmatrix} 7.83150888076929 \\ 0.06030028103283 \\ 0.00001721698357 \\ 0.00000709308327 \\ 0.00000686335677 \\ 0.00000193951690 \end{bmatrix} \\
\rho_1 &= 0.99708144002046 \\
u_1 &= \begin{bmatrix} 3.91446643181680 \\ 1.40489247654198 \\ 0.82572596834002 \\ 0.67996447199948 \\ 0.31729327632275 \\ 0.08943890601040 \end{bmatrix} \\
\rho_2 &= 0.99638675704850 \\
u_2 &= \begin{bmatrix} -0.47426373506062 \\ -0.47230696512167 \\ -0.24726753243728 \\ -0.02466335420981 \\ -0.16608069292878 \\ -0.24902541027738 \end{bmatrix}
\end{aligned}$$

This modification coincides an update with a down-date. Then $S = \text{diag}[\rho_1 - \rho_2]$ and $D^+[S] = 1$. The obtained eigenvalues are

$$\lambda = 1.0e + 004 * \begin{bmatrix} 7.83326428037729 \\ 0.06033464123542 \\ 0.00001722237330 \\ 0.00000708052010 \\ 0.00000694040046 \\ 0.00000194342674 \end{bmatrix} \quad (34)$$

We illustrate that how the spectrum-slicing formula (32) can be used to locate the eigenvalues in Table 1. MATLAB package is used to simulate this example and we use the MATLAB like notation. Let us start with the interval $(d(0) \ d(1))$, where $d(0) = d(1) + \rho|u_1|^2$. The counting formula evaluates at $d(1) + \varepsilon$ and $d(0) - \varepsilon$, then $N_{\hat{R}}(d(0) + \varepsilon) = 5$ and $N_{\hat{R}}(d(0) - \varepsilon) = 6$. The ε is a small constant depending on a required accuracy, which $\varepsilon = 10^{-4}$ in this example. The result of each $N_{\hat{R}}(\lambda)$ indicate that there is one eigenvalue in this interval. A bisection method can be executed until the user-defined accuracy. The iteration step converges in 18 steps. The maximum number of iterations q required to converge can be determined such that $2^q > (u - l)/\varepsilon$ where l and u are lower and upper bound of the interval.

Once the modified eigenvalues are obtained, the corresponding eigenvectors can be computed. We can solve the intermediate vector y from the $k \times k$ equation (30). The eigenvector x can computed explicitly using (28). This two-step procedure is much more efficient than solving the original $n \times n$ system. The explicit expression x relating y is given by

$$\begin{aligned}
x &= -(R - \lambda I)^{-1} U y \\
&= -Q(D - \lambda I)^{-1} Q^T U y \\
&= -\sum_{i=1}^N \frac{q_i^T U y}{(d_i - \lambda)} q_i. \quad (35)
\end{aligned}$$

Table 1: Bisection search for eigenvalues with the spectrum-slicing formula

Interval	Midpoint	$N_{\hat{R}, D_W, N_{\hat{R}}}$
1.0e+004 *	1.0e+004 *	1.0e+004 *
7.83150888076929 , 7.83335842892308	7.83243365484618	6 , 0 , 5
7.83243365484618 , 7.83335842892308	7.83289604188463	6 , 0 , 5
7.83289604188463 , 7.83335842892308	7.83312723540385	6 , 0 , 5
7.83312723540385 , 7.83335842892308	7.83324283216346	6 , 0 , 5
7.83324283216346 , 7.83335842892308	7.83330063054327	6 , 1 , 6
7.83324283216346 , 7.83330063054327	7.83327173135337	6 , 1 , 6
7.83324283216346 , 7.83327173135337	7.83325728175842	6 , 0 , 5
7.83325728175842 , 7.83327173135337	7.83326450655589	6 , 1 , 6
7.83325728175842 , 7.83326450655589	7.83326089415715	6 , 0 , 5
7.83326089415715 , 7.83326450655589	7.83326270035652	6 , 0 , 5
7.83326270035652 , 7.83326450655589	7.83326360345621	6 , 0 , 5
7.83326360345621 , 7.83326450655589	7.83326405500605	6 , 0 , 5
7.83326405500605 , 7.83326450655589	7.83326428078097	6 , 1 , 6
7.83326405500605 , 7.83326428078097	7.83326416789351	6 , 0 , 5
7.83326416789351 , 7.83326428078097	7.83326422433724	6 , 0 , 5
7.83326422433724 , 7.83326428078097	7.83326425255910	6 , 0 , 5
7.83326425255910 , 7.83326428078097	7.83326426667004	6 , 0 , 5
7.83326426667004 , 7.83326428078097	7.83326427372550	6 , 0 , 5

and normalizing x gives the update eigenvectors as follows:

$$\hat{q} = \frac{x}{\|x\|_2} \quad (36)$$

The total computational cost to obtain the eigenvectors $\hat{q}$, that is, an estimate of the extended observability matrix is $O(N^2 k)$, where N is the rank of the covariance matrix and k is the rank of an additive matrix.

3.2 Rank-1 modification

For the deterministic case of the SES procedure, it remarks that the updating form of the Schur complement matrix in (14d) is rank-1 modification. Then we consider the following rank-1 case. Assume that we have *a priori* information of the EVD of the original covariance $\mathcal{R}_N$.

$$\mathcal{R}_N = Q_1 D_1 Q_1^T \quad (37)$$

where D_1 has eigenvalues of $\mathcal{R}_N$ in diagonal elements as in (23). Then we have the W-A matrix as follows:

$$\begin{aligned}
w(\lambda) &= \alpha + e_i^T (\mathcal{R}_N - \lambda I)^{-1} e_i \\
&= \alpha + e_i^T Q_1 (D_1 - \lambda I)^{-1} Q_1^T e_i \\
&= \alpha + \sum_{j=1}^N \frac{|q_j^T e_i|^2}{(d_j - \lambda)} \quad (38)
\end{aligned}$$

We also assume that no deflation is possible, that is, all eigenvalues d_j are distinct and $q_j^T e_i \neq 0$. The interlacing property of the eigenvalues λ_j can be represented as

$$d_N < \lambda_N < d_{N-1} < \dots < \lambda_1 < d_0 \quad (39)$$

where $d_0 = d_1 + \frac{1}{\alpha} |e_i|^2$. The corresponding search intervals for each λ_j can be limited in $I_j = (d_j, d_{j+1})$ for $j = 1, 2, \dots, N$.

The following Newton algorithm [8] can be applied

in an iterative search to obtain the j -th eigenvalue λ .

$$\lambda_* = \lambda_j - \frac{w(\lambda_j)}{w'(\lambda_j)} \quad (40a)$$

$$\lambda_{j+1} = \begin{cases} \lambda_* & \text{if } \lambda_* \in I_j \\ \frac{\lambda_j + u}{2} & \text{if } \lambda_* > u, \lambda_j \text{ replaces } l \\ \frac{l + \lambda_j}{2} & \text{if } \lambda_* < l, \lambda_j \text{ replaces } u \end{cases} \quad (40b)$$

and stopping if $|\lambda_{j+1} - \lambda_j| < \delta \lambda_{j+1}$ where δ is the convergence threshold. To identify the full set of the eigenvalues λ , the above iterative algorithm (40) is used in parallel.

Once the modified eigenvalues are determined, the eigenvectors can be solved in a procedure (35), i.e.,

$$x = - \sum_{j=1}^N \frac{q_j^T e_i}{(d_j - \lambda)} q_j \quad (41a)$$

$$\hat{q} = \frac{x}{\|x\|_2} \quad (41b)$$

3.3 Rank-2 modification

In this section, we consider the recursive update form of the covariance matrix $\tilde{\mathcal{R}}$ in (22). This recursion forms the rank- k modification problem, where $k = l \times i$ is the rank of $\mathcal{R}^*$. To avoid to apply the modification algorithm straightly for reducing the computational cost, here we can represent another form of (22) as follows.

$$\begin{aligned} \tilde{\mathcal{R}}_{N+1} &= \mathcal{R}_{N+1} - \mathcal{R}_{N+1}^* \\ &= \mathcal{R}_N + e_i e_i^T / \alpha - (\mathcal{R}_N^* + e_i^* e_i^{*T} / \beta) \\ &= \tilde{\mathcal{R}}_N + \frac{1}{\alpha} e_i e_i^T - \frac{1}{\beta} e_i^* e_i^{*T} \\ &= \tilde{\mathcal{R}}_N + U_2 S_2 U_2^T \end{aligned} \quad (42)$$

where

$$U_2 = [e_i \ e_i^*] \quad (43)$$

$$S_2 = \begin{bmatrix} 1/\alpha & 0 \\ 0 & -1/\beta \end{bmatrix} \quad (44)$$

This update form is rank-2 modification [7]. Suppose that we have the result of the EVD of $\tilde{\mathcal{R}}_N$ for an initialization of the algorithm.

$$\tilde{\mathcal{R}}_N = Q_2 D_2 Q_2^T \quad (45)$$

Then the Weinstein-Aronszjan matrix is given by

$$W(\lambda) = S_2^{-1} + U_2^T Q_2 (D_2 - \lambda I)^{-1} Q_2^T U_2 \quad (46)$$

The W-A matrix in (46) can be computed easily by using the following determinant:

$$\begin{aligned} w(\lambda) &= \left(\alpha + \sum_{j=1}^N \frac{|\tilde{y}_j|^2}{(d_j - \lambda)} \right) \left(\beta - \sum_{j=1}^N \frac{|\tilde{z}_j|^2}{(d_j - \lambda)} \right) \\ &\quad + \left(\sum_{j=1}^N \frac{\tilde{y}_j^T \tilde{z}_j}{(d_j - \lambda)} \right) \left(\sum_{j=1}^N \frac{\tilde{z}_j^T \tilde{y}_j}{(d_j - \lambda)} \right) \end{aligned} \quad (47)$$

where $\tilde{y} = q_j^T e_i$ and $\tilde{z} = q_j^T e_i^*$. we assume that all deflation procedures have been carried out, then the interlacing property for rank-2 modification are as follows:

$$d_j + 1 < \lambda_j < d_{j-1} \quad (48)$$

where $d_N + 1 = d_N - \frac{1}{\beta} |e_i^*|^2$ and $d_0 = d_1 + \frac{1}{\alpha} |e_i|^2$. Therefore the modified eigenvalues are bounded by the upper and the lower eigenvalues of the original covariance matrix. In this rank-2 modification, each interval may have zero, one or two eigenvalues, and the intervals which the modified eigenvalues exist overlies each other. Thus we cannot apply the nonlinear search algorithm to each interval restricted with the original eigenvalues directly. Fortunately we can use the spectrum-slicing formula to isolate the modified eigenvalues to disjoint intervals. Once the eigenvalues are isolated, we can employ the Newton's method for identifying the modified eigenvalue and this nonlinear search can be implemented in parallel as in the rank-1 modification. The eigenvector can be determined in two steps. First an intermediate vector y can be obtained as the solution of (30), and the eigenvector x can be obtained explicitly in terms of y as in (35). Since $k = 2$, we can specify the solution $y = [1 \ v]^T$. Solving $W(\lambda)y = 0$ gives v

$$v = - \frac{(\alpha + a)}{b} \quad (49)$$

where

$$a = \sum_{j=1}^N \frac{|\tilde{y}_j|^2}{(d_j - \lambda)}$$

and

$$b = \left(\sum_{j=1}^N \frac{\tilde{y}_j^T \tilde{z}_j}{(d_j - \lambda)} \right)$$

We have the following expression for the eigenvectors

$$\begin{aligned} x &= - \sum_{j=1}^N \frac{\tilde{y}_j - v \tilde{z}_j}{(d_j - \lambda)} q_j \\ \hat{q} &= \frac{x}{\|x\|_2}. \end{aligned} \quad (50)$$

4 Conclusion

In this paper, we presented the recursive subspace identification based on rank- k modification algorithm. Using the rank- k modification algorithm involving the spectrum-slicing theorem, we derive the recursive algorithm to estimate the extended observability matrix for the both deterministic and stochastic cases. The algorithm improved the computational complexity from $O(N^3)$ to $O(N^2k)$ class where $N \times N$ is the size of the covariance matrix.

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