A Feedback-Type ANC System with Adaptive Modeling of Secondary Path

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Abstract

This paper presents some convergence results of an adaptive filter algorithm for the feedback-type active noise control (ANC) system with feedback path modeling using a dither signal and that of hearing aids. In these systems there is an adaptive filter in the feedback path so that the conventional method for analysis does not work. A frequency domain method is used for the analysis. It is based on the averaging method and the frequency domain expression of the adaptive algorithm with the causality constraint. The convergence conditions are derived for both problems and a bias expression of the weights for the latter problem is presented. The positive realness of the transfer function of the optimal predictor of the incoming signal is required for stability. Finally, the convergence condition and the bias are explicitly given for simple examples and their validities are shown by some simulations.

1 Introduction

As shown in Fig.1, the conventional feedback type ANC system uses only one microphone to provide necessary signals for adjusting the adaptive filter. In this system an adaptive filter is operating in the feedback path so that the conventional method for analysis does not work. In this paper, we analyze a modified version of the feedback type active noise control (ANC) system suggested in [2] by including adaptive modeling of the secondary path. With the frequency domain technique[2], we first derive the averaged system[5] corresponding to our feedback type ANC system and show a stationary point of the adaptive algorithm. Then we show some stability conditions. As for hearing aids,

![Figure 2: Hearing aids system.](image)

there is an acoustic feedback path from the receiver to the microphone and this causes annoying effects such as whistling and howling (Fig.2). An adaptive filter is used to model the acoustic feedback path. In [4] based on the time domain approach, an approximate expression of this bias in the weight vector of the adaptive filter has been derived by assuming that it is very small. Here, again via the frequency domain approach we derive a corresponding expression of the bias in terms of the prediction error filter of the incoming signal to the hearing-aid. The stability condition is also derived.

2 The Frequency Domain Expression

A block diagram of the feedback-type ANC system in [1] which was analyzed in [2] is shown in Fig.3 where \( d(n) \) is a stationary noise with zero mean and \( S(z) \) is the transfer function of the secondary path. The adaptive filter denoted by \( W(z) \) is used for prediction of \( d(n) \) through \( S(z) \) but the fixed estimate \( \hat{S}(z) \) of \( S(z) \) is used for updating the weights of the adaptive filter via the Filtered-X LMS algorithm. The block diagram of the ANC system treated in this paper is shown in Fig.4. The transfer function of adaptive filter is denoted as \( W(z) \). \( S(z) \) is a transfer function of secondary path, and \( \hat{S}(z) \) is its estimate. In this scheme, we use another adaptive filter with a dither signal for the estimation
of the secondary path. Let \( d(n) \) be a stationary noise with zero mean, \( v(n) \) be the dither signal, and \( e(n) \) be the error signal.

The relations of the signals in Fig. 4 are written as

\[
e_w(n) = d(n) - \sum_{i=0}^{N_w-1} \hat{s}_i n w(n-i) \tag{1}
\]

\[
e(n) = d(n) - \sum_{i=0}^{N_s-1} s_i \{ x'(n-i) + v(n-i) \}
\]

\[
x'(n) = \sum_{i=0}^{N_w-1} w_i(n)x(n-i)
\]

\[
x(n) = e(n) + \sum_{i=0}^{N_s-1} \hat{s}_i n x'(n-i) + v(n-i)
\]

\[
u(n) = \sum_{i=0}^{N_s-1} \hat{s}_i n x(n-i)
\]

where \( N_w \) and \( N_s \) are the numbers of the tap coefficients \( \{w_i(n)\}, \{\hat{s}_i(n)\} \) of the adaptive filters, \( N_s \) is the order of the transfer function of \( S(z) \) whose impulse response is \( \{\hat{s}_i\} \). We assume that \( N_s \ll N_w \). Then the error signal \( e(n) \) is given by

\[
e_w(n) = d(n) - \sum_{i=0}^{N_s-1} \hat{s}_i n x(n-i) + \sum_{i=0}^{N_s-1} (\hat{s}_i n - s_i)(v(n-i)). \tag{2}
\]

By the filtered-X LMS algorithm, the tap weights of each adaptive filters are updated as

\[
w_i(n+1) = w_i(n) + \mu_w u(n-i) e_w(n)
\]

\[
\hat{s}_i(n+1) = \hat{s}_i(n) + \mu_s v(n-i) e_s(n). \tag{3}
\]

Since step sizes \( \mu_w \) and \( \mu_s \) are small positive numbers and the differences between \( w_i(n), \hat{s}_i(n) \) and \( w_i(n-i), \hat{s}_i(n-i) \) are of \( O(\mu_w^2), O(\mu_s^2) \), its effect through \( e(n) \) in (3) is of \( O(\mu_w^2), O(\mu_s^2) \) and can be discarded. Thus the right hand of (2) is approximately expressed as

\[
e_w(n) \simeq d(n) - \sum_{i=0}^{N_s-1} \left( \sum_{i=0}^{N_s-1} s_i w_{n-i}(n) \right) x(n-i) + \sum_{i=0}^{N_s-1} \Delta s_i v(n-i) \tag{4}
\]

with \( \Delta s_i = \hat{s}_i - s_i \). Next, we define the \( N \)-dimensional weight and signal vectors as

\[
w(n) = [w_0(n), \cdots, w_{N_w-1}(n), 0, \cdots, 0]^T
\]

\[
d(n) = [d(n), \cdots, d(n-N_w+1), 0, \cdots, 0]^T \tag{5}
\]

where \( N \geq 2 \max(N_w, N_s) \) respectively. Padding some zeros is required to retrieve the causality of \( \{w_i(n)\} \) through the inverse discrete Fourier transform (DFT)[2]. The other weight and signal vectors are defined in the same way. Then (4) can be written as

\[
e_w(n) = -e_s(n)
\]

\[
\simeq d(n) - (s \otimes \mathbf{w}(n))^\dagger x(n) + \Delta s^\dagger(n)v(n) \tag{6}
\]

where "\( \otimes \)" and "\( ^\dagger \)" denote the convolution and the complex conjugate transpose, respectively. The \( N \)-point DFT vector obtained by applying the DFT matrix

\[
F = \left[\exp\left(-i \frac{2\pi kl}{N}\right)\right] \quad k, l = 0, 1, \cdots, N-1
\]

to each vector is denoted by the corresponding capital bold letter as \( X(n) = Fx(n) \). From the property \( FF^\dagger = NI \), applying \( F \) to (6) we get

\[
e_w(n) = \left\langle d(n) - \frac{1}{N} X^\dagger(n) \Delta_s W(n) + \frac{1}{N} V^\dagger(n) \Delta S(n) \right\rangle \tag{7}
\]

where \( \Delta_s \) is a diagonal matrix given by

\[
\Delta_s = \text{diag}[S_0, S_1, \cdots, S_{N-1}].
\]
With vectors and matrices, the update algorithm of the adaptive filters by the Filtered-X LMS algorithm is expressed as

\[ w(n+1) = w(n) + \mu w u(n) e_w(n) \]  
\[ \hat{s}(n+1) = \hat{s}(n) + \mu s v(n) e_s(n) \]  

(8)

Applying \( F \) to (8), we get

\[ W(n+1) = W(n) + \mu w U(n) e_w(n) \]  
\[ \hat{S}(n+1) = \hat{S}(n) + \mu s V(n) e_s(n). \]  

(9)

From (1), we get the approximate relation

\[ U(n) \simeq A^*(n) X(n) \]  

(10)

where "*" denotes the complex conjugate. Similarly the corresponding approximate expressions for (1) are

\[ E(n) \simeq D(n) - A^*_S(X'(n) + V(n)) \]  
\[ X'(n) \simeq A^*_W(n) X(n) \]  
\[ X(n) \simeq E + A^*_S(n) (X'(n) + V(n)). \]  

(11)

Hence by eliminating \( E(n) \) and \( X'(n) \) in (11), \( X(n) \) is written as

\[ X(n) \simeq Q(n) D(n) + Q(n) A^*_S(n) V(n) \]  

(12)

with

\[ Q(n) = \left[ I - A^*_W(n) A^*_S(n) \right]^{-1} \]  
\[ A^*_S(n) = A^*_S(n) - A_S. \]  

(13)

Substituting (10) and (12) into (9), we get the discrete frequency domain expression of (8) as

\[ W(n+1) = W(n) + \mu w \left[ A^*_S(n) Q(n) P \right] \]  
\[ \hat{S}(n+1) = \hat{S}(n) + \mu s \left[ A^*_S(n) Q(n) P \right] \]  

(18)

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\[ \hat{S}(n+1) = \hat{S}(n) + \mu s \left[ A^*_S(n) Q(n) P \right] \]  

(18)

3 Stationary Point of the Algorithm and its Stability Condition

Here we derive a stationary point of the algorithm by the averaging method and consider its stability.

Since \( N \) is large and the primary noise \( d(n) \) is a zero-mean stationary process, the element of \( D(n) \) is uncorrelated with each other. Hence,

\[ E \left[ D(n) D^T(n) \right] \simeq N \text{diag} [P_0, \cdots, P_{N-1}] \equiv \Lambda_P \]  

where \( P(e^{i\omega}) \) is the spectral density of \( d(n) \) and \( P_l = P(e^{i\omega_l}) \). Let \( V(e^{i\omega}) \) be the spectral density of \( v(n) \) and we have

\[ E[V(n) V^T(n)] \equiv N \sigma_v^2 I \]  

(16)

in the same way. Also, \( d(n) \) is expressed as

\[ d(n) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} d(n-k) e^{i2\pi j k} = \frac{1}{N} D^T(n) \pi \]  

(17)

By taking the average with respect to \( D(n) \) and \( V(n) \) in the right hand side of (14) and replacing \( W(n) \) and \( \Delta S(n) \) with the corresponding deterministic quantities \( \bar{W}(n) \) and \( \bar{S}(n) \), the averaged system is given by

\[ \bar{W}(n+1) = \bar{W}(n) + \mu w \left[ A^*_S(n) \bar{Q}(n) P \right] \]  
\[ -A^*_S(n) \bar{Q}(n) P \bar{Q}^T(n) A_S \bar{W}(n) \]  
\[ + A^*_S(n) \bar{Q}(n) A^*_S(n) \sigma_v^2 \]  
\[ \times \left[ -A_S \bar{Q}^T(n) A_S \bar{W}(n) + \bar{S}(n) \right] \]  
\[ \Delta \bar{S}(n+1) = \Delta \bar{S}(n) \]  
\[ + \mu_s \sigma_v^2 \left[ A_S \bar{Q}^T(n) A_S \bar{W}(n) - \Delta \bar{S}(n) \right]. \]  

(18)

where \( \bar{Q} \) is given by (13) with \( W(n) \) and \( \Delta S(n) \) replaced by \( \bar{W}(n) \) and \( \Delta \bar{S}(n) \). \([ \cdot ]_+ \) operation means that the causal part is taken from the inverse transform of the function in \([ \cdot ] \). This operation is necessary to keep \( \bar{W}(n) \) to be causal. Since all the matrices in (18) are diagonal, the \( l \)-th elements of (18) are written as the following scalar nonlinear difference equation

\[ \bar{W}_i(n+1) = \bar{W}_i(n) + \mu w \left[ \bar{S}_i^T(n) P_i \right] \]  
\[ - \left( 1 - \bar{W}_i^T(n) \Delta \bar{S}_i(n) \right) \left( 1 - \bar{W}_i(n) \Delta \bar{S}_i(n) \right) \]  
\[ + \mu_s \sigma_v^2 \left[ A_S \bar{Q}_i^T(n) A_S \bar{W}_i(n) - \Delta \bar{S}_i(n) \right] \]  
\[ \Delta \bar{S}_i(n+1) = \Delta \bar{S}_i(n) \]  
\[ + \mu_s \sigma_v^2 \left[ A_S \bar{Q}_i^T(n) A_S \bar{W}_i(n) - \Delta \bar{S}_i(n) \right]. \]  

(19)
Thus the stability point of the original filtered-X LMS algorithm in (8) is obtained by solving (19) with $\hat{W}_i(n+1) = \hat{W}_i(n) = W_i$. When $N \to \infty$, we can replaced the discrete frequencies with the continuous one so that instead of $W_i$ we use $W(z)$ where $z = e^{j\omega}$. Then to get the stability point of $W(z)$ we should solve the equation below,

$$
\left[ \hat{S}(z^{-1}) P(z) - \frac{\hat{S}(z^{-1}) P(z) S(z) W(z)}{1 - W(z) \Delta S(z)} \right. \\
+ \hat{S}(z^{-1}) \Delta S(z^{-1}) \sigma^2 \\
\times \left( \frac{\Delta S(z) S(z) W(z)}{1 - W(z) \Delta S(z)} + \Delta S(z) \right) \right]_+ = 0
$$

where $1 - W(z) \Delta S(z^{-1})$ is purely non-causal except the constant term so that from the denominator we can get rid of this.

In order to obtain the optimal $W(z)$, we give two assumptions. First, the spectral factorization of $P(z)$ is

$$
P(z) = R(z) R(z^{-1})
$$

where $R(z)$ is of minimum phase. Second, $S(z)$ and $\hat{S}(z)$ are expressed as

$$
S(z) = z^{-q} C(z) \\
\hat{S}(z) = z^{-q} \hat{C}(z)
$$

where $C(z)$ and $\hat{C}(z)$ are stable polynomials and a positive integer $q$ is the delay. Obviously, the stationary point of $\Delta S(z)$ is zero and the stationary point of $W(z)$ is given by

$$
W_{\text{opt}}(z) = \frac{A(z)}{C(z)}
$$

where

$$
A(z) = \frac{[z^q R(z)]}{R(z)}.
$$

We note that $A(z)$ is the transfer function of the optimal $q$-step ahead linear predictor of $d(n)$. To examine the stability around the stationary point, we calculate the following derivative matrix

$$
J = \\
\begin{pmatrix}
\frac{\partial \hat{W}_i(n+1)}{\partial \hat{W}_i(n)} & \frac{\partial \hat{W}_i(n+1)}{\partial \Delta S_i(n)} \\
\frac{\partial \Delta S_i(n+1)}{\partial \hat{W}_i(n)} & \frac{\partial \Delta S_i(n+1)}{\partial \Delta S_i(n)}
\end{pmatrix} = \\
\begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix}
$$

The complex differentiation suggested in [3] is used to calculate the derivative of $\hat{W}_i(n)$ and $\Delta S_i(n)$. Since noncausal part in (19) is a function of $\hat{W}_i(n)$ and $\partial z^*/\partial z = 0$ from the definition of complex differentiation in [3], we can discard the operation [ ] of (19) in calculating the derivative. Thus each element of $J$ is as follows,

$$
J_{11} = 1 + \mu_w \frac{1}{|1 - W_i(n) \Delta S_i(n)|^2} \\
\left[ \frac{\hat{S}_i^*(n) P_i \hat{S}_i + \hat{S}_i^*(n) \Delta S_i^*(n) \Delta S_i(n) S_i \sigma^2}{1 - W_i^*(n) \Delta S_i^*(n)} \right]
$$

$$
J_{21} = \\
\mu_w \sigma^2 \left[ \frac{S_i W_i^*(n) \Delta S_i^2(n)}{|1 - W_i(n) \Delta S_i(n)|^2} + \frac{\Delta S_i(n) S_i}{(1 - W_i(n) \Delta S_i(n))} \right]
$$

$$
J_{22} = 1 + \mu_w \sigma^2 \left[ \frac{S_i W_i^2(n) \Delta S_i(n)}{|1 - W_i(n) \Delta S_i(n)|^2} + \frac{S_i W_i(n)}{(1 - W_i(n) \Delta S_i(n))} \right] \right] - 1
$$

Substituting the stationary point, we get

$$
J_{11,\text{opt}} = 1 - \mu_w S_i^2 P_i S_i \\
J_{22,\text{opt}} = 1 + \mu_w \sigma^2 \left[ S_i W_{\text{opt}} - 1 \right] \right] \\
J_{12,\text{opt}} = 0.
$$

For the stability around the stationary point, it is required that the absolute values of all the eigenvalues of matrix $J_{\text{opt}}$ are less than 1. Since $0 < \mu_w, \mu_s < 1$ and $|S_i|^2 P > 0$, the stability condition is given by

$$
\text{Re} \left[ 1 - S(z) W_{\text{opt}}(z) \right] > 0.
$$

From (23)(22) this condition is

$$
\text{Re} \left[ 1 - z^{-q} A(z) \right] > 0.
$$

We note that $1 - z^{-q} A(z)$ is just the transfer function of the optimal $q$-step ahead linear prediction error filter of $d(n)$.

If $S(z)$ is a general non-minimum phase transfer function, it is difficult to obtain a simple expression for $W_{\text{opt}}$ like (23) and the corresponding stability condition like (29).

### 4 Analysis of the Adaptive Algorithm in Hearing Aids

Fig. 5 shows the block diagram of a hearing aid plant where $d(n)$ is a stationary incoming signal with zero mean and the spectral density $P(z)$ and the forward path and the feedback path transfer functions are $G(z)$, $H(z)$, respectively. A dither signal $v(n)$ is also added for generality but an explicit stability condition is derived for the case of no dither signal. The weights of the adaptive filter denoted by $W(z)$ is updated by the standard LMS algorithm to cancel the effect of $H(z)$. 

---
The analysis of the system in Fig.5 can be done in the same way as in Section 2.3. The error signal $e(n)$ is given by

$$e(n) = d(n) + h^T x(n) - w^T x(n)$$  \hspace{1cm} (30)

where $h$ is the weight vector corresponding to $H(z)$. The weight vector $\hat{w}(n)$ of the adaptive filter is updated by

$$w(n + 1) = \hat{w}(n) + \mu x(n) e(n)$$  \hspace{1cm} (31)

where $w, x, e$ are defined like (5). The frequency domain expression of $e(n)$ using the DFT matrix is given by

$$E(n) = D(n) + \Lambda_H^* X(n) - \Lambda_{W(n)}^* X(n)$$  \hspace{1cm} (32)

where $D(n), E(n), X(n), \Lambda_H, \Lambda_{W(n)}$ are defined as in Section 2. Also,

$$X(n) \simeq \Lambda_G^* E(n) + V(n).$$  \hspace{1cm} (33)

Hence we have

$$X(n) \simeq Q(n) (\Lambda_{G}^* D(n) + V(n))$$  \hspace{1cm} (34)

with

$$Q(n) = \left[ I + \Lambda_G^* \left( \Lambda_{W(n)}^* - \Lambda_H^* \right) \right]^{-1}.$$  \hspace{1cm} (35)

Applying $F$ to (31) and using (34) we have

$$W(n + 1) = W(n) + \mu Q(n) (\Lambda_G^* D(n) + V(n)) \cdot \left[ d(n) - \frac{1}{N} \left( D^T(n) \Lambda_G + V^T(n) \right) Q^T(n) (W(n) - H) \right]$$

Using (15)-(17) the corresponding averaged system is given by

$$\bar{W}(n + 1) = \bar{W}(n) + \mu \left[ \bar{Q}(n) \Lambda_{G} P \right. \left. - \bar{Q}(n) (\Lambda_G^* \Lambda_P \Lambda_G + \sigma_e^2 I) \bar{Q}^T(n) (\bar{W}(n) - H) \right] +$$

The $l$-th component is written as

$$\bar{W}_l(n + 1) = \bar{W}_l(n) + \mu \left[ G^*_l P_l - \sigma_e^2 (\bar{W}_l(n) - H_l) \right]
\left[ 1 + G_l(\bar{W}_l^T(n) - H_l) \right] +$$

At the stationary point $W(z) = W_{opt}(z)$ we assume that $1 + G(z)(W(z) - H(z))$ is stable. Then from (37) we have

$$\left[ \frac{G(z^{-1}) P(z) - \sigma_e^2 (W(z) - H(z))}{1 + G(z)(W(z) - H(z))} \right] = 0.$$  \hspace{1cm} (38)

In general it is difficult to solve this "generalized" Wiener-Hopf equation. We consider the special case of no dither signal, that is, $\sigma_e^2 = 0$ and we assume that $G(z) = z^{-q} G_c(z)$ where $G_c(z)$ is a stable polynomial. Then, (38) becomes

$$\left[ z^q R(z) \right]_+ = \frac{G_c(z) R(z) B(z)}{1 + G_c(z) z^{-q} B(z)}$$  \hspace{1cm} (39)

where we set the bias $B(z)$ as $B(z) = W_{opt}(z) - H(z)$ and assume that $1 + z^{-q} G_c(z) B(z)$ is stable. Hence, we have

$$B(z) = \frac{A(z)}{G_c(z)(1 - z^{-q} A(z))}$$  \hspace{1cm} (40)

where we assume that the prediction error filter $1 - z^{-q} A(z)$ is stable. Taking the derivative of (37) and substituting this stationary point $W_{opt,l} = H_l + A_l/(G_{c,l} - G_l A_l)$ at discrete frequencies with $\sigma_e^2 = 0$, we have

$$\frac{\partial \bar{W}_l}{\partial \bar{W}_l} = \left[ \frac{\partial \bar{W}_l}{\partial \bar{W}_l} \right] = 1 - z^{-q} A(z)$$  \hspace{1cm} (41)

But from (40)

$$\frac{1}{1 + G(z)(W_{opt}(z) - H(z))} = 1 - z^{-q} A(z)$$  \hspace{1cm} (42)

so that the stability condition is again given by (29).

5 Examples and Simulation Results

Here we assume that $d(n)$ is an $m$-th order AR (autoregressive) process with the innovation variance 1. That is, $R(z)$ is given by

$$R(z) = \frac{1}{1 - a_1 z^{-1} - \cdots - a_m z^{-m}}$$  \hspace{1cm} (43)
For \( m = 1 \), \( A(z) = a_1^2 \) where \( |a_1| < 1 \). Hence (29) is satisfied. For \( m = 2 \) and \( q = 1 \), \( A(z) = a_1 + a_2 z^{-1} \) and (29) becomes \( \text{Re}[1 - a_1 z^{-1} - a_2 z^{-2}] > 0 \). For \( m = 2 \) and \( q = 2 \), \( A(z) = a_1^2 + a_2 + a_1 a_2 z^{-1} \) and (29) becomes \( \text{Re}[(1 + a_1 z^{-1})(1 - a_1 z^{-1} - a_2 z^{-2})] > 0 \). For the latter problem \( (1 + a_1 z^{-1})(1 - a_1 z^{-1} - a_2 z^{-2}) \) must be a stable polynomial. Numerical investigations show that both regions coincide. In Fig. 6 the learning curve shows the empirical variance of the squared error \( e^2(n) \) of the former problem for the case of \( m = 1 \), \( a_1 = 0.9 \), \( S(z) = z^{-1} \) and initial \( \hat{S}(z) = 2.5 z^{-1} \) with \( N_w = 8, N_s = 2, \sigma_v^2 = 0.01, \mu_w = 0.0001, \mu_s = 0.01 \). The variance is obtained by averaging over 50 data sets. In the early stage some instability is seen because initial \( \hat{S}(z) \) is at the instability domain, but it transfers to stability domain by adaptive modeling of the secondary path. Then the steady state variance is 1.0296 which is close to the lower bound 1.01. For the case of \( m = 2, q = 2 \), we see divergence if we set the parameters outside the above stability region. In Table 1, the steady state first 4 weights of the bias for the latter problem are presented together with the theoretical ones in (40) for the case of \( m = 1, q = 1, a_1 = 0.9, H(z) = z^{-1}/10, G(z) = 4z^{-1}, N_w = 16, \mu = 0.0001 \). The agreements are good.

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Table 1: Theoretical and estimated bias in the hearing aid.

6 Conclusion

We have presented the analysis of the adaptive filter algorithms for the feedback-type ANC with on-line modeling of the feedback path and hearing aids with respect to the stationary point of the adaptive algorithm and the stability of the point using the frequency domain expression and the averaging method. The validity of

the theoretical results we have found is shown by the simulation results. A further study is needed to find schemes that do not require the condition (29).

A The Averaging Method

We give a brief explanation of the averaging method[5]. Let us consider a general adaptive algorithm below

\[
\theta(k+1) = \theta(k) + \mu \tilde{h}(\theta(k), x(k))
\]

where \( \theta, x \) and \( \mu \) denote a parameter vector, a stationary input signal and small positive step gain, respectively. The averaged system of (44) is described as

\[
\tilde{\theta}(k+1) = \tilde{\theta}(k) + \mu \tilde{h}(\theta(k))
\]

with

\[
\tilde{h}(\theta) = \text{E}[h(\theta, x(k))]
\]

To find the convergence property of (44), we can examine that of the averaged system (45) under some regularity conditions. That is, we can alternate the input signal with its expectation in the algorithm without changing the convergence property.

References