The Effects of FSD Changes in Multiplicative Background Risk on Risk-Taking Attitude

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Abstract
This paper analyzes the effects of first-degree stochastic dominance (FSD) changes in multiplicative background risk on the risk-taking attitude of a decision maker. First, we consider contractive FSD changes in multiplicative background risk and analyze the effect of these changes. Then we consider general FSD change in multiplicative background risk. Also, within the context of coinsurance, we examine the effects of simple FSD changes and monotone likelihood ratio (MLR) changes in multiplicative background risk.

key words: first-degree stochastic dominance changes, multiplicative background risk, risk-taking attitude, coinsurance

1. Introduction

Usually, background risk is assumed to be additive. There exists a large body of literature concerning additive background risk. To name a few of these papers, Eeckhoudt and Kimball [3] consider the conditions under which the presence of additive background risk increases the optimal amount of insurance against the other risk. Eeckhoudt, Gollier, and Schlesinger [2], and Meyer and Meyer [8] discuss the effects of changes in additive background risk on the risk-taking attitude of a decision maker. As for other studies concerning additive background risk, we refer to Gollier [6].

We assume in this paper that background risk is multiplicative. It seems that the risk-taking attitude of a decision maker in the presence of multiplicative background risk is first considered systematically by Franke, Schlesinger, and Stapleton [4]. In particular, they consider the conditions under which the presence of multiplicative background risk make a decision maker more risk averse. However, they don’t consider the problem of how changes in multiplicative background risk affect the risk-taking attitude of a decision maker.

As stated above, in the case where background risk is additive, this problem is considered by Eeckhoudt, Gollier, and Schlesinger [2], and Meyer and Meyer [8]. Eeckhoudt, Gollier, and Schlesinger consider first-degree stochastic dominance (FSD) changes and second-degree stochastic dominance (SSD) changes in additive background risk. In particular, they give necessary and sufficient conditions for these changes to imply more risk averse attitude of a decision maker. Meyer and Meyer analyze the effects of changes in additive background risk on the demand for coinsurance and deductibles. These changes in additive background risk include strong increases in risk, simple FSD transformations, and simple SSD transformations. Strong increases in risk are included in SSD changes. As for strong increases in risk, see Meyer and Ormiston [9]. Simple FSD transformations and simple SSD transformations are deterministic transformations. The class of simple FSD (SSD) transformations is a subclass of FSD (SSD) transformations. As for these transformations, see Ormiston [13] and Meyer and Ormiston [10]. In Sakagami [15], third-degree stochastic dominance (TSD) transformations are defined, and the comparative statics of TSD transformations is considered.

Meyer and Meyer also state in notes that they have also considered monotone probability ratio (MPR) changes. MPR changes are included in FSD changes, and MPR changes include Monotone Likelihood Ratio (MLR) changes. As for the
MLR order, see Landsberger and Meilijson [7]. Also, as for the MPR order, see Eeckhoudt and Gollier [1]. In Sakagami [15], the comparative statics of an order which generalizes the MPR order is considered.

We focus in this paper on FSD changes in multiplicative background risk and analyze the effects of these changes on the risk-taking attitude of a decision maker. In section 2.1, we consider contractive FSD changes in multiplicative background risk. In section 2.2, we consider general FSD changes in multiplicative background risk. In section 3, we examine the effects of simple FSD changes and MLR changes in multiplicative background risk within the context of coinsurance. In section 4, we state our concluding remarks.

2. Contractive FSD changes and general FSD changes

2.1. Contractive FSD changes

In this subsection, we examine the effect of contractive FSD changes in multiplicative background risk on the risk-taking attitude of a decision maker. The decision maker’s utility function $u(w)$ is assumed to be strictly increasing and twice continuously differentiable. We assume that $w > 0$. Denote the measure of relative risk aversion of $u(w)$ by $R_u(w)$ or $R_u$ throughout this paper. Thus, $R_u(w) = -w' u''(w)/u'(w)$. This measure is useful when we analyze multiplicative risk such as investment rates of returns and exchange rates. Define the derived utility functions by $v_i(w) = Eu(w|w_{yi}) (i = 1, 2)$. Here $w_{y1}$ and $w_{y2}$ are the initial background risk and the background risk after a change, respectively. Let $r_i(i = 1, 2)$ denote the measure of absolute risk aversion of $v_i(i = 1, 2)$. Then $r_i(w) = -v_i''(w)/v_i'(w)$. As for the measure of absolute risk aversion and the measure of relative risk aversion, see Pratt [14]. It should be noted that in the case of additive background risk, the derived utility function is defined by $v_i(w) = Eu(w + w_{yi}) (i = 1, 2)$. See Kihlstrom, Romer, and Williams [5], and Nachman [12].

We now assume that the initial background risk $w_{y1}$ has m-point distribution $(m \geq 2)$ such that $P(w_{yi} = w_{yi}) = p_i (i = 1, 2, \ldots, m)$ and $0 < w_{y1} < w_{y2} < \cdots < y_{1m}$. Assume that $w_{y2} = y_{1} \xi$. Here $\xi$ has n-point distribution $(n \geq 1)$ such that $P(\xi = \xi_i) = q_i (i = 1, 2, \ldots, n)$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_n \leq 1$. Here $w_{y1}$ and $\xi$ are assumed to be stochastically independent. In this paper, this change from $w_{y1}$ to $w_{y2}$ is called a contractive FSD deterioration.

We can prove the following proposition.

**Proposition 2.1.** If relative risk aversion $R_u$ is nonincreasing, then the decision maker does not become less risk averse after any contractive FSD deterioration in multiplicative background risk.

2.2. General FSD changes

We now consider the case of general FSD changes. Let the distribution functions of the initial background risk $w_{y1}$ and the background risk after change $w_{y2}$ be $F_1(y)$, $F_2(y)$, respectively. When $F_1(y) \leq F_2(y)$ for all $y$ with $< for some $y$, we say that $w_{y1}$ first-degree stochastically dominates $w_{y2}$ and represent this relation by $w_{y1} \preceq_{FSD} w_{y2}$. We assume that the support of $w_{y1}$ and the support of $w_{y2}$ are contained in $[a, b]$ where $a > 0$. We also assume that $F_1(a) = F_2(a) = 0$. The decision maker’s utility function $u(w)$ is assumed to be three times continuously differentiable. Moreover, we assume that $u''(w) > 0$, $u''(w) < 0$ and $u'''(w) > 0$. As in subsection 2.1, we also assume that $w > 0$. Denote the measure of relative prudence of $u(w)$ by $P_u(w)$ or $P_u$. Thus, $P_u(w) = -w' u''(w)/u'(w)$. As for the effect of FSD deteriorations in multiplicative background risk, we have the following proposition. We prove this proposition by using the method of Eeckhoudt, Gollier, and Schlesinger [2].

**Proposition 2.2.** Suppose that $R_u(wx) > 1$, $P_u(wx) \geq 2R_u(wx)$ for all $x$ in $[a, b]$ and that $P_u(wx) \geq R_u(wx)$ for all $x, y$ in $[a, b]$. Then $w_{y1} \preceq_{FSD} w_{y2}$ implies $R_{u2}(w) \geq R_{u1}(w)$.

**Example 2.1.** Let $u(w) = (r + \frac{w}{\gamma})^{1-\gamma}$, where $\eta < 0$ and $0 < \gamma < 1$. For this utility function, $R_u(wx) = wx(\eta + \frac{w}{\gamma})^{-1}$ and $P_u(wx) = \frac{1+wx\eta}{\gamma}$. Here we assume that $w < \frac{\eta}{\gamma-1}$. Then $R_u(wx) > 1$. Also, $P_u < 2R_u$. Now define $x^*$ as follows. If there exists a $x^*$ such that $-\gamma x^* < wx^*$ and $P_u(\frac{wx^*}{\gamma}) = R_u(wx^*)$,
then \( x^{**} = x^* \). Otherwise, \( x^{**} = \frac{-b}{w} \).

Then, for all \( x, y \geq x^{**} \), \( P_u(wx) \geq R_u(wx) \). Hence, if supports of \( y_1 \) and \( y_2 \) are contained in \( \{x^{**} = \frac{-b}{w(\beta - 1)}\} \), then \( y_2 \leq \text{FSD} y_1 \) implies \( r_{v_2}(w) \geq r_{v_1}(w) \).

**Remark 2.3.** Suppose \( R_u(wx) > 1 \), \( P_u(wx) \geq 2R_u(wx) \) implies \( P_u(wx) \geq 1 + R_u(wx) \). Thus, it follows from Lemma 3 in Franke, Schlesinger, and Stapleton [4] that \( R_u(wx) < 0 \) for \( x \in [a, b] \). Thus, the set of utility functions which satisfy the assumptions of Proposition 2.2 is the subset of the set of utility functions with nonincreasing relative risk aversion.

**Remark 2.2.** Since \( R_u(wx) \) is the subset of the set of utility functions which satisfy the assumptions of Proposition 2.2, \( y_2 \leq \text{FSD} y_1 \) implies \( r_{v_2}(w) \geq r_{v_1}(w) \).

In the next section, we continue the analysis within the coinsurance framework.

### 3. Simple FSD changes and MLR changes in the coinsurance demand model

Meyer and Meyer [8] analyze the effects of changes in additive background risk on the demand for coinsurance. The changes in additive background risk which they refer to include simple FSD changes and monotone likelihood ratio (MLR) changes. MPR changes are related to the ratio of two cumulative distributions, and MLR changes are related to the ratio of two density functions. MPR changes include monotone likelihood ratio (MLR) changes. As stated earlier, it is MPR changes that they refer to in their notes. These categories are special cases of FSD changes. First, we consider the effect of simple FSD changes in multiplicative background risk on the demand for coinsurance. We use the same coinsurance demand model as Meyer and Meyer [8] and use the method of Meyer and Meyer [8]. The decision-maker with utility function \( u(w) \) is assumed to select covering fraction \( \theta \) of the loss in \([0, 1]\) to maximize \( E u(\tilde{w}) \), where

\[
\tilde{w} = \{M - \tilde{x} + \theta[I(\tilde{x}) - p]\} \bar{y}_1. \tag{1}
\]

Here \( \bar{y}_1 \) is a positive multiplicative background risk which is stochastically independent of \( \tilde{x} \), and the support of \( \bar{y}_1 \) is contained in \([a, b]\) where \( a > 0 \); \( M \) is the value of a risky asset; \( \tilde{x} \) is the insurable risk, and the support of \( \tilde{x} \) is contained in \([0, M]\); \( p \) is the price of one unit of insurance; and \( I(\tilde{x}) \) is the indemnification of \( \tilde{x} \). It is assumed that \( I(x) \) is nondecreasing, and satisfies \( 0 \leq I(x) \leq \tilde{x} \) and \( I'(x) \leq 1 \). The utility function \( u(w) \) is assumed to be strictly increasing, strictly concave and twice continuously differentiable. Moreover, it is assumed that there is an interior solution to above maximization.

We now define a function \( m(y) \) by

\[
m(y) = y E \tilde{x} u'(\tilde{w}) I(\tilde{x}) - p, \]

where \( E \tilde{x} \) represents the expectation with respect to \( \tilde{x} \).

We denote by \( \theta^* \) the value of \( \theta \) that satisfies \( E \bar{y}_1 m(\bar{y}_1) = 0 \). Following Meyer and Meyer [8], we examine the properties of the function \( m(y) \).

**Lemma 3.1.** Suppose that \( R_u(w) \) is nonincreasing. Then, at \( \theta = \theta^* \), there exists a \( y^* \) such that \( m(y) \geq 0 \) for all \( y \leq y^* \) and \( m(y) \leq 0 \) for all \( y \geq y^* \).

Moreover, if we add the assumption that \( R_u(w) \geq (\leq 1) \), then \( m(y) \geq (\leq 0) \) implies \( m'(y) \leq 0 \).

Similarly, in the case where \( R_u(w) \) is nondecreasing, we can prove the following lemma.

**Lemma 3.2.** Suppose that \( R_u(w) \) is nondecreasing. Then, at \( \theta = \theta^* \), there exists a \( y^* \) such that \( m(y) \leq 0 \) for all \( y \leq y^* \) and \( m(y) \geq 0 \) for all \( y \geq y^* \).

Moreover, if we add the assumption that \( R_u(w) \geq (\geq 1) \), then \( m(y) \geq (\geq 0) \) implies \( m'(y) \geq 0 \).

For the most part, above lemmas correspond to Lemma 1 of Meyer and Meyer [8].

We now consider the effect of simple FSD changes in multiplicative background risk on the demand for coinsurance.

We represent the multiplicative background risk as \( \bar{y}_1 + \delta k(\bar{y}_1) \). Here \( 0 \leq \delta \leq 1 \) and \( k(\bar{y}_1) = t(\bar{y}_1) - \bar{y}_1 \), where \( t(y) \) is

\[t(y) = y^{\beta - 1} I(\tilde{x}) - \bar{y}_1.\]

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1 The author could not prove one property, that is, \( E \bar{y}_1 m'(\bar{y}_1) \leq 0 \) which is included in Lemma 1 of Meyer and Meyer [8].
nondecreasing in \( y \). When \( \delta = 0 \), this risk is the initial background risk. On the other hand, when \( \delta = 1 \), this risk is the background risk after a change. This change is called a simple FSD deterioration if \( k(y) \leq 0, k'(y) \geq 0 \). See, Meyer and Meyer [8].

We can now prove the following proposition.

**Proposition 3.1.** Suppose that \( R_u(w) \geq (\leq) 1 \) and that \( R_u(w) \) is nonincreasing (nondecreasing). Then the optimal \( \theta \) when \( \delta = 1 \) does not become smaller after any simple FSD deterioration.

Next we consider the effect of MLR changes. We assume that initial background risk \( \hat{y}_1 \) dominates the background risk after a change \( \hat{y}_2 \) by monotone likelihood ratio, that is, \( f_1(y) \) is nonincreasing in \( y \). Here we denote respective densities of \( \hat{y}_1 \) and \( \hat{y}_2 \) by \( f_1(y) \) and \( f_2(y) \).

**Proposition 3.2.** Suppose that \( R_u(w) \) is nonincreasing. Then the optimal \( \theta \) does not become smaller after any MLR deterioration.

### 4. Concluding Remarks

We analyzed in this paper the effects of first-degree stochastic dominance (FSD) changes in multiplicative background risk on the risk-taking attitude of a decision maker. In section 2, we gave sufficient conditions for a decision maker to become more risk averse in the presence of FSD deteriorations (contractive FSD deteriorations and general FSD deteriorations) in multiplicative background risk. In section 3, we gave sufficient conditions for optimal covering fraction \( \theta \) of the loss not to become smaller after FSD deteriorations (simple FSD deteriorations and MLR deteriorations).

In section 2, we considered FSD changes within a general framework. In section 3, we considered FSD changes within the coinsurance framework. The results in section 2 will be useful for analysis of comparative statics in the insurance demand model such as the coinsurance model in section 3. Also, it should be noted that the content of FSD changes in background risk is different. In particular, we considered general FSD changes in section 2.2. On the other hand, we considered some special well-known FSD changes in section 3. Thus, the conditions for general FSD changes in section 2.2 are stronger than those for special FSD changes in section 3.

This work is a first step for studying changes in multiplicative background risk. Only FSD changes were considered. It will be necessary to consider other changes such as SSD changes. It will be also necessary to consider within the context of portfolio selection. Much work must be done for understanding the risk-taking attitude in face of changes in multiplicative background risk.

### References


