On the Pricing of an Exotic Warrant

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Abstract
The present paper is a survey on our studies on pricing of an exotic warrant. The problem becomes a double stopping problem and is solved numerically by constructing "binary forest".

1 Introduction
We consider pricing of some warrants which have exotic features. Usually, a warrant is a derivative that entitles the holder to buy the stock of a company that issued it at a specified price. In our case, however, the exercise price is not fixed. Instead, fixed is the total amount to pay: we let the amount \( G \) be fixed. The number of the stock holdings will then change and there are some options on the percentage of the stock holdings. If the percentage is large enough, then she/he can liquidate the company at will. Our warrant agreement keeps the buyer from selling the stocks until some maturity date, and hence the right to liquidate before the firm value reaches zero (we regard this event as bankruptcy) becomes important. A detailed/precise description will be given in section 2.

The pricing of this exotic warrant is mathematically an optimal double stopping problem since we must choose optimal times both to get the stocks and to liquidate the company if necessary.

After working in a general setting in section 3 discussing a mathematical framework to deal with the pricing problem, we will then establish a double Bellman principle (Theorem 4.1) and apply it to our pricing in a binary setting. In doing this, we need to construct "trees on a tree", which we call binary forest. This procedure is computationally tractable and some numerical results are presented in section 6.

2 The Description of the Exotic Warrant
Here is a precise description of our exotic warrant:

(I) Total amount to pay is fixed to \( C_t \) which depends on the time \( t \).

(II) The holder must exercise the right definitely until its pre-agreed maturity \( T_1 \).

(III) If the share of the stock goes beyond \( \beta \) (ratio), the holder can liquidate the company at will.

(IV) After exercising the right, the holder is prevented from selling the stock until a pre-agreed date \( T_2 \).

(V) The warrant holder can purchase the stock by some discounted value. We may call it exercise price, which is dependent only on the market price of the stock, let say \( f(S_t) \) for some measurable \( f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \), where \( S_t \) is the price of stock at time \( t \).

(VI) The company goes bankrupt when its capital value plus the payment \( C_{t\wedge T} \) of the warrant holder first hits zero. Where \( \tau \) is the time when the buyer exercises the right.

3 The Model
For the moment we will be working on a general setting with a given filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+})\). Let \( A \) be a continuous positive adapted process which models "the total asset value" of the company in our problem. We assume that \( \mathbb{P} \) is the unique equivalent probability measure under which \( e^{-rt} A_t \) is a martingale. Here \( r \geq 0 \) is the risk-free interest rate. This means the asset value is supposed to be marketable, or in other words, hedgeable in the market.

The capital value \( E_t \) before the warrant is exercised is given by

\[
E_t = A_t - G_t
\]

where \( G_t \) stands for debt profile of the company, which we assume deterministic function of \( t \). For any function \( G \), there can be a term structure of debt. It can be constructed by a similar procedure as the ones in [2] and [3].

The capital value becomes

\[
E_t = A_t - G_t + C_{t\wedge T}
\]

after the exercise time \( \tau \). Note that by the condition (II), we have \( \tau \leq T_1 \).

The price of a unit stock, denoted by \( S \), is given by

\[
S_t = \frac{E_t}{X} 1_{(t < \tau)}.
\]
where $X$ is the total number of the stocks, and
\[ \nu := \inf \{ t > 0 : A_t - G_t + C_{\tau, \lambda t} = 0 \}, \]
the first hitting time at zero of $A_t - G_t + C_{\tau, \lambda t}$. This is compatible with the requirement (VI).

Next, let us consider the stock price after the warrant buyer exercised the right. By (V) of the extra features of the warrant, the number of the stocks obtained by the buyer is
\[ Y := \frac{C_r}{f(S_{\tau^-})}, \quad (3.1) \]
therefore the stock value after exercising becomes
\[ S_{\tau} = \frac{E_{\tau^-} + C_r}{X + Y} \cdot (3.2) \]
It should be noted that
\[ \frac{E_{\tau^-}}{X} = \frac{E_{\tau^-} + C_r}{X + Y} \]
if and only if
\[ f(S_{\tau^-}) = S_{\tau^-} \left(= \frac{E_{\tau^-}}{X} \right). \]
Note also that
\[ \frac{C_r}{f(S_{\tau^-})} = \frac{C_r}{f(A_t - G_t + C_{\tau, \lambda t})}. \]
Next, we consider the event of "intentional" liquidation. First of all, we notice that by (III) of the assignment,
\[ \frac{Y}{X + Y} \geq \beta \iff Y \geq \frac{\beta}{1 - \beta} X \quad (3.3) \]
is required to have the right to liquidate. Note that the condition (3.3) is equivalent to
\[ f \left( \frac{A_{\tau} - G_{\tau}}{X} \right) \leq C_r \left(1 - \beta \right) \frac{X}{1 - \beta} . \]
We denote the event by $B$, which is $\mathcal{F}_{\tau}$-measurable. Suppose that, on the event $B$, the holder liquidates the company at time $\sigma$. Then the holder receives the fraction $\frac{Y}{X + Y}$ of the reduced capital value $\alpha E_{\sigma}$, where $\alpha \in (0, 1)$ may be understood as the friction rate of the cost of the liquidation. Thus, the amount of the money received by the holder at the liquidation is
\[ D_{\sigma}^\gamma := 1_B \frac{\alpha Y E_{\sigma}}{X + Y} 1_{(\sigma < \tau)} = 1_B \alpha Y S_{\tau}. \]
At the event of $B^c$, we may suppose that $D_{\sigma}^\gamma = 0$ for convenience. Note that we have an expression of $D_{\sigma}^\gamma$ in terms of $A$:
\[ D_{\sigma}^\gamma = 1_{A_{\lambda t} \leq \xi} \frac{C_r}{f(A_{\lambda t} + C_{\tau, \lambda t})} \cdot \frac{1_{(\min_{\xi < \xi \leq (A_{\lambda t} - G_{\lambda t} + C_{\tau, \lambda t}) > 0})}}{X f\left( \frac{A_{\lambda t} - G_{\lambda t}}{X} \right) + C_r} \cdot \alpha \frac{C_r}{X f\left( \frac{A_{\lambda t} - G_{\lambda t}}{X} \right) + C_r} \]
Now we may consider $\sigma$ to be a stopping time which is greater than the stopping time $\tau$. More precisely, $(\tau, \sigma) \in A$ where
\[ A := \{(\tau, \sigma) : \text{pair of stopping times such that} \quad \tau \leq T_1 \leq \sigma \leq T_2 \}. \]
We may call the element of $A$ an admissible strategy.
For an admissible strategy $(\tau, \sigma)$, its cash flow is summarized as
- $-C_\tau$ at time $\tau$,
- $D_{\sigma}^\gamma$ at time $\sigma < T_2$,
- $YS_{T_2}$ at time $T_2$ if $\sigma = T_2$.
Note that even if $\nu < \tau$ the holder receives nothing after all though she/he pays $C_{\tau}$ and receives $Y$ stocks at time $\tau$ in our setting. Note also that at time $T_2 < \nu$
\[ D_{T_2}^\gamma < Y S_{T_2}. \]
Namely, the holder will not liquidate the company at $T_2 < \nu$.

Therefore the present value $J_0$ of this exotic warrant is given by
\[ J_0 = \mathbb{E}[-C_\tau e^{-rT} + 1_{(\sigma < T_2)} D_{\sigma}^\gamma e^{-r\sigma} + 1_{(\sigma = T_2)} Y S_{T_2} e^{-rT_2}]. \]
More generally, the present value at time $t$ is given by
\[ J_t = \mathbb{E}[-C_\tau e^{-rT} + 1_{(\sigma < T_2)} D_{\sigma}^\gamma e^{-r\sigma} + 1_{(\sigma = T_2)} Y S_{T_2} e^{-rT_2} | \mathcal{F}_t] \quad \text{if} \ t \leq \tau, \quad (3.4) \]
\[ = -C_\tau e^{-rT} + \mathbb{E}[1_{(\sigma < T_2)} D_{\sigma}^\gamma e^{-r\sigma} + 1_{(\sigma = T_2)} Y S_{T_2} e^{-rT_2} | \mathcal{F}_t] \quad \text{if} \ \tau < t \leq \sigma, \quad (3.5) \]
and
\[ = -C_\tau e^{-rT} + D_{\sigma}^\gamma e^{-r\sigma} \quad \text{if} \ \sigma < t. \quad (3.6) \]
By pricing of the exotic warrant, we mean calculating the value function:
\[ V_0 := \sup_{(\tau, \sigma) \in A} J_0(\tau, \sigma), \quad (3.7) \]
or more generally
\[ V_t := \sup_{(\tau, \sigma) \in A, t < \tau} J_t(\tau, \sigma). \]
Note that, this is the very definition of the pricing in this paper, where we omit a detailed argument about hedging. Note also that the framework we have given is also valid in discrete time frameworks including the one we are going to work with from section 5.
4 Bellman Principle under Markovian Setting

We assume that \( A \) is a Markov process and \( \mathcal{F} \) is its natural filtration. We may consider \( G_t = C_t \) is an absorbing boundary of the stopped process \( A^\tau := A_{\tau=\infty} \), which still is a Markov process. Then our problem becomes a double stopping problem under Markovian setting.

Let us consider the problem in a more general setting where \( M \) is a Markov process and \( \mathcal{G} \) is its natural filtration. Let \( \mathcal{F} \) be a bounded measurable function and put

\[
V_t := \operatorname*{ess sup}_{(r,\sigma)\in A^\tau, t \leq r} \mathbb{E}[F(r, \sigma, M_r, M_\sigma)|\mathcal{G}_t],
\]
and

\[
W^r_t := 1_{t \leq \tau} \operatorname*{ess sup}_{\sigma \leq r} \mathbb{E}[F(r, \sigma, M_r, M_\sigma)|\mathcal{G}_t].
\]

Note that \( V_t \) is \( \mathcal{G}_t \)-measurable and \( W \) is \( \mathcal{G}_\tau \)-measurable. We claim that a kind of Bellman principle holds in this situation:

**Theorem 4.1** (Akahori and Akasaka [1]). Under appropriate conditions, we have

\[
V_t = \operatorname*{ess sup}_{(r,\sigma)\in A^\tau, t \leq \tau} \mathbb{E}[W^r_t|\mathcal{F}_t],
\]

For details, see [1].

Note that this is applicable to our pricing problem. In the next section, we will be working on a binary-tree framework, where \( V_0 \) in (3.7) is calculated using binary trees attached to each node of a binary tree. We may call this binary forest.

5 Binary Forest

The time parameter is now set to be discrete; we let \( t \in \mathbb{Z}_+ \).

Let \( A \) be defined as

\[
A_n(w_1, w_2, \ldots, w_n) = A_0 \prod_{j=1}^{n} w_j, \quad n \in \mathbb{N}, \tag{5.1}
\]

where \( w_i \in \{u, d\} \). Here \( u \) and \( d \) are constants such that \( d < u \) <.

We will be working on the unique equivalence martingale measure defined by the direct product of the risk neutral probability \( (p, q) \) assigned to \( (u, d) \), where

\[
p := \frac{e^r - d}{u - d}, \quad q = 1 - p.
\]

Under the measure, \( A \) is a Markov process, and so is the stopped process \( A^\tau \).

Denote the natural filtration of \( A \) and \( A^\tau \) by \( \mathcal{F} \) and \( \mathcal{F}^\tau \), respectively. Instead of considering the pairs of stopping times in \( A \), we may work on

\[
A^\tau := \{(s, t) \in A : s \text{ and } t \text{ are } \mathcal{F}^\tau \text{-stopping times}\}.
\]

To give a more concrete result, we set \( C_t = C e^{rt} \) for some \( C \leq 0 \) and in this case (3.4)–(3.6) become

\[
J_t = \mathbb{E}[-C + 1_{\{s < T_2\}} D_\sigma^t e^{-r\sigma} + 1_{\{s = T_2\}} Y S_{T_2} e^{-rT_2}|\mathcal{F}_t] \quad \text{if } t \leq \tau, \tag{5.2}
\]

\[
= -C + \mathbb{E}[1_{\{s < T_2\}} D_\sigma^t e^{-r\sigma} + 1_{\{s = T_2\}} Y S_{T_2} e^{-rT_2}|\mathcal{F}_t] \quad \text{if } \tau < t \leq \sigma, \tag{5.3}
\]

and

\[
= -C + D_\sigma^t e^{-r\sigma} \quad \text{if } \sigma < t. \tag{5.4}
\]

**Lemma 5.1.** For \( t \geq 0 \) we have

\[
V_t = \sup_{(r,\sigma)\in A^\tau, t \leq \tau} J^r_t(r, \sigma),
\]

where \( J^r_t \) is the conditional expectation with respect to \( \mathcal{G}_t \) instead of \( J_t \) in \( \mathcal{F}_t \).

**Proof.** The image of the functional

\[
1_{\{s < T_2\}} D_\sigma^t e^{-r\sigma} + 1_{\{s = T_2\}} Y S_{T_2} e^{-rT_2}
\]

is invariant under the shift from \( A \) to \( A^\tau \) since \( D_\sigma^2 = ST_2 = 0 \) on the event \( \{\sigma < \tau\} \) because of \( T_\sigma = 0 \) on the event \( \{\sigma \leq \tau\} \) of \( A^\tau \), then \( \tau \) is trivially optimal. Hence,

\[
V_0 = \sup_{(r,\sigma)\in A^\tau} J^r_0(r, \sigma).
\]

In the same manner, we have

\[
V_t = \sup_{(r,\sigma)\in A^\tau, t \leq \tau} J^r_t(r, \sigma). \tag{5.5}
\]

Now we see that we can apply Theorem 4.1. For our case,

\[
F(t, s, x, y) = -C e^{-rt}
\]

\[
+ e^{-rT_2} 1_{\{(x - G_{T_2} y \leq S_{T_2} \leq x y + G_{T_2})\}} 1_{\{s < T_2\}} \frac{\alpha C_t (y - G_s + C_s)}{X f(x - G_{T_2})}
\]

\[
+ e^{-rT_2} 1_{\{s = T_2\}} \frac{C_t (y - G_s + C_s)}{X f(x - G_{T_2})}
\]

and \( M = A^\tau \).

In this discrete-time setting, we can use so-called backward induction method. In fact, we have inductively

\[
W^{T_2}_{T_2} = \max \left\{ \alpha_1 B S_{T_2} - C_N, Y^{T_2} S_{T_2} - C_N \right\} = Y^N S_{T_2}.
\]

where \( Y^N \) is the number of the stocks obtained by the buyer who exercised the right at time \( N \) and \( S^N \) is the
stock price exercised the right at time $N$. 
for $N \leq T_1$ and $N \leq \forall n < T_2,$

$$W^N_n(w_1, w_2, \ldots, w_n) = \max \left\{ D^N_n(w_1, w_2, \ldots, w_n),
\right.$$ 

$$e^{-r} \left( pW^N_{n+1}(w_1, w_2, \ldots, w_n, U)
+ qW^N_{n+1}(w_1, w_2, \ldots, w_n, D) \right) \right\}. \tag{5.6}$$

Further the value function $V$ can be calculated inductively

$$V_{T_1}(w_1, w_2, \ldots, w_n) = W^T_{T_1}(w_1, w_2, \ldots, w_M), \tag{5.7}$$

and

$$V_n(w_1, w_2, \ldots, w_n) = \max \left\{ W^N_n(w_1, w_2, \ldots, w_n),
\right.$$ 

$$e^{-r} \left( pV_{n+1}(w_1, w_2, \ldots, w_n, U)
+ qV_{n+1}(w_1, w_2, \ldots, w_n, D) \right) \right\}. \tag{5.8}$$

Thus we finally get the value $V_0$ and it is the value of 
this warrant.

To do these calculation,

1. We first construct a binary tree $T$ for $A$ (or $A^v$).
2. For each node of the tree, we construct a tree for 
computation of $W^N_n(w_1, \cdots, w_N)$ using backward induction: we construct a forest $T \times T^m$.
3. By the procedure (5.5) and (5.6), we obtain the 
value $W^N_n$ for each node of the original tree until 
time $T_1$.
4. Using the tree, we can calculate $V$ by the standard 
backward procedure.

6 Numerical Experiments

Below we present a numerical result of the above pro-

\begin{itemize}
  \item $T_1 = 20, T_2 = 200,$
  \item $A_0 = 1000, \, r = 0.02,$
  \item $u = e^{\sqrt{\Delta t}}, \, d = e^{-\sqrt{\Delta t}} = 0.3$, where $\Delta t = 1/400,$
  \item the debt profile is given by $G_t = 490 + 2.5t$,
  \item $X = 40, \, C = 200,$
  \item $\alpha = 0.95, \, \beta = 0.3,$ and
  \item the function $f$ is given by
\end{itemize}

$$f(x) = 1_{(x_1/10 \leq x_2/10 \leq x_3)} \frac{3}{4} S_0$$

$$+ 1_{(x_2/10 < x_3/10 < x_4)} \frac{9}{10} x_2/10$$

$$+ 1_{(x_4 < x_5/10)} 2S_0.$$

To see the contrasts, we also calculated the values for 
the derivatives 

\begin{itemize}
  \item that $\textit{should}$ take $\tau = 0$, where the value function is 
denoted by $V^1$,
  \item that $\textit{should}$ take $\sigma = T_2$, where $V''$ denotes the 
value function, and
  \item that $\textit{should}$ take $\tau = 0$ and $\sigma = T_2$, where the value 
function is denoted by $V'''$.
\end{itemize}

Note that

$$V'_n = W^0_n. \tag{6.1}$$

The figure 1 is the result for $V_0, V'_0, V''_0, \text{ and } V'''_0$ 
against $A_0$ between $[500, 2000]$. We see that as $A_0$ be-
comes smaller (and hence as the default probability in-
creases), $V_n - V''_n$ increases.

The figure 2 is the result for $V_0$ when $\sigma$ is in $[0.1, 0.5]$.

\(^1\)Here dashes never mean a derivative.
The figure 3 is the result of $V_0$ against $G$ between [100, 900].

The figure 4 is the result of $V_0$ and $V'_0$ against $X$ between [10, 90].

The figure 5 is the result of $V_0$, $V'_0$, $V''_0$, and $V'''_0$ against $C$ between [130, 290].

**Remark 6.1.** A detailed analysis of the results of the numerical experiments is given in [1].

**References**

