Some sufficient conditions for laws of large numbers in random linear programs

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Abstract
Large-scale linear programming problems whose coefficients are random variables will be considered in this paper. Stability of the optimum solutions of these problems as the number of variables increases was shown in terms of the laws of large numbers by Kuhn and Quandt (on the weak law) and by Prékopa (on the strong law). The same conclusions were proved for a broader class of problems under weaker assumptions in the previous paper. Here, we prove the laws of large numbers under even weaker assumptions for independent and identically distributed (i.i.d.) random variables by applying an estimate by Brillinger and Petrov. We give a simple example which shows that the assumption \( r \geq 2 \) associated with the moment condition given in Proposition 3.1 cannot be replaced by the weaker assumption \( 0 < r < 2 \).

Keywords: linear programming, random coefficient, law of large numbers.

1 Introduction
We study optimum solutions of large-scale linear programming problems whose numerical computations might be unstable due to accumulation of round-off errors. We also have difficulty in getting rid of discrepancies in practice between our models and the actual data. Thus, we consider linear programming problems whose coefficients are random variables, and study stability of the optimum solutions of these problems.

Stability of them as the number of variables increases was first shown by Kuhn and Quandt ([4]) in terms of the weak law of large numbers. Namely, they showed that the random errors in the original data have diminishing effects on the optimum as the number of variables increases. The weak law by Kuhn and Quandt ([4]) was refined to a strong law for a more general class of problems by Prékopa in [6] (see also [3] and [7]). Their results were further generalized to a broader class and could be proved under even weaker assumptions in the previous paper ([8]).

We now start with the problem formulation. Let us consider the following linear programming problem (LP):

\[
\text{Max } (x_1 + x_2 + \cdots + x_n), \\
\text{subject to } \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq 1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq 1, \\
\ldots \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq 1, \\
x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.
\]

We assume that the coefficients \( a_{ik} \) are random variables. Then, Problem (1) is called a random linear programming problem (RLP).

A weak law of large numbers for Problem (1) was first considered by Kuhn and Quandt ([4]) under the assumption that \( \{a_{ik}\} \) are independent and uniformly distributed random variables in a fixed interval \( [u, v] \) (\( u > 0 \)). They showed that, for any \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} P \left( \left| \frac{\mu_n - 2}{\mu_n + v} \right| \geq \varepsilon \right) = 0,
\]

or equivalently, for any \( \varepsilon' > 0 \)

\[
\lim_{n \to \infty} P \left( \left| \frac{1}{\mu_n} - \frac{u + v}{2} \right| = \left| \frac{1}{\mu_n} - \mathbb{E}(a_{ik}) \right| \geq \varepsilon' \right) = 0,
\]

where \( \mu_n \) (\( 0 < u \leq \mu_n \leq v \)) is the random optimum value of Problem (1) (with \( m = n \)). This result was then generalized to a more general class and also refined to a strong law by Prékopa (see [6] and [7]). In [8], we proved the same conclusion under much weaker assumptions than in [6] and [7].

This paper is organized as follows. Some preliminary results are given in Section 2. In Section 3, we prove the laws of large numbers for independent and identically distributed (i.i.d.) random noises under even
weaker assumptions than in [8] by applying an estimate by Brillinger and Petrov (see Lemma 2.1) (and also [5]). We give a simple example which shows that the assumption $r \geq 2$ is associated with the moment condition given in Proposition 3.1 cannot be replaced by the weaker assumption $0 < r < 2$.

2 Preliminaries

First, we suppose that an array (or a double sequence) of random variables $a_{ik}$ has the form

$$a_{ik} = a_{ik}^{(0)} + \xi_{ik}, \quad i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n,$$

where $a_{ik}^{(0)}$ are real numbers and $\{\xi_{ik}\}$ are independent random variables with $E(\xi_{ik}) = 0$. Here, $E$ denotes expectation. Then, we immediately have $a_{ik} = E(a_{ik})$.

The (random) optimum value of RLP (1) is denoted by $\mu_{mn}$. Let us also consider the following deterministic linear programming problem (DLP):

$$\text{Max } (x_1 + x_2 + \cdots + x_n),$$

subject to

$$a_{11}^{(0)} x_1 + a_{12}^{(0)} x_2 + \cdots + a_{1n}^{(0)} x_n \leq 1,$$

$$a_{21}^{(0)} x_1 + a_{22}^{(0)} x_2 + \cdots + a_{2n}^{(0)} x_n \leq 1,$$

$$\cdots$$

$$a_{m1}^{(0)} x_1 + a_{m2}^{(0)} x_2 + \cdots + a_{mn}^{(0)} x_n \leq 1,$$

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.$$

The optimum value of DLP (5) is denoted by $\mu_{mn}^{(0)}$.

As in [6], [7] and [8], we make the following assumption.

Condition A When $m$ and $n$ are sufficiently large, we assume that there exist bounds $\beta$ and $\gamma$ with $0 < \beta < \gamma$ such that the following condition holds:

$$0 < \beta \leq \frac{m}{n} \leq \gamma < \infty.$$

Let us introduce some notations. Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers with $c_n > 0$. We use the notation $b_n = O(c_n)$ (resp. $b_n = o(c_n)$) if $\lim_{n \to \infty} b_n/c_n = 0$ (resp. $|b_n|/c_n \leq C < \infty$, $n \geq 1$).

For a sequence of random variables $\{Z_i\} = \{Z_1, Z_2, \ldots, Z_i, \ldots\}$, we use the notation $S_n$ for the sum of the first $n$ terms, i.e., $S_n = \sum_{i=1}^{n} Z_i$. For a sequence of independent and identically distributed random variables $\{Z_i\}$, the following estimate was proved for $1 \leq r < 2$ by Brillinger ([2]) and generalized by Petrov ([5]).

Lemma 2.1 (Theorem 28 of Chapter IX in [5]) Let $\{Z_i\}$ be a sequence of independent and identically distributed random variables with $E|Z_1|^r < \infty$ and

$$E(|Z_1|^r) < \infty,$$

for some $r \geq 1$. Then, we have

$$P \left( \left| \frac{S_n}{n} \right| \geq \varepsilon \right) = o(n^{-r+1})$$

for any $\varepsilon > 0$.

We also consider an array of random variables $\{Z_{ik}\}$ ($i = 1, 2, \ldots; k = 1, 2, \ldots$):

$$Z_{11} Z_{12} \cdots Z_{1n} \cdots$$

$$Z_{21} Z_{22} \cdots Z_{2n} \cdots$$

$$\vdots$$

$$Z_{m1} Z_{m2} \cdots Z_{mn} \cdots$$

We denote by $S_{n}^{(0)}$ the partial sums of the rows of the array $\{Z_{ik}\}$:

$$S_{n}^{(0)} := \sum_{k=1}^{n} Z_{ik}.$$

For positive integers $m$ and $n$, we define $\overline{\alpha}_{mn}$ and $\underline{\alpha}_{mn}$ respectively by

$$\overline{\alpha}_{mn} := \max_{1 \leq k \leq m} \frac{1}{n} \sum_{i=1}^{n} a_{ik}^{(0)}$$

and

$$\underline{\alpha}_{mn} := \min_{1 \leq k \leq m} \frac{1}{n} \sum_{i=1}^{n} a_{ik}^{(0)}.$$

In addition to Condition A, we make use of the following assumption as in [8].

Condition B There exists a constant $\alpha > 0$ such that

$$\lim_{m \to \infty} \lim_{n \to \infty} \overline{\alpha}_{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \underline{\alpha}_{mn} = \alpha.$$

For reader's convenience, we give the following simple sufficient conditions for Condition B which were stated in [8].

Lemma 2.2 Each of the following conditions is a sufficient condition for Condition B.

(1) The double sequence $\{a_{mn}^{(0)}\}$ converges to $c > 0$ as $m$ and $n$ tend to $\infty$ in the following sense: For any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left| a_{mn}^{(0)} - c \right| \leq \varepsilon$$

holds for $n \geq N$ and $1 \leq m \leq n\gamma,$

and also for $m \geq N$ and $1 \leq n \leq m/\beta$. (10)

(2) Each $a_{mn}^{(0)}$ is equal to one of $q_1, q_2, \ldots, q_r$, and they appear regularly such as:

$$a_{mn}^{(0)} = q_k \quad \text{if} \quad n \equiv k + m - 1 \pmod{r}.$$ (11)
3 Laws of large numbers for i.i.d. random variables

First, we have the following weak convergence of \( \{ Z_k \} \) by applying the estimate by Brillinger (Lemma 2.1).

**Proposition 3.1** Let \( \{ Z_k \} \) be an array of independent and identically distributed random variables with \( \mathbb{E}(Z_{11}) = 0 \) and

\[
\mathbb{E}(|Z_{11}|^r) < \infty
\]

for some \( r \geq 2 \). Then, for arbitrary \( \varepsilon > 0 \), as \( m \to \infty \) and \( n \to \infty \) with (6), we have

\[
P \left( \max_{1 \leq k \leq m} \frac{1}{n} \sum_{k=1}^{n} Z_{ik} > \varepsilon \right) \to 0
\]

and

\[
P \left( \min_{1 \leq k \leq m} \frac{1}{m} \sum_{i=1}^{m} Z_{ik} < -\varepsilon \right) \to 0,
\]

and hence

\[
P \left( \max_{1 \leq k \leq m} \frac{1}{n} \sum_{k=1}^{n} Z_{ik} > \varepsilon \right) \to 0
\]

and

\[
P \left( \min_{1 \leq k \leq m} \frac{1}{m} \sum_{i=1}^{m} Z_{ik} > \varepsilon \right) \to 0.
\]

**Proof** For any \( \varepsilon > 0 \) fixed, we have

\[
P \left( \max_{1 \leq k \leq m} \frac{1}{n} \left| S_n^{(t)} \right| \geq \varepsilon \right)
\]

\[
\leq \sum_{l=1}^{m} P \left( \frac{1}{n} \left| S_n^{(t)} \right| \geq \varepsilon \right)
\]

\[
\leq m \rho(n) n^{-r-1} = \gamma \rho(n) n^{2-r}
\]

with \( \rho(n) = o(1) \) by applying Lemma 2.1 and Condition A. Thus, we have for \( r \geq 2 \)

\[
P \left( \max_{1 \leq k \leq m} \frac{1}{n} \left| S_n^{(t)} \right| \geq \varepsilon \right) \to 0
\]

(as \( n \to \infty, m \to \infty \))

and hence

\[
P \left( \max_{1 \leq k \leq m} \frac{1}{n} \sum_{k=1}^{n} Z_{ik} > \varepsilon \right) \to 0
\]

(as \( n \to \infty, m \to \infty \)).

The rest of Proof is done by the same way as in Proposition 2.4 in [8].

The following simple example shows that the assumption \( r \geq 2 \) associated with the moment condition given in Proposition 3.1 cannot be replaced by the weaker assumption \( 0 < r < 2 \).

**Example** Let \( \{ Z_{ik} \} \) be an array of i.i.d. random variables, where

\[
Z_{11} = \begin{cases} 
\pm n & \text{(with probability } c_0/(2n^3)\text{)} \\
0 & \text{(with probability } 1 - \sum_{n=1}^{\infty} (c_0/n^3)\text{)}
\end{cases}
\]

for some constant \( c_0 > 0 \) satisfying \( \sum_{n=1}^{\infty} (c_0/n^3) < 1 \). In view of the equality

\[
P (|Z_{11}| > n) = \sum_{k=n+1}^{\infty} c_0/k^3
\]

and the simple inequality

\[
\int_{n}^{\infty} \frac{c_0}{(x+1)^3} dx < \sum_{k=n+1}^{\infty} \frac{c_0}{k^3} < \int_{n}^{\infty} \frac{c_0}{x^3} dx,
\]

i.e.,

\[
\frac{c_0}{2(n+1)^3} < \sum_{k=n+1}^{\infty} \frac{c_0}{k^3} < \frac{c_0}{2n^2},
\]

we have

\[
n^2 P (|Z_{11}| > n) \to \frac{c_0}{2} \left( \neq 0 \right) \quad (\text{as } n \to \infty).
\]

And it is easy to observe that

\[
\mathbb{E} \left( Z_{11} I_{\{|Z_{11}| < n\}} \right) = 0
\]

holds for any \( n > 0 \), where \( I_B \) denotes the indicator function of a set \( B \). Then, for any \( \varepsilon > 0 \), owing to Theorem 4 in [1], we obtain

\[
\limsup_{n \to \infty} n P \left( \frac{1}{n} \left| S_n^{(t)} \right| > \varepsilon \right) > 0
\]

for \( t = 1, 2, \ldots \). Let \( \varepsilon > 0 \) be fixed and \( D_n^{(t)} \) be defined by

\[
D_n^{(t)} = \left\{ \omega \mid \frac{1}{n} \left| S_n^{(t)} \right| > \varepsilon \right\}.
\]

Noting that \( P \left( D_n^{(t)} \right) \) is independent of \( t \) since \( \{ Z_k \} \) is an array of i.i.d. random variables, we use the notations \( u_n \) and \( v_n \) for the following probabilities:

\[
u_n = P \left( D_n^{(t)} \right) \quad \text{and} \quad v_n = P \left( \bigcup_{t=1}^{n} D_n^{(t)} \right).
\]

Applying the simple equality

\[
v_n = P \left( \bigcup_{t=1}^{n} D_n^{(t)} \right)
\]

\[
\text{—128—}
\]
\[
\begin{align*}
&= \sum_{\ell=1}^{n} P \left( D_{\ell}^{(t)} \right) - \sum_{\ell<j} P \left( D_{\ell}^{(t)} \cap D_{j}^{(t)} \right) \\
&+ \sum_{\ell<j<k} P \left( D_{\ell}^{(t)} \cap D_{j}^{(t)} \cap D_{k}^{(t)} \right) - \cdots \\
&+ (-1)^{n-1} P \left( D_{1}^{(t)} \cap D_{2}^{(t)} \cap \cdots \cap D_{n}^{(t)} \right) - \cdots \\
&= \sum_{\ell=1}^{n} P \left( D_{\ell}^{(t)} \right) - \sum_{\ell<j} P \left( D_{\ell}^{(t)} \right) P \left( D_{j}^{(t)} \right) \\
&+ \sum_{\ell<j<k} P \left( D_{\ell}^{(t)} \right) P \left( D_{j}^{(t)} \right) P \left( D_{k}^{(t)} \right) - \cdots \\
&+ (-1)^{n-1} P \left( D_{1}^{(t)} \right) P \left( D_{2}^{(t)} \right) \cdots P \left( D_{n}^{(t)} \right) \\
&= \left( \frac{n}{1} \right) u_{n} - \left( \frac{n}{2} \right) u_{n}^{2} + \left( \frac{n}{3} \right) u_{n}^{3} - \cdots \\
&+ (-1)^{n-1} \left( \frac{n}{n} \right) u_{n}^{n}.
\end{align*}
\]

we have
\[
1 - u_{n} = \left( \frac{n}{1} \right) - \left( \frac{n}{2} \right) u_{n} + \left( \frac{n}{3} \right) u_{n}^{2} - \cdots \\
+ (-1)^{n-1} \left( \frac{n}{n} \right) u_{n}^{n}.
\quad (25)
\]

It follows from (22) that
\[
\limsup_{n \to \infty} n u_{n} \geq c_{1} \quad (26)
\]
holds for some \( c_{1} > 0 \). We then have
\[
\liminf_{n \to \infty} \left( 1 - u_{n} \right)^{n} = \liminf_{n \to \infty} \left( 1 - \frac{n u_{n}}{n} \right)^{-n/n} \leq e^{-c_{1}} \quad (27)
\]
and hence
\[
\limsup_{n \to \infty} P \left( \max_{1 \leq \ell \leq m} \frac{1}{n} \left| S_{\ell}^{(t)} \right| \geq \varepsilon \right) = \limsup_{n \to \infty} v_{n}
\]
\[
= \limsup_{n \to \infty} \left[ 1 - \left( 1 - u_{n} \right)^{n} \right] = 1 - \liminf_{n \to \infty} \left( 1 - u_{n} \right)^{n} \geq 1 - e^{-c_{1}} > 0. \quad (28)
\]

Namely, the conclusion of Proposition 3.1 does not hold for this example. Moreover, we can easily observe that
\[
E \left( \left| Z_{11}^{(t)} \right|^{r} \right) = \sum_{n=1}^{\infty} C_{0} n^{r} = \sum_{n=1}^{\infty} C_{0} n^{3-3r} < \infty \quad (29)
\]
holds for any \( r \) with \( 0 \leq r < 2 \) and that
\[
E \left( \left| Z_{11}^{(t)} \right|^{2} \right) = \sum_{n=1}^{\infty} C_{0} n = \infty \quad (30)
\]
holds. Thus, \( \{ Z_{ik} \} \) does not satisfy the condition (12) in Proposition 3.1.

Remark 3.2 For independent (but not necessarily i.i.d.) random variables, the same conclusions were obtained in [8] under the condition
\[
E \left( \left| Z_{ik}^{(t)} \right|^{r} \right) \leq M \quad (31)
\]
for \( i = 1, 2, \ldots \), and \( k = 1, 2, \ldots \) with \( r > 2 \) and \( M > 0 \) (see also Theorem 1 (p. 280) in [6]).

Under a slightly stronger condition than (12), we can show almost sure convergence as follows.

Proposition 3.3 Let \( \{ Z_{ik} \} \) be an array of independent and identically distributed random variables with \( E \left( Z_{11}^{(t)} \right) = 0 \) and
\[
E \left( \left| Z_{11}^{(t)} \right|^{r} \right) < \infty \quad (32)
\]
for some \( r > 3 \). Then, as \( m \to \infty \) and \( n \to \infty \) with (6), we have
\[
\max_{1 \leq k \leq m} \frac{1}{n} \sum_{k=1}^{n} Z_{ik} \to 0 \quad a.s. \quad (33)
\]
and
\[
\min_{1 \leq k \leq m} \frac{1}{m} \sum_{i=1}^{m} Z_{ik} \to 0 \quad a.s. \quad (34)
\]

Proof We fix an arbitrary \( \varepsilon > 0 \), and define the set \( U_{n, \nu} \) by
\[
U_{n, \nu} := \left\{ \omega : \max_{1 \leq \ell \leq m} \frac{1}{n} \left| S_{\ell}^{(t)} \right| \geq \varepsilon \right\}
\]
with \( \nu = \left[ n \gamma \right] \), where \( \gamma \) was defined in Condition A. Here, \( \left[ c \right] \) denotes the largest integer not exceeding \( c \). Then, we immediately obtain
\[
P \left( U_{n, \nu} \right) \leq \sum_{t=1}^{\nu} P \left( \frac{1}{n} \left| S_{t}^{(t)} \right| \geq \varepsilon \right). \quad (35)
\]
Applying Lemma 2.1 and Condition A, we have
\[
P \left( U_{n, \nu} \right) \leq \gamma n \rho(n) n^{-r+1} = \gamma \rho(n) n^{2-r} \quad (36)
\]
with \( \rho(n) = o(1) \). It then follows that for \( r > 3 \)
\[
\sum_{n=1}^{\infty} P \left( U_{n, \nu} \right) < \infty. \quad (37)
\]
The rest of Proof is done by the same way as in Proposition 2.6 in [8].

Remark 3.4 For independent (but not necessarily i.i.d.) random variables, the same conclusions were obtained in [8] under the condition
\[
E \left( \left| Z_{ik}^{(t)} \right|^{r} \right) \leq M \quad (38)
\]
for \( i = 1, 2, \ldots \), and \( k = 1, 2, \ldots \) with \( r > 4 \) and \( M > 0 \) (see also Theorem 2 (p. 281) in [6]).
The weak law of large numbers (Theorem 3.5) and the strong law of large numbers (Theorem 3.6) for i.i.d. random variables respectively follow from Proposition 3.1 and Proposition 3.3. Proofs of these theorems are omitted here.

**Theorem 3.5** Let $\mu_{mn}$ be the optimum value of RLP (1) and let the random variables $\{a_{ik}\}$ have the form

$$a_{ik} = a_{ik}^{(0)} + \xi_{ik},$$

where $\{a_{ik}^{(0)}\}$ are real numbers and $\{\xi_{ik}\}$ are independent and identically distributed random variables with $E(\xi_{11}) = 0$. Suppose that $a_{ik}$ are positive-valued with probability 1 and

$$E(|\xi_{11}|^r) < \infty$$

for some $r \geq 2$. Then, for any $\varepsilon > 0$, we have

$$P \left( \left| \frac{1}{\mu_{mn}} - \alpha \right| > \varepsilon \right) \to 0$$

(as $m \to \infty$ and $n \to \infty$). \hspace{1cm} (41)

Moreover, we obtain

$$\frac{1}{\mu_{mn}^{(0)}} \to \alpha \hspace{1cm} \text{(as $m \to \infty$ and $n \to \infty$)}$$

and hence

$$P \left( \left| \frac{1}{\mu_{mn}} - \frac{1}{\mu_{mn}^{(0)}} \right| > \varepsilon \right) \to 0$$

(as $m \to \infty$ and $n \to \infty$). \hspace{1cm} (43)

**Theorem 3.6** Let $\mu_{mn}$ be the optimum value of RLP (1) and let the random variables $\{a_{ik}\}$ satisfy the assumption of Theorem 3.5. Suppose furthermore that inequality (40) holds for some $r > 3$. Then, we have

$$P \left( \lim_{m \to \infty} \lim_{n \to \infty} \left| \frac{1}{\mu_{mn}} - \alpha \right| = 0 \right)$$

$$= P \left( \lim_{m \to \infty} \lim_{n \to \infty} \left| \frac{1}{\mu_{mn}} - \frac{1}{\mu_{mn}^{(0)}} \right| = 0 \right) = 1. \hspace{1cm} (44)$$

**References**


