Influence of Random Noises on Spatio-temporal Patterns in Predator-prey Systems

Masaaki Ishikawa and Minori Yokoo

Graduate School of Science and Engineering, Yamaguchi University
2-16-1 Tokiwadai, Ube 755-8611 Japan

E-mail: ishi@yamaguchi-u.ac.jp

Abstract

Physical, chemical and social phenomena contain some kind or another of uncertainties to a greater or lesser extent. The analysis of influence of such uncertainties on phenomena is very important as a basic problem in various fields including design and planning of controlled systems in control engineering and analysis of option pricing in economics. In this paper, focusing on biological communities, we study the influence of the random uncertainties on predator-prey systems with diffusion. Noting that interaction of phytoplankton and zooplankton is the basis of a food chain in the ocean, we firstly consider the stochastic predator-prey systems consists of phytoplankton and zooplankton. Secondly, a modeling of the stochastic phytoplankton-zooplankton-fish system is considered. We analyze the influence of the random uncertainties on the spatio-temporal patterns generated by two types of the predator-prey systems with diffusion by the numerical simulations.

1 Introduction

Many spatio-temporal patterns are observed in various fields of engineering and the analysis of such patterns has been one of challenging problems in pattern formation problems. So far, many excellent works have been done in analysis of the spatio-temporal pattern formations [1], [2], however, most of such works are deterministic analyses. Although the deterministic analysis is effective for real phenomena, there exists some cases that the deterministic approach is not applicable. For example, random fluctuations have a great influence on the Turing structure [1], [2] and we often experience that the random uncertainties in natural world change the ecological situations. In this way, an analysis of influence of random uncertainties on the phenomena is very important. Among many phenomena, we especially consider the formation of the spatio-temporal patterns in biological systems and we study the influence of the random uncertainties on the spatio-temporal patterns generated by predator-prey systems with diffusion. In this paper, as the predator-prey system, two types of systems are studied: one is the predator-prey

system consists of phytoplankton and zooplankton, the other is the phytoplankton-zooplankton-fish system.

2 Stochastic Zooplankton-Phytoplankton Model without Fish Predation

Letting $u(t, x)$ and $v(t, x)$ be the densities of prey (phytoplankton) and predator (zooplankton) at time t and position x , consider the stochastic predator-prey model with diffusion such that

$$\begin{aligned} u_t(t, x) = & d_u u_{xx}(t, x) + au(t, x)(1 - u(t, x)) \\ & - \frac{u(t, x)}{u(t, x) + h} v(t, x) + r_1 u(t, x) \dot{w}_x(t, x), \end{aligned} \quad (1)$$

$$(t, x) \in \Theta \times G,$$

$$\begin{aligned} v_t(t, x) = & d_v v_{xx}(t, x) + \alpha \frac{u(t, x)}{u(t, x) + h} v(t, x) \\ & - \beta v(t, x) + r_2 v(t, x) \dot{q}_x(t, x), \end{aligned} \quad (2)$$

$$(t, x) \in \Theta \times G,$$

with the initial and boundary conditions

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in G, \quad (3)$$

$$\frac{\partial u(t, x)}{\partial \nu} = \frac{\partial v(t, x)}{\partial \nu} = 0, \quad (t, x) \in \Theta \times \partial G, \quad (4)$$

where subscripts denote partial derivatives corresponding to each subscript, $\Theta = (0, T)$ (T is a positive constant), $\partial(\cdot)/\partial \nu$ denotes the exterior normal derivative on the boundary ∂G of G and all parameters are constants and $\dot{w}_x$ and $\dot{q}_x$ are mutually independent space-time white noises [3]. The locally growth of prey is logistic and predator shows the Holling-type II functional response [4]

The pattern formation problems in the predator-prey systems without random uncertainties in Eqs.(1) and (2) have already been studied by Petrovskii and Malchow [5], [6] and they found that an irregular pattern appears in some conditions in the initial predator distribution $v_0(x)$ (or the prey distribution $u_0(x)$). Main feature of this paper is to study the influence of the random uncertainties on spatio-temporal patterns

in predator-prey systems, which is impossible by the model in Petrovskii and Malchow [5], [6]. In order to study the dominant behavior of Eqs.(1) and (2), consider the system without random noises and diffusion:

$$\frac{du(t)}{dt} = au(t)(1 - u(t)) - \frac{u(t)}{u(t) + h}v(t), \quad (5)$$

$$\frac{dv(t)}{dt} = \alpha \frac{u(t)}{u(t) + h}v(t) - \beta v(t). \quad (6)$$

It is easily shown that Eqs.(5) and (6) have three spatially uniform equilibrium states $(u^*, v^*) = (0, 0), (1, 0)$ and (u_e, v_e) (co-existence of the prey and the predator) such that

$$u_e = \frac{ph}{1 - p}, \quad v_e = (1 - u_e)(h + u_e), \quad (p = \frac{\beta}{\alpha}). \quad (7)$$

Although the linear stability of the equilibrium state (u^*, v^*) depends on the values h, p and α , it follows from the linear stability analysis that the equilibrium states $(0, 0)$ and $(1, 0)$ are always saddle points and the equilibrium state (u_e, v_e) are unstable (stable) if $h < (1 - p)/(1 + p)$ ($h > (1 - p)/(1 + p)$).

3 Stochastic Zooplankton-phytoplankton Model under Fish Predation

In this section, we consider the predator-prey model consisting of three species of zooplankton, phytoplankton and fishes [7]-[9]. Fishes feed on the zooplankton and their predation rate is given by the so-called Holling type III [4]. It follows from Eqs. (1) and (2) that the stochastic predator-prey model under predation by fishes is given in such a way that

$$\begin{aligned} u_t(t, x) = & d_u u_{xx}(t, x) + au(t, x)(1 - u(t, x)) \\ & - \frac{u(t, x)}{u(t, x) + h}v(t, x) + r_1 u(t, x)\dot{w}_x(t, x), \end{aligned} \quad (8)$$

$$(t, x) \in \Theta \times G,$$

$$\begin{aligned} v_t(t, x) = & d_v v_{xx}(t, x) + \alpha \frac{u(t, x)}{u(t, x) + h}v(t, x) \\ & - \beta v(t, x) - f \frac{nv(t, x)^2}{n^2 + v(t, x)^2} \\ & + r_2 v(t, x)\dot{q}_x(t, x), \end{aligned} \quad (9)$$

$$(t, x) \in \Theta \times G.$$

The initial and the boundary conditions are given by Eqs. (3) and (4).

The 4th term of right-side hand of Eq. (9) denotes the predation of zooplankton by fishes, the predation rate is given by the Holling type III and a parameter f and n are positive constants.

The dependence of the spatially uniform steady state of Eqs. (8) and (9) without the noise on the fish predation rate f is shown in Fig. 1. As shown in Fig. 1, at the high predation rate f , phytoplankton-dominated state is a steady state. As lowering f , two steady states appear by the saddle-node bifurcation, so that the system

becomes bistable. Further lowering f , the two steady states disappear by the saddle-node bifurcation and the system goes back to the single stable steady state. At the point indicated by a filled circle in Fig. 1, a Hopf bifurcation occurs. On the left-side region of the Hopf bifurcation, the steady state is unstable and a stable limit cycle is generated.

Fig. 1: Dependence of the steady states on the fish predation rate f

4 Simulations

4.1 Simulations for Zooplankton-phytoplankton Model without Fish Predation

Taking parameters so as for the equilibrium state (u_e, v_e) to be the unstable focus and changing the initial distribution of the prey, we study the influence of the random uncertainties on the predator-prey system by numerical simulations. Setting parameters as $\alpha = 2.0, \beta = 0.6, d_u = d_v = 1, h = 0.4$ and taking the noise coefficients and the spatial region G as $r_1 = r_2 = 0.008$ and $G = (0, 1200)$, under the initial densities $u_0(x) = u_e$ and $v_0(x) = v_e + 0.0000025(x - 900)^2 + \delta$ with two different δ , simulations are performed.

(Case-1) $\delta = 0.005$: Results of simulations are shown in Fig. 2 and Figs. 2(a) and 2(b) are time evolution of densities of the prey $u(t, x)$ and the predator $v(t, x)$ under the no noise and the noise. From Fig. 2, we can see that in the noise case, the irregular region appears earlier than the no noise case. In order to see this more clearly, time evolution of the density of the prey $u(t, x)$ in the (t, x) -plain is shown in Fig. 3. In Fig. 3, value of the density of $u(t, x)$ is given by the gray scale indicated at the right-hand side bar. From Fig. 3, in both cases of the no noise and noise, the irregular region propagates at the constant speed as shown by the lines AB and AC and the onset of the irregular pattern in the noise case is earlier than the no noise case. Figure 4 is the (u, v) -phase plain at $x = 250$ for $1000 < t < 2000$ under the no noise and the noise. Since Fig. 4 captures only the characteristics of the behavior of the prey and the predator at the special

(a) Time evolution under the no noise

(b) Time evolution under the noise

Fig. 2: Time evolution of densities of the prey $u(t, x)$ and the predator $v(t, x)$ in Case-1

spatial point $x = 250$, in order to see the global characteristics, define

$$u_A(t) = \frac{1}{L} \int_0^L u(t, x) dx, \quad v_A(t) = \frac{1}{L} \int_0^L v(t, x) dx. \quad (10)$$

The $(u_A(t), v_A(t))$ phase plain is depicted in Fig. 5. It follows from Figs. 4 and 5 that the predator and the prey randomly move in some restricted region in the phase plain in the no noise and the noise cases.

(a) No noise case

(b) Noise case

Fig. 3: Behavior of the prey $u(t, x)$ in (t, x) -plain in Case-1

(a) No noise case

(b) Noise case

Fig. 4: (u, v) -phase plane at $x = 250$ for $1000 < t < 2000$

(a) No noise case

(b) Noise case

Fig. 5: Phase plane $(u_A(t), v_A(t))$ of spatially averaged densities for $1000 < t < 2000$

(a) Time evolution under the no noise

Time evolution under the noise

Fig. 6: Time evolution of densities of the prey $u(t,x)$ and the predator $v(t,x)$ in Case-2

(Case 2) $\delta = 0.015$: It should be noted that the distance between minimum value of the initial predator distribution $v_0(x)$ and the equilibrium state v_e is longer than one in Case-1. It follows from Fig. 6(a) that behavior of the densities of the prey and the predator under the no noise have some constant period so that the limit cycle is formed as shown in Fig. 7(a). Figure 7 denotes the time evolution of the prey and the predator in the (u,v) -phase plane at $x = 250$ for $1000 < t < 5000$. In the noise case, the behavior of the prey and

the predator becomes very irregular as shown in Figs. 6(b), 7(b) and 8(b). In this case, the prey and predator move randomly inside of the limit cycle. From Fig. 9, we can see that the irregular region propagates at the constant speed as shown by the lines AB and AC in the noise cases. In this way, random uncertainties can drastically change the behavior of the predator and the prey. In the real biological systems, we often observe that the predator and the prey form patchy inhabitable regions. One of reason of generation of patchy region seems randomness in the environmental situations.

Fig. 7: (u,v) -phase plane at $x = 250$ for $1000 < t < 5000$ in Case-2

Fig. 8: Phase plane $(u_A(t), v_A(t))$ of spatially averaged densities for $1000 < t < 5000$

4.2 Simulations for the Predator-prey Model under Predation by Fishes

In this section, behavior analyses of phytoplankton-zooplankton-fish model (8) and (9) are performed by numerical simulations. The influence of the random fluctuation in the considered system on the spatio-temporal behavior is studied.

Let the spatial region G be $G = (0,30)$ which consists of two patches: the region $(0,15)$ is inhabitable for fish and the region $(15,30)$ is fish-free (a fish predation $f=0$). We study the influence of the fish predation rate f and the random fluctuation in the system on the spatio-temporal behavior by simulations.

(Case 3) $f = 0.180$: Spatio-temporal behavior of the zooplankton density $v(t,x)$ for the fish predation rate

(a) No noise case

(b) Noise case

Fig. 9: Behavior of the prey $u(t, x)$ in (t, x) -plain in Case-2

$f = 0.180$ under the no noise and the noise is shown in Fig. 10. The fish predation rate $f = 0.180$ keeps away to the left from the Hopf bifurcation point in Fig. 1. From Fig. 10, we can see that the periodic behavior in time in both fish-habitant and fish-free regions occurs in the no noise case, on the other hand, in the noise case, the influence of the random noise on the system behavior is observed.

(Case 4) $f = 0.395$: Spatio-temporal behavior of the zooplankton density $v(t, x)$ for the fish predation rate $f = 0.395$ under the no noise and the noise is shown in Fig. 11. The fish predation rate $f = 0.395$ is slightly displaced to the right from the Hopf bifurcation point in Fig. 1. In this case, in the fish-habitant region, spatially uniform state is maintained independently of the existence of the noise, however, in the fish-free regions, the periodic behavior in time is disturbed in both of the no noise and the noise cases. It follows from Fig. 3 that the influence of the random noise on the system behavior in the whole region is little.

In order to see the dependence of the behaviors of the zooplankton on the fish predation rate, its dependence is depicted in Fig. 12. From Fig. 12, we can see that the influence of the random noise on the system behaviors is little in the low or high fish predation rate.

(a) No noise case

(b) Noise case

Fig. 10: Behavior of zooplankton $v(t, x)$ in (t, x) -plain in Case-3 ($f = 0.180$)

(a) No noise case

(b) Noise case

Fig. 11: Behavior of zooplankton density $v(t, x)$ in (t, x) -plain in Case-4 ($f = 0.395$)

(a) No noise case

(b) Noise case

Fig. 12: Dependence of the zooplankton density $u(5000, x)$ on the fish predation rate f

5 CONCLUSIONS

In this paper, the influence of the random uncertainties on the spatio-temporal pattern formation in the two types of predator-prey systems with diffusion, the zooplankton-phytoplankton system and the zooplankton-phytoplankton-fish system, has been studied by numerical simulations. From the simulation results for the former system, we can see that random uncertainties can drastically change the behavior of the predator and the prey and random uncertainties caused by the fluctuations of environmental situations seem to be part of the reason that patchy region is generated in ecosystems.

For the latter system, we have clarified that the influence of the random noise on the system behaviors is little in the low or high fish predation rate and the influence appears in the lower vicinity of the Hopf bifurcation point.

In all simulations in this research, even if the sample paths of the random fluctuation are different, on the condition that the amplitude of the noise is the same value, the characteristic properties of the simulation results are invariant.

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