Performance analysis of recursive maximum filter

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Abstract

The recursive maximum filter (RMF) is an algorithm devised to solve the problem of detecting small moving objects in noisy image sequences. This problem arises in such applications as small target detection for infrared image sensors, and dim star detection for spaceborne star sensors. In this paper, we analyze the performance of the RMF. First, we formulate the RMF as a Bellman equation for the dynamic programing. Then, false alarm probability is evaluated using the extreme value theory. Finally, detection probability is obtained as a function of input signal to noise ratios (SNRs). It is shown that input SNRs required for target detection can be reduced to less than half by the RMF.

1. Introduction

The problem of detecting small moving objects in noisy image sequences arises in such applications as small target detection for infrared image sensors, and dim star detection for spaceborne star sensors. The difficulty of the problem lies in the fact that targets cannot be detected from each frame due to a high background noise level. The recursive maximum filter (RMF) is an algorithm devised to solve this problem[1], [2]. We have applied the RMF to various image sequences and found it effective for enhancing dim targets. However, the algorithm was derived heuristically, and its principle was not clear. In this paper, we show that under an adequate model, the RMF is derived as a Bellman equation of dynamic programming (DP).

DP-based dim target detection algorithms have already been proposed[3], and their performances have been analyzed[4], [5], [6]. Their aim was to estimate both position and velocity of a target on the assumption that the target moves with constant velocity. The algorithms are complicated and the amount of calculation required is huge. The difference between the RMF and these DP-based algorithms is that the RMF assumes a target with a kind of a Brownian motion. In some applications, apparent target motion on images includes random motion caused by the vibration of a camera, therefore the constant velocity assumption is too restrictive.

Recently, a method to evaluate the performance of DP-based algorithms for small target detection was proposed[6]. This method uses the extreme value theory in the field of order statistics. The accuracy of the approximation for the false alarm probability ($P_{FA}$) and the detection probability ($P_{D}$) is superior to that of the conventional method[4], [5], which uses an assumption of independence of distribution for all stages and states, in addition to Gaussian approximation.

In this paper, we evaluate $P_{FA}$ for the RMF by applying the method of [6]. We obtain the detection probability of a target as a function of input SNRs.

2. Recursive Maximum Filter

2.1 Model of small target detection

Let $Y_{ij}(k), i, j = 1, \ldots, n, k = 1, 2, \ldots$ be the image sequences to be processed, where $(i, j)$ denotes the pixel number, and $k$, the frame number at time $t_k$. The image data contain targets, a structured background, clutter and noise. Here we assume that the backgrounds except for noise are negligible, or have already been whitened by pre-processing.

We assume that the targets move randomly from frame to frame. This assumption does not exclude the possibility of constant velocity movement, but implies a lack of knowledge about the motion of the targets. We also assume that the maximum velocity of a target is known, which is denoted by $v_{\text{max}}$. For a pixel $(i, j)$, we define the neighborhood $D(i, j)$ as such a region that the target, which is present at the pixel $(i, j)$ in one image, may have existed in the preceding image. Let $\nu$ be the number of pixels contained in $D(i, j)$. Here $D(i, j)$ and $\nu$ are determined by the maximum velocity of the targets. For example, when $v_{\text{max}} \leq 1$ pixels/frame, $D(i, j)$ is given by

$$D(i, j) = \{(i', j') ; i' = i, i \pm 1, j' = j, j \pm 1\}, \quad (1)$$

and $\nu = 9$. 

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We assume that the background noise $N_{ij}(k)$ is Gaussian with a mean of $\mu_N$ and a variance of $\sigma_N^2$, and it is white both in time and space, i.e.,

$$E(N_{ij}(k)) = \mu_N,$$  \hspace{1cm} (2)  

$$E((N_{ij}(k) - \mu_N)(N_{ij'}(k') - \mu_N)) = \sigma_N^2 \delta_{ii'} \delta_{jj'} \delta_{kk'},$$  \hspace{1cm} (3)  

where $E(\cdot)$ denotes the expectation, and $\delta_{ii'}$ is the Kronecker’s delta.

Let $A$ be the intensity of a target. Then the image data are represented as

$$Y_{ij}(k) = \begin{cases}  A + N_{ij}(k), & \text{target is present} \\ N_{ij}(k), & \text{otherwise} \end{cases},$$  \hspace{1cm} (4)  

We define the input SNR of a target as

$$\text{SNR}_{\text{in}} = \frac{A - \mu_N}{\sigma_N}.$$

If we detect targets in a single frame of image data by thresholding, then $\text{SNR}_{\text{in}}$ must be greater than 6 or 7.

The problem treated in this paper is the detection of targets that cannot be detected in a single frame of image data. Therefore, the $\text{SNR}_{\text{in}}$ value we are concerned with is, say, less than 4 or 5.

2.2 Algorithm of Recursive Maximum Filter

The algorithm of the RMF is represented as$[1]$

$$X_{ij}(0) = 0$$  \hspace{1cm} (6)  

$$X_{ij}(k) = Y_{ij}(k) + \alpha \max_{(i',j') \in D(i,j)} X_{i',j'}(k - 1), \quad k = 1, 2, \cdots,$$  \hspace{1cm} (7)  

where $X_{ij}(k), k = 0, 1, 2, \cdots$ is the output image sequences of the RMF, $\alpha$ is a forgetting coefficient, that takes a value somewhat less than 1 (for example 0.95 or 0.98) and has a function to avoid the divergence of the RMF with elapsed time.

The algorithm of (7) can be implemented by the architecture shown in Fig. 1. The RMF in Fig. 1 has a local maximum filter in its feedback loop. Targets are enhanced by the RMF with elapsed time and detected by thresholding with an adequate threshold; although the RMF is very simple, it is very good at detecting dim moving targets. In Fig. 2, we show an example of an RMF application: dim star tracking.

2.3 Formulation of the RMF as a dynamic programming algorithm

Although the RMF was originally derived in a heuristic manner$[1]$, it can be formulated as a dynamic programming algorithm. In this formulation, the state space is the set of all pixels in an image $\{(i, j)\}_{i,j=1,\cdots,N}$, and the stages are frames of an image sequence. If we denote the trajectory of a target as $(i_k, j_k), k = 1, 2, \cdots$, then the target’s motion is modeled using the transition probability:

$$p(i_{k+1}, j_{k+1} | i_k, j_k, k) = \begin{cases} 1/\nu, & \text{if } (i_k, j_k) \in D(i_{k+1}, j_{k+1}), \\ 0, & \text{otherwise} \end{cases},$$

which represents a kind of a Brownian motion.

The merit function is a summation of the intensities along a possible track, i.e.,

$$J_{ij}(k) = Y_{ij}(k) + \sum_{\ell=1}^{h-1} \alpha^\ell Y_{i_{k-\ell}, j_{k-\ell}}(k-\ell) \rightarrow \max,$$  \hspace{1cm} (9)  

where $(i_k, j_k), (i_{k-1}, j_{k-1}), (i_{k-2}, j_{k-2}), \cdots; (i_1, j_1)$ is a backward sequence of states starting from $(i, j) \equiv (i_k, j_k)$ and has to satisfy the restriction: $(i_{k-\ell}, j_{k-\ell}) \in D(i_{k-\ell+1}, j_{k-\ell+1})$. Let $\pi$ be a set of state sequences.
that satisfy the above restriction. The goal of dynamic programing is to find the state sequence in $\pi$ that maximizes the merit function, i.e., to find $X_{ij}(k) = \max_{x} J_{ij}(k)$. The Bellman equation for the above dynamic programing problem is none other than the RMF algorithm of (7).

3. Distribution of output image intensities of RMF

In general, the false alarm probability and the detection probability are used for performance analysis of target detection algorithms. The probability that the maximum intensity of image pixels exceeds a threshold is the false alarm probability in the absence of a target, and the probability that the intensity of the pixel in which a target is present exceeds the threshold is the detection probability. To evaluate these probabilities, it is necessary to analyze the distribution of intensities of the RMF’s output image.

Hereafter, for simplicity, we assume $\mu_{N} = 0$ and $\sigma_{N} = 1$, which does not lose the generality.

3.1 Growth of mean and variance in the absence of a target

To evaluate the false alarm probability, it is necessary to know the distribution of output image intensities from the RMF in the absence of a target. However, since this distribution is that of the maximum of correlated variables as shown in (7), it is difficult to derive it analytically. Therefore, we obtain the approximate distribution with the aid of numerical result of simulations.

We define a variable $\{Z_{ij}(k)\}$ by

$$Z_{ij}(k) = \max_{(i',j') \in D(i,j)} X_{i',j'}(k).$$

Then (7) can be written as

$$X_{i,j}(k) = Y_{i,j}(k) + \alpha Z_{i,j}(k - 1).$$

If we obtain the mean and the variance of $Z_{i,j}(k)$, we can derive the mean and the variance of $X_{i,j}(k)$ easily from (11). Therefore we determine the parameters of the mean and the variance of $Z_{i,j}(k)$ by simulation results.

In Fig. 3, we show the mean and the standard deviation obtained from a simulation when $\alpha = 1$. As is apparent from the figure, the mean and the fourth power of the standard deviation (= square of the variance) are growing linearly. These properties are not obvious, and different results concerning the growth of the variance are obtained by analytical approximation. For example, the variance grows linearly by the approximation of [1], but saturates by that of [5]. At the present time, the cause of this phenomena is not clear, however we employ the above property and approximate the mean and the standard deviation in cases of $\alpha \neq 1$, such that

$$\mu_{Z}(k) = \frac{1 - \alpha^{k}}{1 - \alpha} \mu_{0},$$

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Fig. 3 Mean and variance of the output distribution when input contains only noise.

Fig. 4 Model fitting to the output distribution when input contains only noise.

\[
\sigma_z^2(k) = \sqrt{\frac{1 - \alpha^2k}{1 - \alpha^2} - \sigma_0^2},
\]

(13)

where \(\mu_0, \sigma_0\) are unknown parameters that are determined by the fitting to simulation results. The stationary values when \(k \to \infty\) become

\[
\mu_z = \frac{\mu_0}{1 - \alpha},
\]

(14)

\[
\sigma_z^2 = \frac{\sigma_0^2}{1 - \alpha^2}.
\]

(15)

In Fig. 4 we show an example of the simulation results and fitting curves by (12) and (13). The fitting also gave good results in other examples.

The values of \(\mu_0\) and \(\sigma_0\) depend on the parameters \(\alpha, \nu\). Figure 5 shows the estimate of \(\mu_0\) and \(\sigma_0\) for different parameter values of \(\alpha, \nu\).

3.2 Mean and variance of the pixel in which a target exists

Since the intensity of the pixel in which a target exists becomes a local maximum in almost all frames, we may
approximate the mean and the variance of the pixel's intensity such that
\[
\mu_T(k) = (1 + \alpha + \cdots + \alpha^{k-1})A + \alpha^k \mu_z = 1 - \alpha^k \frac{A + \alpha^k \mu_z}{1 - \alpha},
\]
\[
\sigma_T^2(k) = 1 + \alpha^2 + \cdots + \alpha^{2(k-1)} + \alpha^{2k} \sigma_z^2 = \frac{1 - \alpha^{2k}}{1 - \alpha^2} + \alpha^{2k} \sigma_z^2,
\]
where \( k \) denotes the number of frames from the appearance of the target and we assume that the RMF attained a steady state when the target appeared.

4. Performance analysis

4.1 SNR improvement effect

We define the output SNR of the RMF as
\[
\text{SNR}_{\text{out}}(k) = \frac{\mu_T(k) - \mu_z}{\sigma_z},
\]
and substituting (16), (14) and (15) into (19) yields
\[
\text{SNR}_{\text{out}}(k) = \frac{(1 - \alpha^2)^{1/4}(1 - \alpha^k) A - \mu_0}{1 - \alpha} \sigma_0.
\]

Assuming that the mean and the variance of input noise are 0 and 1, respectively, the intensity \( A \) is equal to the input SNR. Therefore, equation (19) represents the SNR improvement effect. As \( k \to \infty \), the output SNR becomes
\[
\text{SNR}_{\text{out}}(\infty) = \frac{(1 - \alpha^2)^{1/4} A - \mu_0}{1 - \alpha} \sigma_0.
\]
Furthermore, as \( \alpha \to 1 \), it becomes
\[
\text{SNR}_{\text{out}}(\infty) \sim \frac{2^{1/4} A - \mu_0}{(1 - \alpha)^{3/4} \sigma_0},
\]
which implies that if \( A = \text{SNR}_{\text{in}} \) is greater than \( \mu_0 \), then \( \text{SNR}_{\text{out}} \) can be increased by approaching \( \alpha \) closer to 1.

Figure 6 shows the SNR improvement effect of the RMF when \( \alpha = 0.96 \).

3.2 False alarm probability

To evaluate the false alarm probability, it is necessary to know the distribution of the sample maximum of output from the RMF when the input contains only noise. Johnston[6] approximates the distribution by using the limiting distribution of sample maxima.

Suppose that \( U_i, i = 1, \ldots, n \) are independent identically distributed (IID) random variables with distribution function \( F \), and let the sample maximum be denoted by
\[
M_n = \max_{i=1,\ldots,n} U_i.
\]
Then the distribution function of \( M_n \) is equal to \( F^n(x) \). If \( c_n > 0, d_n, n = 1, 2, \ldots \) exist and as \( n \to \infty \)
\[
F^n((x - d_n)/c_n) \to G(x),
\]
then \( G(x) \) is called the extreme value distribution. The extreme value distributions are restricted to three types; Gumbel, Weibull and Frechet. When the support
of \( F(x) \) is \((\infty, \infty)\), the only extreme value distribution is Gumbel distribution [7, pp. 113-179];

\[
\Lambda(x) = \exp\{-\exp(-x)\}. \tag{24}
\]

Even if \( U_i \)'s correlate, a limit distribution exists that is the same for the case of IID variables under the kind of mixing condition with respect to the correlation [7, pp. 113-179]. In this paper, we employ the Gumbel distribution to approximate the sample maximum.

Let \( H \) be the threshold for detection, and normalize it by the mean and the standard deviation of output noise as

\[
h = \frac{H - \mu_z}{\sigma_z}. \tag{25}
\]

The false alarm probability is defined in terms of \( X_{i,j}(k) \), the output of the RMF, by

\[
P_{FA} = \mathcal{P}\left( \max_{i,j} X_{i,j} \geq H \right). \tag{26}
\]

By the definition of (10), it holds \( \max_{i,j} Z_{i,j} = \max_{i,j} X_{i,j} \). We define

\[
Z_{\text{max}} = \max_{i,j} Z_{i,j}, \tag{27}
\]

and normalize it as

\[
\tilde{Z}_{\text{max}} = \frac{Z_{\text{max}} - \mu_z}{\sigma_z}. \tag{28}
\]

Then the false alarm probability is represented as

\[
P_{FA} = \mathcal{P}(\tilde{Z}_{\text{max}} \geq h). \tag{29}
\]

We approximate the distribution of \( \tilde{Z}_{\text{max}} \) using the Gumbel distribution. The mean and the variance of the Gumbel distribution of (24) are \( \gamma = 0.577216 \) (Euler’s constant) and \( \pi^2/6 \), respectively. Therefore, if the mean \( \mu \) and the variance \( \sigma^2 \) are given, the Gumbel distribution becomes

\[
\Lambda(z; \mu, \sigma) = \exp\left\{ -\exp\left(-\frac{\pi}{\sqrt{6}\sigma}(z - \mu - \gamma) \right) \right\}. \tag{30}
\]

Using this representation, if we write the mean and the variance of \( \tilde{Z}_{\text{max}} \) as \( \mu_{\text{max}} \) and \( \sigma_{\text{max}} \), respectively, then the false alarm probability is given by

\[
P_{FA}(h) = 1 - \Lambda(h; \mu_{\text{max}}, \sigma_{\text{max}}). \tag{31}
\]

We assume that the distribution of \( \tilde{Z}_{\text{max}} \) does not depend on \( \alpha \). When \( \alpha = 0 \), the distribution of \( \tilde{Z}_{\text{max}} \) is obtained analytically. Consequently, \( \mu_{\text{max}} \) and \( \sigma_{\text{max}} \) are obtained analytically. Figure 7 shows the false alarm probability calculated by (31).

Conversely, the threshold for a given false alarm probability becomes

\[
h(P_{FA}) = \mu - \frac{\sqrt{6}\sigma}{\pi} \left\{ \gamma + \log[-\log(1 - P_{FA})] \right\}. \tag{32}
\]

### 3.3 Detection probability

If we assume that the intensity distribution of a pixel in which a target exists is normal, then the probability density function becomes

\[
p_T(x) = \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left\{ -\frac{(x - \mu_T)^2}{2\sigma_T^2} \right\}. \tag{33}
\]

Under this assumption, the detection probability for the threshold \( H \) is given by

\[
P_d = \int_H^\infty \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left\{ -\frac{(x - \mu_T)^2}{2\sigma_T^2} \right\} dx = 1 - \Phi\left( \frac{H - \mu_T}{\sigma_T} \right), \tag{34}
\]

where

\[
\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\} dx \tag{35}
\]

is the cumulative distribution function for the standard normal distribution. Substituting (25) and (19) into (34) yields

\[
P_d = 1 - \Phi\left( \frac{h - \text{SNR}_{\text{out}}(k)}{\sigma_z/\sigma_T} \right), \tag{36}
\]

which implies that when the output SNR is equal to the threshold, the detection probability becomes 50%. Figure 8 shows the detection probability calculated by (34).

### 5. Conclusion

We formulated the RMF as a Bellman equation of the dynamic programming and analyzed the performance. First, we determined the parameters of the RMF's output distribution with the aid of the numerical result of simulation. Next, we evaluated the false alarm probability of the RMF by using the extreme value theory. We then determined the threshold for a given false
alarm rate. Finally, detection probability was obtained as a function of input signal to noise ratios (SNRs). It was shown that input SNRs required for target detection can be reduced to less than half by the RMF.

References


