On a smoother of discrete-time linear stochastic systems with unknown disturbances

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Abstract

We consider discrete-time linear stochastic systems with unknown disturbances and study a smoothing problem for those systems. We derive a smoothing algorithm by applying the optimal filter obtained in our previous work to an augmented system.

Keywords: stochastic systems, smoother, optimal filter, unknown inputs.

1 Introduction

We consider optimal filtering and smoothing problems for discrete-time linear stochastic systems. It is well known that a standard Kalman filter is the optimal linear filter since it minimizes the mean-square errors in an appropriate class of linear filters (see e.g., [14], [16], [17], [18] and [19]). We need to recall that the Kalman filter can work well only when we have an accurate mathematical modelling of the monitored system. However, in practice, models derived by engineers often contain modelling errors which greatly increase the state estimation error as if the models have unknown disturbances.

In order to develop reliable filtering algorithm which are robust with respect to unknown disturbances and modelling errors, many research works based on the use of the disturbance decoupling principle have been published. Pioneering works were done by Darouach et al. ([8] and [9]), Chang and Hsu ([4]) and Hou and Müller ([11]). They utilized some transformations to make the original systems with unknown inputs into some singular systems without unknown inputs. The most important previous study related to this paper was done by Chen and Patton ([5]). They proposed a simple and useful optimal filtering algorithm, ODDO (Optimal Disturbance Decoupling Observer), with unknown input decoupling property and showed excellent simulation results. See also the papers such as [3], [12], [13] and [22]. Their algorithm recently has been modified by the authors in [23].

The optimal filtering problem is to investigate the optimal estimate of the state x_t or x_{t+1} with min-

imum variance based on the observation $\mathcal{Y}_t$ of the states $\{y_0, y_1, \dots, y_t\}$, i.e., $\mathcal{Y}_t = \sigma\{y_s, s = 0, 1, \dots, t\}$ (the smallest σ -field generated by $\{y_0, y_1, \dots, y_t\}$ (see e.g., Katayama [19], Chapter 4)), but we here consider smoothing problems which allow us time-lags for computing state estimates. Namely, we study the optimal estimate $\hat{x}_{t-L/t}$ of the state x_{t-L} based on the observation $\mathcal{Y}_t$ with $L > 0$. The smoothing problems are usually classified into the following three types. The first problem is the fixed-point smoothing for which we investigate the optimal estimate of the state x_k for a fixed k based on the observations $\{\mathcal{Y}_t, t = k+1, k+2, \dots\}$. Algorithms to compute $\hat{x}_{k/t}, t = k+1, k+2, \dots$, recursively are called fixed-point smoothers. The second problem is to study the optimal state estimate of x_{t-L} based on the observation $\mathcal{Y}_t$ for a given $L \geq 1$. Algorithms to compute $\hat{x}_{t-L/t}, t = L+1, L+2, \dots$, recursively are called fixed-lag smoothers. The third problem is to study the optimal state estimate $\hat{x}_{t/N}$ of the states x_t at all times $t = 0, 1, \dots, N$ based on the observation $\mathcal{Y}_N$ of all the states $\{y_0, y_1, \dots, y_N\}$. Algorithms to compute $\hat{x}_{t/N}, t = N, N-1, \dots, 0$ recursively are called fixed-interval smoothers. See also [1], [2], [15], [19] and [20] for smoothers.

We state some preliminary results in Section 2, and propose a fixed-point smoother in Section 3. This smoothing algorithm is derived by applying the optimal filter (see Proposition 2.2) obtained in [23] to an augmented system. We note that the fixed-point smoother is rather important also due to the reason why the fixed-interval smoother can be readily derived from the fixed-point smoother. It will be discussed in the subsequent paper.

2 Preliminaries

Consider the following discrete-time linear stochastic system for $t = 0, 1, 2, \dots$:

$$x_{t+1} = A_t x_t + B_t u_t + E_t d_t + \zeta_t, \quad (1)$$

$$y_t = C_t x_t + \eta_t, \quad (2)$$

where

$x_t \in \mathbf{R}^n$ the state vector

$y_t \in \mathbf{R}^m$ the output vector
 $u_t \in \mathbf{R}^r$ the known input vector
 $d_t \in \mathbf{R}^q$ the unknown input vector

ζ_t and η_t are independent zero mean white noise sequences with covariance matrices Q_t and R_t . A_t, B_t, C_t and E_t are known matrices with appropriate dimensions.

In [23], we studied the optimal state estimate $\hat{x}_{t+1/t+1}$ of the state x_{t+1} proposed by Chen and Patton ([5]) with the following structure:

$$z_{t+1} = F_{t+1} z_t + T_{t+1} B_t u_t + K_{t+1} y_t, \quad (3)$$

$$\hat{x}_{t+1/t+1} = z_{t+1} + H_{t+1} y_{t+1}, \quad (4)$$

for $t = 0, 1, 2, \dots$. Let $\hat{x}_{0/0}$ be chosen to be z_0 for a fixed z_0 . We denote the state estimation error by e_t , i.e., $e_t = x_t - \hat{x}_{t/t}$ for $t = 0, 1, 2, \dots$. Its covariance matrix (i.e., the error covariance matrix of state estimate (3)-(4)) is denoted by P_t , i.e., $P_t = \mathbf{E}\{e_t e_t^T\}$, where $\mathbf{E}$ denotes expectation and T denotes transposition of a matrix. In this paper, we assume that random variables $e_0, \{\eta_t\}, \{\zeta_t\}$ are independent. Following Chen and Patton ([5]), we considered state estimate (3)-(4) with the matrices $F_{t+1}, T_{t+1}, H_{t+1}$ and K_{t+1} of the forms ([23]):

$$K_{t+1} = K_{t+1}^1 + K_{t+1}^2, \quad (5)$$

$$E_t = H_{t+1} C_{t+1} E_t, \quad (6)$$

$$T_{t+1} = I - H_{t+1} C_{t+1}, \quad (7)$$

$$F_{t+1} = A_t - H_{t+1} C_{t+1} A_t - K_{t+1}^1 C_t, \quad (8)$$

$$K_{t+1}^2 = F_{t+1} H_t. \quad (9)$$

The next lemma on equality (6) was obtained and used by Chen and Patton [5]. Before stating it, we assume that E_k is a full column rank matrix which is not an essential restriction.

Lemma 2.1 *Equality (6) holds if and only if*

$$\text{rank}(C_{t+1} E_t) = \text{rank}(E_t). \quad (10)$$

When this condition holds true, matrix H_{t+1} which satisfies (6) must have the form

$$H_{t+1} = E_t \left\{ (C_{t+1} E_t)^T (C_{t+1} E_t) \right\}^{-1} \times (C_{t+1} E_t)^T. \quad (11)$$

Hence, we have

$$C_{t+1} H_{t+1} = C_{t+1} E_t \left\{ (C_{t+1} E_t)^T (C_{t+1} E_t) \right\}^{-1} (C_{t+1} E_t)^T \quad (12)$$

which is a non-negative definite symmetric matrix. ■

We found out the explicit form of the matrix K_{t+1}^1 which makes the variance of the state estimation error e_{t+1} minimum and its sufficient conditions in ([23]). Thus, we considered the matrix K_{t+1}^1 given by

$$K_{t+1}^1 = A_{t+1}^1 (P_t C_t^T - H_t R_t) \times (C_t P_t C_t^T + R_t)^{-1}, \quad (13)$$

where

$$A_{t+1}^1 = A_t - H_{t+1} C_{t+1} A_t. \quad (14)$$

We here state the main result (Theorem 2.7 in ([23]) on the optimal filtering algorithm as follows.

Proposition 2.2 *If $C_t H_t$ and R_t are commutative, i.e.,*

$$C_t H_t R_t = R_t C_t H_t, \quad (15)$$

then the optimal gain matrix K_{t+1}^1 which makes the variance of the state estimation error e_{t+1} minimum is determined by (13). Hence, we obtain the optimal filtering algorithm:

$$\hat{x}_{t+1/t+1} = A_{t+1}^1 \left\{ \hat{x}_{t/t} + G_t (y_t - C_t \hat{x}_{t/t}) \right\} + H_{t+1} y_{t+1} + T_{t+1} B_t u_t, \quad (16)$$

$$P_{t+1} = A_{t+1}^1 M_t A_{t+1}^{1T} + T_{t+1} Q_t T_{t+1}^T + H_{t+1} R_{t+1} H_{t+1}^T, \quad (17)$$

where

$$G_t = (P_t C_t^T - H_t R_t) (C_t P_t C_t^T + R_t)^{-1}, \quad (18)$$

and

$$M_t = P_t - G_t (C_t P_t - R_t H_t^T). \quad (19)$$

Remark 2.3 If the matrix R_t has the form

$$R_t = r_t I$$

with some positive number r_t for each $t = 1, 2, \dots$, then condition (15) holds. ■

Finally, we remark that the standard Kalman filter is a special case of the optimal filter proposed in this section (see e.g., Theorem 5.2 (page 90) in Katayama [19]).

Proposition 2.4 *Suppose that $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero). Then, Lemma 2.1 cannot be applied directly. But, we can choose $H_t \equiv O$ for all t in this case, and the optimal filter given in Proposition 2.2 reduces to the standard Kalman filter.* ■

3 The fixed-point smoothing

Let k be a fixed time. We study an iterative algorithm to compute the optimal estimate $\hat{x}_{k/t}$ of the state x_k based on the observation $\mathcal{Y}_t, t = k+1, k+2, \dots$, where $\mathcal{Y}_t = \sigma\{y_s, s = 0, 1, \dots, t\}$. We define state vectors $\theta_t, t = k, k+1, \dots$, by

$$\theta_{t+1} = \theta_t, t = k, k+1, \dots; \theta_k = x_k. \quad (20)$$

It is easy to observe that the optimal estimate $\hat{\theta}_{t/t}$ of the state θ_t based on the observation $\mathcal{Y}_t$ is identical to the optimal smoother $\hat{x}_{k/t}$ in view of the equalities $\theta_t = x_k, t = k, k+1, \dots$.

In order to derive the optimal fixed-point smoother, we consider the following augmented system for $t =$

$k, k+1, \dots$:

$$\begin{bmatrix} x_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} A_t & O \\ O & I \end{bmatrix} \begin{bmatrix} x_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} B_t \\ O \end{bmatrix} u_t + \begin{bmatrix} E_t \\ O \end{bmatrix} d_t + \begin{bmatrix} I \\ O \end{bmatrix} \zeta_t, \quad (21)$$

$$y_{t+1} = [C_{t+1} \ O] \begin{bmatrix} x_{t+1} \\ \theta_{t+1} \end{bmatrix} + \eta_{t+1}. \quad (22)$$

Denote these equations respectively by

$$\widetilde{x}_{t+1} = \widetilde{A}_t \widetilde{x}_t + \widetilde{B}_t u_t + \widetilde{E}_t d_t + \widetilde{J}_t \zeta_t, \quad (23)$$

$$y_{t+1} = \widetilde{C}_{t+1} \widetilde{x}_{t+1} + \eta_{t+1}, \quad (24)$$

where

$$\begin{aligned} \widetilde{x}_t &= \begin{bmatrix} x_t \\ \theta_t \end{bmatrix}, \quad \widetilde{A}_t = \begin{bmatrix} A_t & O \\ O & I \end{bmatrix}, \quad \widetilde{B}_t = \begin{bmatrix} B_t \\ O \end{bmatrix}, \\ \widetilde{E}_t &= \begin{bmatrix} E_t \\ O \end{bmatrix}, \quad \widetilde{J}_t = \begin{bmatrix} I \\ O \end{bmatrix} \text{ and } \widetilde{C}_{t+1} = [C_{t+1} \ O]. \end{aligned}$$

Here, I and O are the identity matrix and the zero matrix respectively with appropriate dimensions. By making use of the notations

$$\widetilde{H}_{t+1} = \begin{bmatrix} H_{t+1} \\ O \end{bmatrix}, \quad \widetilde{T}_{t+1} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} - \widetilde{H}_{t+1} \widetilde{C}_{t+1},$$

we have the equalities:

$$\begin{aligned} \widetilde{C}_{t+1} \widetilde{E}_t &= C_{t+1} E_t, \\ \widetilde{T}_{t+1} &= \begin{bmatrix} T_{t+1} & O \\ O & I \end{bmatrix}, \\ \widetilde{A}_{t+1}^1 &= \widetilde{T}_{t+1} \widetilde{A}_t = \begin{bmatrix} A_{t+1}^1 & O \\ O & I \end{bmatrix}. \end{aligned}$$

We introduce the covariance matrix $\widetilde{P}_t$ of the state estimation error of the augmented system (23)-(24):

$$\begin{aligned} \widetilde{P}_t &= \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} \\ &= \mathbf{E} \left\{ \begin{bmatrix} x_t - \hat{x}_{t/t} \\ \theta_t - \hat{\theta}_{t/t} \end{bmatrix} \begin{bmatrix} x_t - \hat{x}_{t/t} \\ \theta_t - \hat{\theta}_{t/t} \end{bmatrix}^T \right\}. \quad (25) \end{aligned}$$

Notice that $P_t^{(1,1)}$ is equal to P_t . Applying the optimal filter given in Proposition 2.2 to the augmented system (21)-(22), we obtain the following optimal fixed-point smoother.

Theorem 3.1 *If $C_t H_t$ and R_t are commutative, i.e.,*

$$C_t H_t R_t = R_t C_t H_t, \quad (26)$$

then we have the optimal fixed-point smoother for (21)-(22) as follows:

(i) *the fixed-point smoother*

$$\hat{x}_{k/t+1} = \hat{x}_{k/t} + D_t(k) [y_t - C_t \hat{x}_{t/t}], \quad (27)$$

(ii) *the gain matrix*

$$D_t(k) = P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1}, \quad (28)$$

(iii) *the covariance matrix of the mean-square error*

$$P_{t+1}^{(2,1)} = \left\{ P_t^{(2,1)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \right. \\ \left. \times (C_t P_t - R_t H_t^T) \right\} A_{t+1}^{1,T}, \quad (29)$$

$$P_{t+1}^{(2,2)} = P_t^{(2,2)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ \times C_t P_t^{(2,1)T}. \quad (30)$$

Here, we note that $P_k^{(2,1)} = P_k^{(2,2)} = P_k$. We notice that $\hat{x}_{t/t}$ is the optimal filter of the original system (1)-(2) given in [23].

Proof Applying the optimal filter given by (16)-(17) in Proposition 2.2 to the augmented system (23)-(24), we have

$$\begin{aligned} \widehat{x}_{t+1/t+1} &= \widetilde{A}_{t+1}^1 \left\{ \widehat{x}_{t/t} + \widetilde{G}_t (y_t - C_t \widehat{x}_{t/t}) \right\} \\ &\quad + \widetilde{H}_{t+1} y_{t+1} + \widetilde{T}_{t+1} \widetilde{B}_t u_t. \quad (31) \end{aligned}$$

This can be rewritten as

$$\begin{aligned} &\begin{bmatrix} \hat{x}_{t+1/t+1} \\ \hat{\theta}_{t+1/t+1} \end{bmatrix} \\ &= \begin{bmatrix} A_{t+1}^1 & O \\ O & I \end{bmatrix} \left\{ \begin{bmatrix} \hat{x}_{t/t} \\ \hat{\theta}_{t/t} \end{bmatrix} + \begin{bmatrix} P_t^{(1,1)} C_t^T - H_t R_t \\ P_t^{(2,1)} C_t^T \end{bmatrix} \right. \\ &\quad \left. \times (C_t P_t C_t^T + R_t)^{-1} (y_t - C_t \hat{x}_{t/t}) \right\} \\ &\quad + \begin{bmatrix} H_{t+1} y_{t+1} \\ O \end{bmatrix} + \begin{bmatrix} T_{t+1} B_t u_t \\ O \end{bmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \hat{x}_{t+1/t+1} &= A_{t+1}^1 \left\{ \hat{x}_{t/t} + (P_t^{(1,1)} C_t^T - H_t R_t) \right. \\ &\quad \left. \times (C_t P_t C_t^T + R_t)^{-1} (y_t - C_t \hat{x}_{t/t}) \right\} \\ &\quad + H_{t+1} y_{t+1} + T_{t+1} B_t u_t \quad (32) \end{aligned}$$

and

$$\begin{aligned} \hat{\theta}_{t+1/t+1} &= \hat{\theta}_{t/t} + P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ &\quad \times (y_t - C_t \hat{x}_{t/t}). \quad (33) \end{aligned}$$

Here, we used the equalities

$$\begin{aligned} &\widetilde{C}_t \widetilde{P}_t \widetilde{C}_t^T + R_t \\ &= [C_t \ O] \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} \begin{bmatrix} C_t^T \\ O \end{bmatrix} + R_t \\ &= C_t P_t C_t^T + R_t \quad (34) \end{aligned}$$

and

$$\begin{aligned} \widetilde{G}_t &= \left(\widetilde{P}_t \begin{bmatrix} C_t^T \\ O \end{bmatrix} - \widetilde{H}_t R_t \right) (\widetilde{C}_t \widetilde{P}_t \widetilde{C}_t^T + R_t)^{-1} \\ &= \left(\begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} \begin{bmatrix} C_t^T \\ O \end{bmatrix} - \begin{bmatrix} H_t \\ O \end{bmatrix} R_t \right) \\ &\quad \times (\widetilde{C}_t \widetilde{P}_t \widetilde{C}_t^T + R_t)^{-1} \\ &= \begin{bmatrix} P_t^{(1,1)} C_t^T - H_t R_t \\ P_t^{(2,1)} C_t^T \end{bmatrix} (C_t P_t C_t^T + R_t)^{-1}. \quad (35) \end{aligned}$$

Thus, equalities (27)-(28) can be obtained from (33) due to $\hat{\theta}_{t/t} = \hat{x}_{k/t}$.

By using the notation $\widetilde{M}_t$ for the augmented system (23)-(24) which corresponds to the matrix M_t in Proposition 2.2, we have

$$\begin{aligned}\widetilde{M}_t &= \begin{bmatrix} M_t^{(1,1)} & M_t^{(1,2)} \\ M_t^{(2,1)} & M_t^{(2,2)} \end{bmatrix} \\ &= \widetilde{P}_t - \widetilde{G}_t \left(\widetilde{C}_t \widetilde{P}_t - R_t [H_t^T \ O] \right) \\ &= \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} - \begin{bmatrix} P_t^{(1,1)} C_t^T - H_t R_t \\ P_t^{(2,1)} C_t^T \end{bmatrix} \\ &\quad \times (C_t P_t C_t^T + R_t)^{-1} \left([C_t \ O] \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} \right. \\ &\quad \left. - [R_t H_t^T \ O] \right).\end{aligned}$$

Thus, we have

$$\begin{aligned}M_t^{(1,1)} &= P_t^{(1,1)} - (P_t^{(1,1)} C_t^T - H_t R_t) \\ &\quad \times (C_t P_t C_t^T + R_t)^{-1} (C_t P_t^{(1,1)} - R_t H_t^T),\end{aligned}\quad (36)$$

$$\begin{aligned}M_t^{(1,2)} &= P_t^{(1,2)} - (P_t^{(1,1)} C_t^T - H_t R_t) \\ &\quad \times (C_t P_t C_t^T + R_t)^{-1} C_t P_t^{(1,2)},\end{aligned}\quad (37)$$

$$\begin{aligned}M_t^{(2,1)} &= P_t^{(2,1)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ &\quad \times (C_t P_t^{(1,1)} - R_t H_t^T),\end{aligned}\quad (38)$$

and

$$\begin{aligned}M_t^{(2,2)} &= P_t^{(2,2)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ &\quad \times C_t P_t^{(1,2)}.\end{aligned}\quad (39)$$

It follows from (17) in Proposition 2.2 that

$$\begin{aligned}\widetilde{P}_{t+1} &= \widetilde{A}_{t+1}^T \widetilde{M}_t \widetilde{A}_{t+1}^T + \widetilde{T}_{t+1} \widetilde{J}_{t+1}^T Q_{t+1} \\ &\quad \times \widetilde{J}_{t+1}^T \widetilde{T}_{t+1} + \widetilde{H}_{t+1} R_{t+1} \widetilde{H}_{t+1}^T \\ &= \begin{bmatrix} A_{t+1}^1 & O \\ O & I \end{bmatrix} \begin{bmatrix} M_t^{(1,1)} & M_t^{(1,2)} \\ M_t^{(2,1)} & M_t^{(2,2)} \end{bmatrix} \begin{bmatrix} A_{t+1}^1{}^T & O \\ O & I \end{bmatrix} \\ &\quad + \begin{bmatrix} T_{t+1} & O \\ O & I \end{bmatrix} \begin{bmatrix} I \\ O \end{bmatrix} Q_{t+1} [I \ O] \begin{bmatrix} T_{t+1}^T & O \\ O & I \end{bmatrix} \\ &\quad + \begin{bmatrix} H_{t+1} \\ O \end{bmatrix} R_{t+1} [H_{t+1}^T \ O].\end{aligned}\quad (40)$$

Equalities (29)-(30) follow from (38)-(40). Finally, we have equalities $P_k^{(2,1)} = P_k^{(2,2)} = P_k^{(1,1)} = P_k$ by the definition of $\widetilde{P}_k$. ■

Remark 3.2 Suppose that $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero) and that $H_t \equiv O$ for all t (as in Proposition 2.4). In this case, it follows from Theorem 3.1 that

$$\hat{x}_{t+1/t+1} = A_t \left\{ \hat{x}_{t/t} + P_t C_t^T (C_t P_t C_t^T + R_t)^{-1} \right.$$

$$\left. \times (y_t - C_t \hat{x}_{t/t}) \right\} + B_t u_t, \quad (41)$$

$$\begin{aligned}\hat{\theta}_{t+1/t+1} &= \hat{\theta}_{t/t} + P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ &\quad \times (y_t - C_t \hat{x}_{t/t}),\end{aligned}\quad (42)$$

$$\begin{aligned}P_{t+1}^{(2,1)} &= \left\{ P_t^{(2,1)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \right. \\ &\quad \left. \times C_t P_t \right\} A_t^T,\end{aligned}\quad (43)$$

and

$$\begin{aligned}P_{t+1}^{(2,2)} &= P_t^{(2,2)} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} \\ &\quad \times C_t P_t^{(2,1)T}.\end{aligned}\quad (44)$$

Here, we note that the state estimate $\hat{x}_{t+1/t+1}$ reduces to the state estimate $\hat{x}_{t+1/t}$ in [19] when $H_t \equiv O$ holds. Moreover, Equalities (37)-(40) with the state estimates $\hat{x}_{t+1/t+1}$ and $\hat{x}_{t/t}$ replaced respectively by $\hat{x}_{t+1/t}$ and $\hat{x}_{t/t-1}$ are identical to those for the pair of the standard Kalman filter and the optimal fixed-point smoother in [19]. ■

Let us introduce some notations:

$$\nu_t = y_t - C_t \hat{x}_{t/t}, \quad (45)$$

$$L_t = A_{t+1}^1 (I - G_t C_t), \quad (46)$$

$$\Psi(t, \tau) = \begin{cases} L_{t-1} L_{t-2} \cdots L_\tau & , \ t > \tau \\ I & , \ t = \tau, \end{cases} \quad (47)$$

where the matrix G_t was defined by (18), i.e.,

$$G_t = (P_t C_t^T - H_t R_t) (C_t P_t C_t^T + R_t)^{-1}. \quad (48)$$

We then have the following results due to (27).

Corollary 3.3 We have the equalities:

$$\begin{aligned}\hat{x}_{k/t+1} &= \hat{x}_{k/k} + \sum_{i=k}^t D_i(k) \nu_i \\ &= \hat{x}_{k/k} + P_k \sum_{i=k}^t \Psi(i, k)^T C_i^T (C_i P_i C_i^T + R_i)^{-1} \nu_i.\end{aligned}\quad (49)$$

Proof It is straightforward to show the first equality from (27). For the second equality, it is sufficient to prove the equality

$$D_t(k) = P_k \Psi(t, k)^T C_t^T (C_t P_t C_t^T + R_t)^{-1} \quad (50)$$

for $t \geq k$. By virtue of (46), equality (29) can be rewritten as

$$P_t^{(2,1)} = P_{t-1}^{(2,1)} (I - C_{t-1}^T G_{t-1}^T) A_t^1{}^T = P_{t-1}^{(2,1)} L_{t-1}^T. \quad (51)$$

By using this equality recursively, we have

$$\begin{aligned}P_t^{(2,1)} &= P_{t-2}^{(2,1)} L_{t-2}^T L_{t-1}^T = \cdots \\ &= P_k^{(2,1)} L_k^T L_{k+1}^T \cdots L_{t-1}^T \\ &= P_k \Psi(t, k)^T.\end{aligned}\quad (52)$$

Substituting this equality into (28), we obtain

$$D_t(k) = P_k \Psi(t, k)^T C_t^T (C_t P_t C_t^T + R_t)^{-1}, \quad (53)$$

i.e., (50). ■

Finally, we study the reduction of the estimation error by the fixed-point smoothing over the optimal filtering. Due to (27), we have

$$P_t^{(2,2)} = \mathbf{E} \left[(x_k - \hat{x}_{k/t}) (x_k - \hat{x}_{k/t})^T \right]. \quad (54)$$

Denote this matrix simply by $P_{k/t}$. It then follows from (30) that

$$P_{k/t+1} = P_{k/t} - P_t^{(2,1)} C_t^T (C_t P_t C_t^T + R_t)^{-1} C_t P_t^{(2,1)T} \quad (55)$$

Summing up these equalities for $t = k, k+1, \dots, s$, we have

$$\begin{aligned} P_{k/k} - P_{k/s+1} \\ = \sum_{i=k}^s P_i^{(2,1)} C_i^T (C_i P_i C_i^T + R_i)^{-1} C_i P_i^{(2,1)T}. \end{aligned} \quad (56)$$

Thus, the right hand side indicates the amount of the reduction of the estimation error by the fixed-point smoothing over the optimal filtering.

4 Conclusion

In this paper, the fixed-point smoothing problem for a discrete-time linear stochastic system with unknown disturbances has been considered. A new smoothing algorithm which is robust to unknown disturbances has been derived from the optimal filter for linear systems with unknown inputs in the author's previous work. It has been shown that this algorithm reduces to the well known optimal smoother derived from the Kalman filter when unknown inputs disappear.

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