Pricing of an Exotic Forward Contract

Jirô Akahori, Yuji Hishida and Maho Nishida
Dept. of Mathematical Sciences, Ritsumeikan University
1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577, Japan

E-mail: {akahori, rp003994, rs028990}@se.ritsumei.ac.jp

Abstract

In this paper we study a pricing problem of an *exotic* Forward contract. Unlike the standard Forward, the contract is not fair, and like an option, it is compensated by the premium. Using standard arguments in the Black-Scholes economy, an explicit formula for hedging as well as pricing is obtained. This is possible because of an exotic way of settlement, which is another focus of this paper. Contrary to our intuition, simpler ways of settlement do not necessarily imply a simpler formula.

1 Introduction

A Forward contract is an agreement between two parties to buy or sell an asset (which can be of any kind) on a pre-agreed future day at a pre-agreed price which is called *Forward price*. Usually the Forward price is set to make the contract fair, and so no payment takes place on the agreement day. For details of standard Forward contracts, see e.g. [2].

If an extra agreement that benefits one party only is added to the contract, as is the situation that we will study, then an adjustment is needed to keep the contract fair. There are two possibilities: one is by a change of the Forward price, and the other is by *payment* at the agreement day.

In this paper we will study such a contract which we call *exotic* Forward. We adopt the latter adjustment and study the premium — how much should be paid for the contract.

The Exotic Forward we will study is defined by the following

Extra Agreement 1. A buyer has a right to choose the price-date at the delivery day between N_1 days before the delivery day and N_2 days after,

with the

Extra Agreement 2 (Settlement of Difference). A settlement day is the delivery day in both cases. If a buyer chooses the N_2 days after, the payment is taken by the way of **settlement of difference**; she/he is paid a pre-agreed price at the delivery day T . and then is paid the difference at $T + N_2$.

We will give an explicit formula for the premium and the corresponding hedging strategy under a Black-Scholes type model. What interests us most is that the formula is very much alike the Black-Scholes formula for the plain European option. The reduction is caused by the exotic way of settlement (Extra Agreement 2).

2 The Setting and Notations

We denote the delivery day by T , $T_1 := T - N_1$, and $T_2 = T + N_2$. We assume the Black-Scholes Economy; i.e., a spot price $S = \{S_t\}_{0 \leq t \leq T_2}$ and risk-free bond $B = \{B_t\}_{0 \leq t \leq T_2}$ satisfy the following stochastic differential equations:

$$\begin{aligned} dS_t &= \sigma S_t dW_t + \mu S_t dt \\ dB_t &= r B_t dt \end{aligned} \quad (1)$$

where $\sigma > 0, r \geq 0, \mu \in \mathbb{R}$, and W is a standard 1-dimensional Brownian motion. For simplicity, we assume that $B_0 = 1$.

Under these settings, the Exotic Forward contract in question is restated as follows.

1. The agreement day is time 0, when the Forward price K is quoted in the market.
2. The buyer pays the premium π on the agreement day.
3. On the delivery day T , the buyer chooses the price day; T_1 or T_2 .
4. If she/he chooses T_1 , she/he is paid $S_{T_1} - K$.
5. If she/he chooses T_2 , she/he is paid Y_T for the moment.
6. And then on T_2 , she/he is paid the difference $S_{T_2} - K - Y_T$.

Though studying general Y_T is possible, we concentrate on the case $Y_T = S_T - K$ in this paper.

Before stating the results, we introduce some notations. Let

$$d_{\pm}(t, x) = \frac{(r \pm \frac{1}{2}\sigma^2)(T - t) + \log(2 - e^{-r(T_2 - T)})x/S_{T_1}}{\sigma\sqrt{T - t}} \quad (2)$$

for $t \in [T_1, T)$ and $x > 0$. We write simply $d_{\pm}$ for $d_{\pm}(T_1, S_{T_1})$. That is,

$$d_{\pm} = \frac{(r \pm \frac{1}{2}\sigma^2)(T - T_1) + \log(2 - e^{-r(T_2-T)})}{\sigma\sqrt{T - T_1}}. \quad (3)$$

Let also

$$A := \{S_{T_1} \leq (2 - e^{-r(T_2-T)})S_T\}. \quad (4)$$

Finally, we use the following conventions:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (5)$$

and

$$\Phi(x) = \int_{-\infty}^x \phi(y) dy, \quad (6)$$

the density and the distribution function of the standard Gaussian random variable.

3 The Results

The first result which corresponds to the *pricing* is the following.

Theorem 1. *Under the hypothesis of No-Arbitrage, the fair premium is given by*

$$\pi = S_0 e^{-r(T-T_1)} \Phi(-d_-) + S_0 (2 - e^{-r(T_2-T)}) \Phi(d_+) - e^{-rT} K. \quad (7)$$

From the above result, we notice that the premium of this contract is represented by a mono-integral, even though this contract has several exotic aspects. We stress that when $Y_T = 0$, i.e. no settlement of difference, the premium is no more mono-integral; a little far away from Black-Scholes formula.

Further, we can also obtain the explicit hedging strategy for the seller as follows.

Theorem 2. *The contract is perfectly hedged by holding*

$$\eta_t = \begin{cases} e^{-r(T-T_1)} \Phi(-d_-) + (2 - e^{-r(T_2-T)}) \Phi(d_+) =: \eta_0, & (0 \leq t < T_1) \\ \eta_0 + \left\{ (2 - e^{-r(T_2-T)}) \phi(d_+(t, S_t)) - \frac{B_t S_{T_1}}{B_T S_t} \phi(-d_-(t, S_t)) \right\} / \sigma\sqrt{T-t}, & (T_1 \leq t < T) \\ 1_A & (T \leq t < T_2) \end{cases} \quad (8)$$

amount of the risky asset at time $t \in [0, T_2)$, and

$$\nu_t = \begin{cases} -K/B_T =: \nu_0, & (0 \leq t < T_1) \\ \nu_0 + (2 - e^{-r(T_2-T)}) \frac{S_t \phi(d_+(t, S_t))}{B_t^{-1} \sigma\sqrt{T-t}} + S_{T_1} B_T^{-1} \left\{ \Phi(-d_-(t, S_t)) - \frac{\phi(-d_-(t, S_t))}{\sigma\sqrt{T-t}} \right\}, & (T_1 \leq t < T) \\ \nu_0 + 1_A S_T (B_T^{-1} - B_{T_2}^{-1}) + 1_{\Omega \setminus A} S_{T_1} B_T^{-1} & (T \leq t < T_2) \end{cases} \quad (9)$$

amount of the bond.

Remark. The hedging strategy is statistical unless $t \in [T_1, T)$, as is the case with standard Forward contracts; known as *cost of carry* (see e.g. [2]). The exotic aspect of our contract forces the seller to hedge dynamically, though it is needed only for $t \in [T_1, T)$.

4 No-Arbitrage Hypothesis

Firstly, we review the consequences of the No-Arbitrage hypothesis on the Black-Scholes economy (1).

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which W is defined, and $\{\mathcal{F}_t\}$ be the natural filtration of W . Put $\theta := (\mu - r)/\sigma$. The P - $\{\mathcal{F}\}$ -martingale $Z_t := \exp(\theta W_t - \frac{1}{2}\theta^2 t)$ defines an equivalent probability measure $\hat{P}$ by

$$\frac{d\hat{P}}{dP} \Big|_{\mathcal{F}_t} = Z_t, \quad t \in [0, T_2]. \quad (10)$$

The probability measure $\hat{P}$ is usually called *equivalent martingale measure*, EMM for short, and plays a central role in the pricing of options and other derivatives.

The Black-Scholes economy (1) under the EMM is

$$\begin{aligned} dS_t &= \sigma S_t d\hat{W}_t + r S_t dt \\ dB_t &= r B_t dt, \end{aligned} \quad (11)$$

or equivalently

$$\begin{aligned} S_t &= S_0 \exp\{\sigma \hat{W}_t - \frac{\sigma^2 t}{2}\}, \\ B_t &= B_0 \exp\{rt\}. \end{aligned} \quad (12)$$

where $\hat{W}_t := W_t - \theta t$ is a standard Brownian motion on $(\{\mathcal{F}_t\}_{0 \leq t \leq T_2}, \hat{P})$, as the Maruyama-Girsanov theorem says.

The fair value at time t of a derivative whose pay-off at time T is H , which is the unique one that excludes arbitrage opportunities, is given by

$$C_t = e^{-r(T-t)} \mathbf{E}^{\hat{P}}[H | \mathcal{F}_t], \quad (0 \leq t \leq T) \quad (13)$$

the discounted conditional expectation of the pay-off random variable with respect to the EMM.

The hedging strategy for the derivative is obtained via the stochastic differential expression of C ;

$$dC_t = \eta_t dS_t + \nu_t dB_t \quad (14)$$

which means holding η amounts of the asset and ν amounts of the non-risky asset perfectly hedges the derivative.

For details of the arguments in this section, see e.g. [1] or [2] and the references therein.

5 Pricing/A Proof of Theorem 1

In this section, we will give the value of the Exotic Forward following the framework of the previous section.

To apply (13), we unify the cash flows as an $\mathcal{F}_{T_2}$ measurable random variable. Let us define an $\mathcal{F}_T$ measurable set

$$\begin{aligned} A(Y_T) &:= \\ \{S_{T_1} - K \leq Y_T + e^{-r(T_2-T)} \mathbf{E}^{\hat{P}}[S_{T_2} - K - Y_T | \mathcal{F}_T]\} \\ &= \{S_{T_1} - K \leq (1 - e^{-r(T_2-T)})Y_T + S_T - e^{-r(T_2-T)}K\} \\ &= \{S_{T_1} - S_T \leq (1 - e^{-r(T_2-T)})(Y_T + K)\}. \end{aligned} \quad (15)$$

On $A(Y_T)$, the buyer is sure to choose T_2 as the price-date, while on $\Omega \setminus A(Y_T)$ she/he will not fail to choose T_1 .

The cash flow on $\Omega \setminus A(Y_T)$ is $S_{T_1} - K$ at T , and will be

$$e^{r(T_2-T)}(S_{T_1} - K) \quad (16)$$

at T_2 , if all is invested to the bond. While on $A(Y_T)$, applying the same argument as above, the cash flow at T_2 is

$$\begin{aligned} &e^{r(T_2-T)}Y_T + S_{T_2} - K - Y_T \\ &= (e^{r(T_2-T)} - 1)Y_T + S_{T_2} - K. \end{aligned} \quad (17)$$

Hence, the pay-off at T_2 of the exotic forward can be seen as

$$\begin{aligned} H(Y_T) &= 1_{\Omega \setminus A(Y_T)} e^{r(T_2-T)}(S_{T_1} - K) \\ &\quad + 1_{A(Y_T)} \{(e^{r(T_2-T)} - 1)Y_T + S_{T_2} - K\}. \end{aligned} \quad (18)$$

Therefore, applying (13), we have

$$\begin{aligned} C_t(Y_T) &= e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[H(Y_T) | \mathcal{F}_t] \quad (0 \leq t \leq T_2) \\ &= e^{-r(T_2-t)} e^{r(T_2-T)} \mathbf{E}^{\hat{P}}[1_{\Omega \setminus A(Y_T)}(S_{T_1} - K) | \mathcal{F}_t] \\ &\quad + e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[1_{A(Y_T)} \{(e^{r(T_2-T)} - 1)Y_T\} | \mathcal{F}_t] \\ &\quad + e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[1_{A(Y_T)}(S_{T_2} - K) | \mathcal{F}_t], \end{aligned} \quad (19)$$

and if $t \leq T_1$,

$$\begin{aligned} &= e^{-r(T-t)} \mathbf{E}^{\hat{P}}[1_{\Omega \setminus A(Y_T)}(S_{T_1} - K) | \mathcal{F}_t] \\ &\quad + e^{-r(T-t)} \mathbf{E}^{\hat{P}}[1_{A(Y_T)} \{(1 - e^{-r(T_2-T)})Y_T\} | \mathcal{F}_t] \\ &\quad + \mathbf{E}^{\hat{P}}[1_{A(Y_T)}(e^{rT} \mathbf{E}^{\hat{P}}[e^{-rT_2} S_{T_2} | \mathcal{F}_T] \\ &\quad \quad - e^{-r(T_2-t)} K) | \mathcal{F}_t] \\ &= e^{-r(T-t)} (\mathbf{E}^{\hat{P}}[1_{\Omega \setminus A(Y_T)}(S_{T_1} - K) | \mathcal{F}_t] \\ &\quad + \mathbf{E}^{\hat{P}}[1_{A(Y_T)} \{(1 - e^{-r(T_2-T)})Y_T \\ &\quad \quad + S_T - e^{-r(T_2-T)}K\} | \mathcal{F}_t]) \\ &= e^{-r(T-t)} \mathbf{E}^{\hat{P}}[\max(S_{T_1} - K, \\ &\quad (1 - e^{-r(T_2-T)})Y_T + S_T - e^{-r(T_2-T)}K) | \mathcal{F}_t]. \end{aligned} \quad (20)$$

In particular, we see that

$$\begin{aligned} C_t(S_T - K) &= e^{-r(T-t)} \mathbf{E}^{\hat{P}}[\max(S_{T_1}, (2 - e^{-r(T_2-T)})S_T) | \mathcal{F}_t] \\ &\quad - e^{-r(T-t)}K, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \pi &= C_0(S_T - K) \\ &= e^{-rT} \mathbf{E}^{\hat{P}}[\max(S_{T_1}, (2 - e^{-r(T_2-T)})S_T)] - e^{-rT}K. \end{aligned} \quad (22)$$

We will calculate the expectation in (22), to complete the proof of Theorem 1.

The most remarkable property is that $A(S_T - K)$ is independent of $\mathcal{F}_t$ for any $t \leq T_1$, since

$$\begin{aligned} &A(S_T - K) \\ &= \{S_{T_1} - K \leq (2 - e^{-r(T_2-T)})S_T - K\} \\ &= \{S_{T_1} \leq (2 - e^{-r(T_2-T)})S_T\} = A \\ &= \{\log S_0 + \sigma \hat{W}_{T_1} + (r - \frac{1}{2}\sigma^2)T_1 \leq \\ &\quad \log S_0 + \sigma \hat{W}_T + (r - \frac{1}{2}\sigma^2)T + \log(2 - e^{-r(T_2-T)})\} \\ &= \{\hat{W}_T - \hat{W}_{T_1} \geq \\ &\quad - \frac{(r - \frac{1}{2}\sigma^2)(T - T_1) + \log(2 - e^{-r(T_2-T)})}{\sigma}\} \\ &= \{\hat{W}_T - \hat{W}_{T_1} \geq -d_- \sqrt{T - T_1}\}. \end{aligned} \quad (23)$$

Therefore,

$$\begin{aligned} \pi &= e^{-rT} \mathbf{E}^{\hat{P}}[S_{T_1}; \Omega \setminus A] \\ &\quad + (2 - e^{-r(T_2-T)})e^{-rT} \mathbf{E}^{\hat{P}}[S_T; A] - Ke^{-rT} \\ &= e^{-rT} \mathbf{E}^{\hat{P}}[S_{T_1}] \{\hat{P}(\Omega \setminus A) \\ &\quad + (2 - e^{-r(T_2-T)})\mathbf{E}^{\hat{P}}[S_T/S_{T_1}; A]\} \\ &= e^{-r(T-T_1)} S_0 \{\Phi(-d_-) \\ &\quad + (2 - e^{-r(T_2-T)})\mathbf{E}^{\hat{P}}[S_T/S_{T_1}; A]\}. \end{aligned} \quad (24)$$

Noting that

$$S_T/S_{T_1} = \exp\{\sigma(\hat{W}_T - \hat{W}_{T_1}) - \frac{\sigma^2}{2}(T - T_1)\}, \quad (25)$$

we have

$$\begin{aligned} &\mathbf{E}^{\hat{P}}[S_T/S_{T_1}; A] \\ &= e^{(r - \frac{\sigma^2}{2})(T - T_1)} \int_{y \geq -d} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{y\sigma\sqrt{T-T_1}} dy \\ &= e^{r(T-T_1)} \int_{y \geq -d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \sigma\sqrt{T-T_1})^2} dy \\ &= e^{r(T-T_1)} \Phi(d_+). \end{aligned} \quad (26)$$

By (24) and (26), we obtain (7). $\square$

Remark. For the cases of $Y_T = 0$, the set $A(0)$ defined in (15) is no longer independent of $\mathcal{F}_{T_1}$, and therefore the pricing formula is somehow more complicated than (7). On the other hand, if we have

$$Y_T = cS_T - K \quad (27)$$

for any deterministic c , then $A(Y_T)$ is again independent of $\mathcal{F}_{T_1}$. Further generalizations of (27) are possible and will be studied elsewhere.

6 Hedging/A Proof of Theorem 2

As was mentioned in section 4, the expression (14) gives the hedging strategy.

Let us start with (19), the expression for all $t \in [0, T_2]$:

$$\begin{aligned} C_t &\equiv C_t(S_T - K) \\ &= e^{-r(T-t)} \mathbf{E}^{\hat{P}}[1_{\Omega \setminus A}(S_{T_1} - K) | \mathcal{F}_t] \\ &\quad + e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[1_A \{(e^{r(T_2-T)} - 1)(S_T - K)\} | \mathcal{F}_t] \\ &\quad + e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[1_A(S_{T_2} - K) | \mathcal{F}_t] \\ &= e^{-r(T-t)} \mathbf{E}^{\hat{P}}[1_{\Omega \setminus A} S_{T_1} | \mathcal{F}_t] - e^{-r(T-t)} K \\ &\quad + e^{-r(T_2-t)} \mathbf{E}^{\hat{P}}[1_A \{S_{T_2} + (e^{r(T_2-T)} - 1)S_T\} | \mathcal{F}_t]. \end{aligned} \quad (28)$$

First, let us consider the case when $t \in [T, T_2]$. Since all terms except S_{T_2} are $\mathcal{F}_t$ measurable, we have

$$\begin{aligned} C_t &= e^{-r(T-t)} 1_{\Omega \setminus A} S_{T_1} - e^{-r(T-t)} K \\ &\quad + (e^{-r(T-t)} - e^{-r(T_2-t)}) 1_A S_T \\ &\quad + 1_A e^{rt} \mathbf{E}^{\hat{P}}[e^{-rT_2} S_{T_2} | \mathcal{F}_t] \\ &= e^{-r(T-t)} 1_{\Omega \setminus A} S_{T_1} - e^{-r(T-t)} K \\ &\quad + (e^{-r(T-t)} - e^{-r(T_2-t)}) 1_A S_T + 1_A S_t. \end{aligned} \quad (29)$$

Therefore, we have

$$\begin{aligned} dC_t &= 1_A dS_t + \{1_A S_T (e^{-rT} - e^{-rT_2}) \\ &\quad + 1_{\Omega \setminus A} S_{T_1} e^{-rT} - e^{-rT} K\} dB_t. \end{aligned} \quad (30)$$

Next, when $t \in [T_1, T]$,

$$\begin{aligned} C_t &= e^{-r(T-t)} S_{T_1} \mathbf{E}^{\hat{P}}[1_{\Omega \setminus A} | \mathcal{F}_t] - e^{-r(T-t)} K \\ &\quad + e^{-r(T-t)} (2 - e^{r(T_2-T)}) S_t \mathbf{E}^{\hat{P}}[1_A S_T / S_t | \mathcal{F}_t]. \end{aligned} \quad (31)$$

Since we have

$$\begin{aligned} A &= \{S_{T_1} \leq (2 - e^{-r(T_2-T)}) S_T\} \\ &= \{S_{T_1} / S_t \leq (2 - e^{-r(T_2-T)}) S_T / S_t\} \\ &= \{-\log(S_T / S_t) \leq \\ &\quad \log(2 - e^{-r(T_2-T)}) + \log(S_t / S_{T_1})\} \\ &= \{\hat{W}_T - \hat{W}_t \geq \\ &\quad - \frac{(r - \frac{1}{2}\sigma^2)(T-t) + \log(2 - e^{-r(T_2-T)}) S_t / S_{T_1}}{\sigma}\} \end{aligned} \quad (32)$$

from (23), the conditional expectations in (31) are explicitly calculated as

$$\mathbf{E}^{\hat{P}}[1_{\Omega \setminus A} | \mathcal{F}_t] = \Phi(-d_-(t, S_t)), \quad (33)$$

and

$$\mathbf{E}^{\hat{P}}[1_A S_T / S_t | \mathcal{F}_t] = e^{r(T-t)} \Phi(d_+(t, S_t)). \quad (34)$$

Noting that $\Phi(\pm d_{\pm}(t, x)) \in C^{1,2}([T_1, T] \times \mathbb{R})$, we can apply Itô's formula to get

$$\begin{aligned} d\Phi(\pm d_{\pm}(t, S_t)) &= \partial_x \Phi(\pm d_{\pm}(t, x))|_{x=S_t} dS_t \\ &\quad + (\partial_t + 2^{-1}\sigma^2 x^2 \partial_{xx}) \Phi(\pm d_{\pm}(t, x))|_{x=S_t} dt \\ &= \pm \frac{\phi(d_{\pm}(t, S_t))}{\sigma \sqrt{T-t}} [S_t^{-1} dS_t - (r + 2^{-1}\sigma^2 \pm 2^{-1}\sigma^2)] dt. \end{aligned} \quad (35)$$

Hence

$$\begin{aligned} dC_t &= e^{-rT} \{S_{T_1} \Phi(-d_-(t, S_t)) - K\} dB_t \\ &\quad + e^{-rT} B_t S_{T_1} d\Phi(-d_-(t, S_t)) \\ &\quad + (2 - e^{r(T_2-t)}) [\Phi(d_+(t, S_t)) dS_t + S_t d\Phi(d_+(t, S_t)) \\ &\quad \quad + d\langle \Phi(d_+), S \rangle_t] \\ &= e^{-rT} \{S_{T_1} \Phi(-d_-(t, S_t)) - K\} dB_t \\ &\quad - e^{-rT} B_t S_{T_1} \frac{\phi(d_-(t, S_t))}{\sigma \sqrt{T-t}} (S_t^{-1} dS_t - r dt) \\ &\quad + (2 - e^{r(T_2-t)}) [\Phi(d_+(t, S_t)) dS_t + \\ &\quad \quad \frac{S_t \phi(d_+(t, S_t))}{\sigma \sqrt{T-t}} \{S_t^{-1} dS_t - (r + \sigma^2) dt\} \\ &\quad \quad \quad + \sigma S_t \frac{\phi(d_+(t, S_t))}{\sqrt{T-t}} dt] \\ &= [(2 - e^{r(T_2-t)}) \left(\Phi(d_+(t, S_t)) + \frac{\phi(d_+(t, S_t))}{\sigma \sqrt{T-t}} \right) \\ &\quad \quad - \frac{B_t S_{T_1} \phi(-d_-(t, S_t))}{B_T S_t \sigma \sqrt{T-t}}] dS_t \\ &\quad + \left[\frac{S_{T_1}}{B_T} \left(\Phi(-d_-(t, S_t)) + \frac{\phi(d_-(t, S_t))}{\sigma \sqrt{T-t}} \right) \right. \\ &\quad \quad \left. - (2 - e^{r(T_2-t)}) \frac{S_t \phi(d_+(t, S_t))}{B_t \sigma \sqrt{T-t}} - \frac{K}{B_T} \right] dB_t \end{aligned} \quad (36)$$

Finally, consider the cases when $t \in [0, T_1]$. Since A is independent of $\mathcal{F}_t$, from (28) we have

$$\begin{aligned} C_t &= e^{-r(T-T_1)} S_t \hat{P}(\Omega \setminus A) - e^{-r(T-t)} K \\ &\quad + e^{-r(T-t)} (2 - e^{r(T_2-T)}) \mathbf{E}^{\hat{P}}[S_{T_1} | \mathcal{F}_t] \mathbf{E}^{\hat{P}}[1_A S_T / S_{T_1}] \\ &= e^{-r(T-T_1)} S_t \Phi(-d_-) - e^{-r(T-t)} K \\ &\quad + (2 - e^{r(T_2-T)}) \Phi(d_+) S_t. \end{aligned} \quad (37)$$

Therefore,

$$\begin{aligned} dC_t &= -K e^{-rT} dB_t + \{e^{-r(T-T_1)} \Phi(-d_-) \\ &\quad + (2 - e^{r(T_2-T)}) \Phi(d_+)\} dS_t. \end{aligned} \quad (38)$$

By (30), (36) and (38), we get (8) and (9). $\square$

References

- [1] J. Akahori: Stochastic Analysis in Mathematical Finance, (in Japanese) *Systems, Control and Information*, 44, pp.429-433, 2000.
- [2] J. C. Hull *Options, Futures, and Other Derivatives*, (5th Edition), Prentice Hall, 2002.