High and Low Type One-dimensional Function with Multi-Step Uniform Invariant Density

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Abstract

Nonlinear systems described by simple mathematical models often exhibit extremely complicated behavior called chaos. In this paper, chaotic behavior, exhibited by the one-dimensional difference equation, is investigated. A method for constructing nonlinear functions with the invariant density is newly demonstrated. Furthermore, the method for finding the invariant density of the nonlinear function is also presented. Nonlinear functions, treated in this study, are piecewise linear ones and the invariant density is piecewise uniform.

1 Introduction

In various research fields, it is well known that nonlinear systems with a simple mathematical model often exhibit extremely complicated behavior called chaos [1-12]. In this paper, chaotic behavior, exhibited by the following deterministic system described by the one-dimensional difference equation, is discussed:

$$\begin{aligned} x_{n+1} &= f(x_n), \quad n = 0, 1, 2, \dots, \\ f: I &\rightarrow I, \quad I = [0, 1]. \end{aligned} \quad (1)$$

Concerned with the nonlinear function $f(x)$, which determines the characteristic of the nonlinear system (1), if a function $p(x)$, $x \in [0, 1]$ satisfies the following Frobenius-Perron equation

$$p(y) = \sum_{x \in f^{-1}(y)} \frac{p(x)}{|f'(x)|}, \quad f'(x) = \frac{df(x)}{dx}, \quad (2)$$

then the function $p(x)$ is called the invariant density of the nonlinear function $f(x)$. Furthermore, if the invariant density $p(x)$ satisfies the following normalized condition

$$\int_I p(x) dx = 1,$$

then we especially call the function $p(x)$ as the invariant

probability density of the nonlinear function $f(x)$. Important statistical information of chaotic behavior of the system (1) are included in the invariant probability density $p(x)$. For example, for any subset $A \subset R^1$, the equation

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(x_k) = \int_I \chi_A(x) p(x) dx \quad (3)$$

holds, where

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

The equation (3) indicates that the invariant density gives the frequency that solution processes $\{x_n, n = 0, 1, 2, \dots\}$ visit a set A . Namely, the invariant density of chaotic system is equivalent to the stationary probability density function of stochastic systems. For example, it is well-known that Tent Map has invariant density with uniform distribution [5]. Nonlinear systems described by nonlinear map with invariant density can be used as pseudorandom number generators [9, 13, 14]. Recently, there exists a trial to utilize chaotic sequences, generated by one-dimensional piecewise linear maps with uniform invariant density, for built-in self-test of LSI [16]. As mentioned above, theoretical studies and applications of chaos are significant from both scientific and engineering viewpoints and there exist many researches relevant to chaos [15-19]. The invariant density of deterministic chaotic systems is extremely interesting and is the first step of research on statistical aspect of chaos. Therefore, studies on the invariant density of simple dynamical systems are important in theory and in practice. In previous work, a class of chaotic systems, whose invariant density is the piecewise uniform distribution, was demonstrated [20, 21]. In this investigation, a class of nonlinear discrete system with the invariant density, which is multi-step type piece-wise uniform, is newly introduced. Results, presented in this paper, are a generalization of previous investigations [20, 21], concerned with piecewise linear functions with the piecewise uniform invariant probability density.

2 Method for Constructing Piecewise Linear Function with Invariant Measure

In this section, we show the method for constructing piecewise linear type nonlinear function with the invariant density.

(Step 1) A interval $[0,1] = \{x : 0 \leq x \leq 1\}$ is divided into n intervals with dividing points $0 = p_0 < p_1 < \dots < p_n = 1$. The following intervals are defined as

$$\begin{aligned} I_1 &= [0, p_1[= [p_0, p_1[= \{x : 0 = p_0 \leq x < p_1\}, \\ I_2 &= [p_1, p_2[= \{x : p_1 \leq x < p_2\}, \\ &\vdots \\ I_{n-1} &= [p_{n-2}, p_{n-1}[= \{x : p_{n-2} \leq x < p_{n-1}\}, \\ I_n &= [p_{n-1}, 1] = [p_{n-1}, p_n] = \{x : p_{n-1} \leq x \leq p_n = 1\}. \end{aligned}$$

(See Fig.1(a))

(Step 2) As shown in Fig.1(b), we draw vertical lines through points $(p_i, 0)$ $i = 1, 2, \dots, n$ and horizontal lines through points $(0, p_i)$ $i = 1, 2, \dots, n$. In this step, many rectangles are created in the area $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

(Step 3) For convenience of explanations, in this paper, it is assumed that,

(A1) On each interval I_i included in x -axis, there exists only one segment, which construct piecewise linear function.

First, on the interval I_n , we draw a segment with terminal points (p_{n-1}, p_{n-1}) and $(p_n, 0)$. (See Fig.1(c))

(Step 4) On remained intervals I_i , $i = 1, 2, \dots, n-1$, we draw segments and we yield a piecewise linear function $f(x)$ according to following manners (See Fig.1(d)):

(C1) The nonlinear function $f(x)$ is a mapping from an interval I_i onto another interval I_j satisfying that

$$f(I_i) = I_j, \quad i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, n.$$

with $f(I_i) \neq I_i$, where $f(I_n) = \{f(x) : x \in I_n\}$. Each segment is a diagonal line of a rectangle, constructed in previous Step 2.

$$(C2) \quad f(I_n) = I_1 \cup I_2 \cup \dots \cup I_{n-1} \quad (4)$$

$$(C3) \quad \bigcup_{k=1}^{n-1} f^{-k}(I_n) = I_1 \cup I_2 \cup \dots \cup I_{n-1}, \quad (5)$$

where $f^{-1}(I_i) = \{x : f(x) = I_i\}$.

Fig.1 Steps constructing a piecewise linear function

(d) Step 4

Fig.1 Steps constructing a piecewise linear function (continued)

3 Method for Finding Invariant Density

In this section, the method for finding the invariant density $p(x)$ of the nonlinear function $f(x)$ constructed in Sec.2 is shown.

(Step 1) Assume that, a candidate of the invariant density $p(x)$ satisfies that

$$p(x) = 1, \quad x \in I_n. \quad (6)$$

(Step 2) Note that, from an assumption (A1) and from (Step 3) in Sec.2 and conditions (C1) to (C3), if $y \in I_n$, then $f^{-1}(y)$ is a point satisfying that

$$f^{-1}(y) \in f^{-1}(I_n).$$

Applying Frobenius-Perron equation (2) to any point $y \in I_n$ and using Eq.(6), we obtain

$$1 = p(y) = \sum_{x \in f^{-1}(y)} \frac{p(x)}{|f'(x)|} = \frac{p(x)}{\frac{L(I_n)}{L(f^{-1}(I_n))}}, \quad (7)$$

where $L(I_i)$ is the length of the interval I_i , namely,

$$L(I_i) = p_i - p_{i-1}.$$

(See Fig.1(d)) Remembering that $L(I_j)$ is the length of the interval I_j and referring Fig.2, it is easy to obtain that, if $y \in I_n$ and $y \in I_n$, then

$$|f'(x)| = \frac{L(I_n)}{L(f^{-1}(I_n))}.$$

From Eq.(7), we obtain

Fig.2 Finding process of the invariant density

$$p(x) = \frac{L(I_n)}{L(f^{-1}(I_n))}, \quad x \in f^{-1}(I_n). \quad (8)$$

(Step 3) Note that, under assumption (A1) and conditions (C1) to (C3), if $y \in f^{-1}(I_n)$, then

$$f^{-1}(y) = \{x_1, x_2 : x_1 \in f^{-2}(I_n), x_2 \in I_n\}$$

(See Fig.2) Applying Frobenius-Perron equation (2) to any point $y \in f^{-1}(I_n)$, we obtain

$$p(y) = \frac{p(x_1)}{\frac{L(f^{-1}(I_n))}{L(f^{-2}(I_n))}} + \frac{p(x_2)}{1 - L(I_n)}. \quad (9)$$

Noting that $y \in f^{-1}(I_n)$ and $x_2 \in I_n$, from Eqs.(6) and (8), we obtain that, in Eq.(9),

$$p(y) = \frac{L(I_n)}{L(f^{-1}(I_n))},$$

$$p(x_2) = 1.$$

Substituting these equation into Eq.(9), we obtain

$$\begin{aligned} p(x_1) &= \frac{L(f^{-1}(I_n))}{L(f^{-2}(I_n))} \left(\frac{L(I_n)}{L(f^{-1}(I_n))} - \frac{L(I_n)}{1 - L(I_n)} \right) \\ &= \frac{L(I_n)}{L(f^{-2}(I_n))} \left(\frac{1 - L(I_n) - L(f^{-1}(I_n))}{1 - L(I_n)} \right) \cdot (10) \\ &= \frac{L(I_n)}{1 - L(I_n)} \cdot \frac{1 - L(I_n) - L(f^{-1}(I_n))}{L(f^{-2}(I_n))} \end{aligned}$$

$$x_1 \in f^{-2}(I_n).$$

(Step 4) Similarly, if $y \in f^{-2}(I_n)$, then

$$f^{-1}(y) = \{x_1, x_2 : x_1 \in f^{-3}(I_n), x_2 \in I_n\}.$$

Hence, we obtain

$$p(y) = \frac{p(x_1)}{\frac{L(f^{-2}(I_n))}{L(f^{-3}(I_n))}} + \frac{p(x_2)}{\frac{1-L(I_n)}{L(I_n)}}. \quad (11)$$

From Eqs.(6) and (10), we obtain

$$p(x_1) = \frac{L(I_n)}{1-L(I_n)} \cdot \frac{1-L(I_n)-L(f^{-1}(I_n))-L(f^{-2}(I_n))}{L(f^{-3}(I_n))}, \quad x_1 \in f^{-3}(I_n). \quad (12)$$

Repeating above process, we obtain that, on any interval Γ , $k=2,3,\dots,n-1$, the invariant density is given by

$$p(x) = \frac{L(I_n)}{1-L(I_n)} \cdot \frac{1-\sum_{k=0}^{n-1} L(f^{-k}(I_n))}{L(f^{-n}(I_n))}. \quad (13)$$

The invariant density $p(x)$ is normalized by

$$p_{norm}(x) = \frac{p(x)}{\int p(x) dx}.$$

4 Examples

In order to show the validity of results demonstrated in previous chapter, three illustrative examples are demonstrated here.

4.1 Example 1

First, we show the nonlinear system, which is slightly modified example of previous one, presented in Ref.[21]. This system has three dividing points

$$p_1 = 0.1, \quad p_2 = 0.3 \quad \text{and} \quad p_3 = 0.7.$$

The difference equation and the piecewise linear function $f_1(x)$ is given by

$$x_{n+1} = f_1(x_n), \quad (14)$$

$$f_1(x) = \begin{cases} 2x + 0.1, & 0 \leq x < 0.1 \\ 2x + 0.1, & 0.1 \leq x < 0.3 \\ \frac{3}{4}x + \frac{19}{40}, & 0.3 \leq x < 0.7 \\ -\frac{7}{3}x + \frac{7}{3}, & 0.7 \leq x \leq 1 \end{cases}.$$

Fig.3 Nonlinear function $f_1(x)$ of example 1

Figure 3 shows the shape of the nonlinear function $f_1(x)$. The invariant probability density of the nonlinear function $f_1(x)$ is given by

$$\tilde{p}_1(x) = \begin{cases} \frac{5}{9}, & 0 \leq x < 0.1 \\ \frac{5}{6}, & 0.1 < x < 0.3 \\ \frac{35}{36}, & 0.3 < x < 0.7 \\ \frac{35}{27}, & 0.7 < x \leq 1 \end{cases}. \quad (15)$$

The shape of the invariant probability density $\tilde{p}_1(x)$ of the nonlinear function $f_1(x)$ and a solution process are plotted in Fig.4.

(b) The invariant density $\tilde{p}_1(x)$

(b) A solution process

Fig.4 Some characteristics of the system described by the nonlinear function $f_1(x)$

4.2 Example 2

Secondly, we consider the 2nd nonlinear system, which is also three dividing points $p_1 = 0.1$, $p_2 = 0.3$ and $p_3 = 0.7$. The difference equation and the piecewise linear function $f_2(x)$ is given by

$$x_{n+1} = f_2(x_n), \quad (16)$$

$$f_2(x) = \begin{cases} 4x + 0.3, & 0 \leq x < 0.1 \\ \frac{3}{2}x + \frac{11}{20}, & 0.1 \leq x < 0.3 \\ \frac{1}{2}x - \frac{1}{20}, & 0.3 \leq x < 0.7 \\ -\frac{7}{3}x + \frac{7}{3}, & 0.7 \leq x \leq 1 \end{cases}$$

Figure 5 shows the shape of the nonlinear function $f_2(x)$. The invariant probability density $\tilde{p}_2(x)$ of the nonlinear function $f_2(x)$ is given by

$$\tilde{p}_2(x) = \begin{cases} \frac{15}{23}, & 0 \leq x < 0.1 \\ \frac{105}{75}, & 0.1 < x < 0.3 \\ \frac{46}{92}, & 0.3 < x < 0.7 \\ \frac{35}{23}, & 0.7 < x \leq 1 \end{cases} \quad (17)$$

The shape of the invariant probability density $\tilde{p}_2(x)$ of the nonlinear function $f_2(x)$ is plotted in Fig.6.

Fig.5 Nonlinear function $f_2(x)$ of example 2

Fig.6 The invariant density $\tilde{p}_2(x)$ of the nonlinear function $f_2(x)$

4.3 Example 3

Final example is described by the following difference equation and the piecewise linear function $f_3(x)$

$$x_{n+1} = f_3(x_n), \quad (18)$$

$$f_3(x) = \begin{cases} 3x + 0.7, & 0 \leq x < 0.1 \\ 2x + 0.1, & 0.1 \leq x < 0.3 \\ \frac{1}{4}x - \frac{3}{40}, & 0.3 \leq x < 0.7 \\ -\frac{7}{3}x + \frac{7}{3}, & 0.7 \leq x \leq 1 \end{cases}$$

Figure 7 shows the shape of the nonlinear function $f_3(x)$.

Fig.7 Nonlinear function $f_3(x)$ of example 3

The invariant probability density $\tilde{p}_3(x)$ of the nonlinear function $f_3(x)$ is given by

$$\tilde{p}_3(x) = \begin{cases} \frac{5}{9}, & 0 \leq x < 0.1 \\ \frac{5}{6}, & 0.1 < x < 0.3 \\ \frac{35}{36}, & 0.3 < x < 0.7 \\ \frac{35}{27}, & 0.7 < x \leq 1 \end{cases} \quad (19)$$

The shape of the invariant probability density $\tilde{p}_3(x)$ of the nonlinear function $f_3(x)$ is plotted in Fig.8.

These three examples show the validity of the results demonstrated in this investigation.

Fig.8 The invariant density $\tilde{p}_3(x)$ of the nonlinear function $f_3(x)$

5 Conclusions

In this paper, chaotic behavior, exhibited by the one-dimensional difference equation, has been discussed. Methods for constructing nonlinear functions and finding the invariant density are demonstrated. Some nonlinear discrete systems with the invariant density, which is multi-step type piece-wise uniform, have been newly introduced. Numerical studies show the validity of this study. In previous results, the position of segments constructing the nonlinear function on intervals I_j , $i = 1, 2, \dots, n-1$ uniformly climb upward. Nonlinear functions, introduced in this studies, has up and down type shapes.

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