On a Class of One-dimensional Chaotic Discrete Dynamical System with Multi Step Piecewise Uniform Invariant Density

Toshihiko Yasuda

Dept. of Mechanical System Engineering
The University of Shiga Prefecture,
2500 Hassaka-cho, Hikone, Shiga, 522-8533, Japan

E-mail: yasuda@mech.usp.ac.jp

Abstract

Nonlinear systems described by the simple mathematical model often exhibit extremely complicated behavior called chaos. In this paper, chaotic behavior, exhibited by the one-dimensional difference equation, is discussed. A class of nonlinear discrete system with the invariant density, which is multi step type piece-wise uniform, is newly introduced.

1 Introduction

In various research fields, it is well known that nonlinear systems with a simple mathematical model often exhibit extremely complicated behavior called chaos [1~12]. In this paper, chaotic behavior, exhibited by the following deterministic system described by the one-dimensional difference equation, is discussed:

$$\begin{aligned} x_{n+1} &= f(x_n), \quad n = 0, 1, 2, \dots, \\ f: I &\rightarrow I, \quad I = [0, 1]. \end{aligned} \quad (1)$$

Concerned with the nonlinear function $f(x)$, which determines the characteristic of the nonlinear system (1), if a function $p(x)$, $x \in [0, 1]$ satisfies the following Frobenius-Perron equation

$$p(y) = \sum_{x \in f^{-1}(y)} \frac{p(x)}{|f'(x)|}, \quad f'(x) = \frac{df(x)}{dx} \quad (2)$$

and satisfies the following normalized condition

$$\int_I p(x) dx = 1,$$

then the function $p(x)$ is called the invariant density of the nonlinear function $f(x)$. Important statistical information of chaotic behavior of the system (1) are included in the invariant density $p(x)$. For example, for any subset $A \subset R^1$, the equation

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(x_k) = \int_I \chi_A(x) p(x) dx \quad (3)$$

holds, where

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

The equation (3) indicates that the invariant density gives the frequency that solution processes $\{x_n, n = 0, 1, 2, \dots\}$ visit a set A . Namely, the invariant density of chaotic system is equivalent to the stationary probability density function of stochastic systems. For example, it is well-known that Tent Map has invariant density with uniform distribution [5]. Nonlinear systems described by nonlinear map with invariant density can be used as pseudorandom number generators [9, 13, 14]. Recently, there exists a trial to utilize chaotic sequences, generated by one-dimensional piecewise linear maps with uniform invariant density, for built-in self-test of LSI [16]. As mentioned above, theoretical studies and applications of chaos are significant from both scientific and engineering viewpoints and there exist many researches relevant to chaos [15~19]. The invariant density of deterministic chaotic systems is extremely interesting and is the first step of research on statistical aspect of chaos. Therefore, studies on the invariant density of simple dynamical systems are important in theory and in practice. In previous work, a class of chaotic systems, whose invariant density is one step type piecewise uniform distribution, was demonstrated [20]. In this investigation, a class of nonlinear discrete system with the invariant density, which is multi step type piece-wise uniform, is newly introduced.

2 System Model

2.1 Type 1

The mathematical model of the first type nonlinear system is given by the following nonlinear difference equation described by a piecewise linear function $f_{abc}(x)$:

$$x_{n+1} = f_{abc}(x_n), \quad 0 < a < b < c < 1, \quad (4)$$

$$f_{abc}(x) = \begin{cases} \frac{b-a}{a}x + a, & 0 \leq x < a \\ \frac{c-b}{b-a}x + \frac{b^2-ac}{b-a}, & a \leq x < b \\ \frac{1-c}{c-b}x + \frac{c^2-b}{c-b}, & b \leq x < c \\ \frac{c}{1-c}x - \frac{c^2}{1-c}, & c \leq x \leq 1 \end{cases}$$

The invariant density of the nonlinear function $f_{abc}(x)$ is given by

$$p_{abc}(x) = \begin{cases} \frac{1}{a+b+2c}, & 0 \leq x < a \\ \frac{b}{(a+b+2c)(b-a)}, & a \leq x < b \\ \frac{c}{(a+b+2c)(c-b)}, & b \leq x < c \\ \frac{c}{(a+b+2c)(1-c)}, & c \leq x \leq 1 \end{cases} \quad (5)$$

Figure 1 shows the shape of the nonlinear function $f_{abc}(x)$ and Figure 2 shows the shape of the invariant density $p_{abc}(x)$ of the nonlinear function $f_{abc}(x)$. As shown in Fig.2, the invariant density $p_{abc}(x)$ is uniform on intervals $[0, a]$, $[a, b]$, $[b, c]$ and $[c, 1]$ respectively,

Fig.1 Nonlinear function $f_{abc}(x)$ (Type 1)

Fig.2 Invariant density $p_{abc}(x)$ of type 1,2 and 3 nonlinear functions

where $[0, a] = \{x : 0 \leq x \leq a\}$ and $]a, b[= \{x : a < x < b\}$, and so on.

Now, the method for constructing the nonlinear function $f_{abc}(x)$ is explained:

- (i) First, points a, b and c are freely selected. (See Fig.3(a).)
- (ii) Then, dotted lines with following terminal points are depicted: (See Fig.3(b).)
 - terminal points $(a, 0)$ and $(a, 1)$
 - terminal points $(b, 0)$ and $(b, 1)$
 - terminal points $(c, 0)$ and $(c, 1)$
 - terminal points $(0, a)$ and $(1, a)$
 - terminal points $(0, b)$ and $(1, b)$
 - terminal points $(0, c)$ and $(1, c)$
- (iii) Finally,
 - segment with terminal points $(0, a)$ and (a, b)
 - segment with terminal points (a, b) and (b, c)
 - segment with terminal points (b, c) and $(c, 1)$

Fig.3 Method for constructing nonlinear function $f_{abc}(x)$

segment with terminal points $(c, 0)$ and $(1, c)$ are depicted. (See Fig.3(c))

The method presented in this subsection is easily extended to the case where the number of points in step 1 increase.

2.2 Type 2

The mathematical model of the 2nd type nonlinear system is given by the following nonlinear difference equation:

$$x_{n+1} = f_{abc}(x_n), \quad 0 < a < b < c < 1, \quad (6)$$

$$f_{abc}(x) = \begin{cases} \frac{a-b}{a}x + b, & 0 \leq x < a \\ \frac{b-a}{c-1}x + \frac{b(c-a)}{c-1}, & a \leq x < b \\ \frac{c-1}{c-b}x + \frac{c(1-b)}{c-b}, & b \leq x < c \\ \frac{-c}{1-c}x + \frac{c}{1-c}, & c \leq x \leq 1 \end{cases}$$

The invariant density of the nonlinear function $f_{abc}(x)$ is given by the Eq.(5). Figure 4 shows the shape of the nonlinear function $f_{abc}(x)$.

Fig.4 Nonlinear function $f_{abc}(x)$ (Type 2)

2.3 Type 3

The mathematical model of the 3rd type nonlinear system is given by the following nonlinear difference equation

$$x_{n+1} = f_{abc-q}(x_n), \quad 0 < a < b < q < c < 1, \quad (7)$$

$$f_{abc-q}(x) = \begin{cases} \frac{b-a}{a}x + a, & 0 \leq x < a \\ \frac{c-b}{q-b}x + \frac{b^2-ac}{q-b}, & a \leq x < b \\ \frac{b-a}{1-c}x + \frac{cq-b}{q-b}, & b \leq x < q \\ \frac{c-1}{c-q}x + \frac{c(1-q)}{c-q}, & q \leq x < c \\ \frac{-c}{1-c}x + \frac{c}{1-c}, & c \leq x \leq 1 \end{cases}$$

The invariant density of the nonlinear function $f_{abc-q}(x)$ is given by the Eq.(5). Figure 5 shows the shape of the nonlinear function $f_{abc-q}(x)$.

Fig.5 Nonlinear function $f_{abc-q}(x)$ (Type 3)

4 Examples and Numerical Studies

In order to show the validity of results demonstrated in previous chapter, three illustrative examples are demonstrated here.

4.1 Example 1: Type 1

First, we consider the first type nonlinear system with parameters $a=0.1$, $b=0.3$ and $c=0.7$. In this case, the difference equation and the piecewise linear function $f_{abc}(x)$ is given by

$$x_{n+1} = f_{abc}(x_n), \quad (8)$$

$$f_{abc}(x) = \begin{cases} 2x+0.1, & 0 \leq x < 0.1 \\ 2x+0.1, & 0.1 \leq x < 0.3 \\ \frac{3}{4}x + \frac{19}{40}, & 0.3 \leq x < 0.7 \\ \frac{7}{3}x - \frac{49}{30}, & 0.7 \leq x \leq 1 \end{cases}$$

Figure 6 shows the shape of the nonlinear function $f_{abc}(x)$. The behavior of chaotic solution process with an initial condition $x_0=0.01$ is presented in Fig.7.

Fig.6 Nonlinear function $f_{abc}(x)$ of example 1 (Type 1)

Fig.7 The chaotic solution process of a type 1 system with $x_0=0.01$

The invariant density of the nonlinear function $f_{abc}(x)$ is given by

$$p_{abc}(x) = \begin{cases} \frac{5}{9}, & 0 \leq x < 0.1 \\ \frac{5}{6}, & 0.1 < x < 0.3 \\ \frac{35}{36}, & 0.3 < x < 0.7 \\ \frac{35}{27}, & 0.7 < x \leq 1 \end{cases} \quad (9)$$

The shape of the invariant density of the nonlinear function $f_{abc}(x)$ is plotted in Fig.8. A histogram of a solution process to (8) is plotted in Fig.9, where an initial condition is selected as $x_0=0.01$ and 1,000,000 numbers of solutions $\{x_n, n=0,1,\dots, 1,000,000\}$ are calculated by using the digital computer. As shown in Fig.9, the histogram shows a good fitting to theoretical one described by Eq.(9).

Fig.8 The invariant density $p_{abc}(x)$ of the nonlinear function $f_{abc}(x)$

Fig.9 A histogram of a solution process x_n of (8)

4.2 Example 2: Type 2

Secondly, we consider the 2nd type nonlinear system with $a=0.1$, $b=0.3$ and $c=0.7$. In this case, the difference equation and the piecewise linear function $f_{abc}(x)$ is given by

$$x_{n+1} = f_{abc}(x_n), \quad (10)$$

$$f_{abc}(x) = \begin{cases} -2x+0.3, & 0 \leq x \leq 0.1 \\ -2x+0.9, & 0.1 < x \leq 0.3 \\ -\frac{3}{4}x + \frac{49}{40}, & 0.3 < x \leq 0.7 \\ -\frac{7}{3}x + \frac{7}{3}, & 0.7 < x \leq 1 \end{cases}$$

Figure 10 shows the shape of the nonlinear function $f_{abc}(x)$. The behavior of chaotic solution process with an initial condition $x_0=0.01$ is presented in Fig.11. The invariant density of the nonlinear function $f_{abc}(x)$ is same as that of example 1.

Fig.10 Nonlinear function $f_{abc}(x)$ of example 2 (Type 2)

Fig.11 A chaotic solution process of a type 2 system (10) with $x_0 = 0.01$

A histogram of a solution process to (10) is plotted in Fig.11, where an initial condition is selected as $x_0 = 0.01$. As shown in Fig.12, the histogram shows a good fitting to theoretical one described by Eq.(9).

Fig.12 A histogram of a solution process x_n of (10)

4.3 Example 3: Type 3

In this subsection, we consider the 3rd type nonlinear system with $a=0.1$, $b=0.3$, $c=0.7$ and $q=0.5$. In this case, the difference equation and the piecewise linear function $f_{abc_q}(x)$ is given by

$$x_{n+1} = f_{abc_q}(x_n), \quad (11)$$

$$f_{abc_q}(x) = \begin{cases} 2x+0.1, & 0 \leq x < 0.1 \\ 2x+0.1, & 0.1 \leq x < 0.3 \\ \frac{3}{2}x + \frac{1}{4}, & 0.3 \leq x < 0.5 \\ -\frac{3}{2}x + \frac{7}{4}, & 0.5 \leq x < 0.7 \\ -\frac{3}{7}x + \frac{3}{7}, & 0.7 \leq x \leq 1 \end{cases}$$

Figure 13 shows the shape of the nonlinear function $f_{abc}(x)$. The behavior of chaotic solution process with an initial condition $x_0 = 0.01$ is presented in Fig.14. The invariant density of the nonlinear function $f_{abc_q}(x)$ is same as that of example 1.

Fig.13 Nonlinear function $f_{abc_q}(x)$ of example 3 (Type 3)

Fig.14 A chaotic solution process of a type 3 system (11) with $x_0 = 0.1$

Fig.15 A histogram of a solution process x_n of (11)

A histogram of a solution process to (11) is plotted in Fig.15, where an initial condition is selected as $x_0=0.01$. As shown in Fig.15, the histogram shows a good fitting to theoretical one described by Eq.(9).

These three examples show the validity of the results demonstrated in this investigation.

5 Conclusions

In this paper, chaotic behavior, exhibited by the one-dimensional difference equation, has been discussed. Some types of nonlinear discrete systems with the invariant density, which is multi step type piece-wise uniform, have been introduced. Numerical studies show the validity of this study.

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