Some Simulation Results on the Computation of Delta of Path-Dependent Options Using a Discrete Version of Clark-Ocone Formula

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Abstract

In this paper, presented are some numerical computations of the hedging portfolio of plain, look-back and average options respectively, using a discrete version of Clark-Ocone formula proposed by the authors themselves in [1].

1 Introduction

1.1 A discrete version of Clark-Ocone formula

Let W be a Brownian motion and X be a Wiener functional¹. Then it holds that

$$X = E[X] + \int_0^T E[D_s X | \mathcal{F}_s] dW_s, \quad (1)$$

where D_s means the Malliavin derivative. The formula (1) is known as Clark-Ocone formula ([2],[5],[6]). In the context of mathematical finance, the formula gives an alternative description of the hedging portfolio in terms of Malliavin derivatives. However, explicit expressions of the Malliavin derivatives of a Wiener functional are not available in general, except for some special cases: see [7].

To overcome this difficulty, we have introduced a discrete version of (1), which is stated as follows. Define

$$\mathcal{G}_k^n := \sigma(\Delta W_1, \dots, \Delta W_k),$$

and for $F \in L^2(\mathcal{W}, \mathcal{G}_n^n, \nu)$, where $\mathcal{W}$ and ν denote the Wiener space and Wiener measure, respectively,

$$F^n := E[F | \mathcal{G}_n^n],$$

where $n = 2^L$ ($L \in \mathbb{N}$), $\Delta t := T/n$, $t_k := k \cdot \Delta t$, $k = 0, 1, \dots, n$ and $\Delta W_k = W_{t_k} - W_{t_{k-1}}$.

Remark 1.1. Let μ^n be a probability measure on $\mathbb{R}^n$ defined by

$$\mu^n(dx) := \frac{1}{(2\pi\Delta t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2\Delta t}} dx \quad (x \in \mathbb{R}^n).$$

¹For the definition of $\mathbb{D}^{1,2}$ in $\mathbb{D}^{1,2}$, see e.g. [4].

Then

$$L^2(\mathcal{W}, \mathcal{G}_n^n, \nu) \simeq L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n),$$

and

$$F \in \mathbb{D}^{1,2} \Rightarrow F^n \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n).$$

Here $\mathcal{W}^{1,2}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n)$ consists of those functions whose first order derivatives in the distribution sense are in $L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n)$, and its norm is given by

$$\|X\|_{\mathcal{W}^{1,2}}^2 = \|X\|_{L^2}^2 + \sum_l \|\partial_l X\|^2,$$

where $\partial_l F^n$ means the partial derivative in l -th variable in the distribution sense.

The following is established in [1].

Theorem 1.2 ([1]). *If $F \in \mathbb{D}^{1,2}$, then*

$$E\left[(F - E[F]) - \sum_{l=1}^n E[\partial_l F^n | \mathcal{G}_{l-1}^n] \Delta W_l\right]^2 \rightarrow 0 \quad (\Delta t \rightarrow 0). \quad (2)$$

The above theorem reads: we have a discrete version of Clark-Ocone formula as

$$\begin{aligned} F &\simeq E[F] + \sum_{l=1}^n E[\partial_l F^n | \mathcal{G}_{l-1}^n] \Delta W_l \\ &= E[F] + \int_0^T E[\partial_l F^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n})}(s) dW_s, \end{aligned}$$

which suggest the use of $E[\partial_l F^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n})}(s)$ as an approximation of $E[D_s F | \mathcal{F}_s]$ (where for a set A , $1_A(\cdot)$ is the indicator function). However, in most cases it is not easy to obtain F^n out of a given F . In such a case, we may use an explicit function $X^n \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n)$ as a substitute. We established an error estimate for X^n in [1]:

Theorem 1.3 ([1]). *For $X^n \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^n)$, we have*

$$\begin{aligned} E\left[(X^n - E[X^n]) - \int_0^T E[\partial_l X^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n})}(s) dW_s\right]^2 \\ \leq \Delta t \|X^n\|_{\mathcal{W}^{1,2}}^2. \end{aligned}$$

This means, a sequence $\{X^n\}$ with $E(F - X^n)^2 \sim O(n^{-1})$ will provide an efficient/tractable “order 1/2” computation scheme for $E[D_s F | \mathcal{F}_s]$ if we could compute $E[\partial_l X^n | \mathcal{G}_l^n]$ properly. In fact, we have

$$\begin{aligned} & E[\{\int_0^T (E[D_s F | \mathcal{F}_s] - E[\partial_l X^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n}]}(s)) dW_s\}^2]^{\frac{1}{2}} \\ & \leq \|F - X^n\| + |E[F] - E[X^n]| \\ & + \left\| X^n - E[X^n] - \int_0^T E[\partial_l X^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n}]}(s) dW_s \right\| \\ & \leq 2 \|F - X^n\| \\ & + \left\| X^n - E[X^n] - \int_0^T E[\partial_l X^n | \mathcal{G}_l^n] 1_{[\frac{l-1}{n}, \frac{l}{n}]}(s) dW_s \right\| \\ & \leq 2 \|F - X^n\| + \sqrt{\frac{T}{n}} \sim O(n^{-\frac{1}{2}}). \end{aligned}$$

In this paper we will construct X^n out of a given F as follows:

1. Let $\eta_n : \mathbb{R}^n \rightarrow C[0, T]$ be the embedding defined by $(x_1, \dots, x_n) \mapsto \{\sum_{k=0}^{\lfloor nt/T \rfloor} x_k + \frac{n}{T}(t - \frac{nt}{T})x_{\lfloor \frac{nt}{T} \rfloor + 1}\}$, where $C[0, T]$ denotes the space of piecewise linear paths and x_0 is constant.
2. We set $X^n = F \circ \eta_n$, where we regard F as a function of $C[0, T]$ in which the Wiener space is embedded.

(The error estimate of $E(F - X^n)^2$ is left behind.)

1.2 The computation of delta under Black-Scholes economy

In the present paper, we will compute the “delta” of some path-dependent options under the Black-Scholes assumptions using the above discrete Clark-Ocone scheme.

Assume that the price S of risky asset follows a geometric Brownian motion:

$$S_t = S_0 \exp(\sigma W_t - \frac{\sigma^2}{2}t), \quad t > 0, \quad (3)$$

where $S_0 > 0$ is a given initial data, σ is the volatility and W is a one dimensional Brownian motion under the risk neutral measure. Here we assume that the interest rate is zero for simplicity.

The pay-off of a path-dependent option written on the risky asset is described as a functional of S , which is also a functional of W since S itself is a functional of W . Let $X \in \mathbb{D}^{1,2}$ be the corresponding Wiener functional of a path-dependent option. The hedging strategy θ , or in other words, the “delta” (see(Hull)) is given by

$$\theta_s = \frac{E[D_s X | \mathcal{F}_s]}{\sigma S_s}$$

since $dS_s = \sigma S_s dW_s$, or more precisely

$$X = E[X] + \int_0^T \frac{E[D_s X | \mathcal{F}_s]}{\sigma S_s} dS_s.$$

In this paper we will deal with two typical examples of path-dependent options:

1. Look back option: its pay-off is given by $\max_{0 \leq u \leq T} S_u - S_T$. It is approximated by $X^n := \max_{0 \leq k \leq n} e^{\sigma \sum_{i=1}^k \Delta W_i - \frac{\sigma^2}{2n} kT} - e^{\sigma \sum_{i=1}^n \Delta W_i - \frac{\sigma^2}{2} T}$. Since we know a closed form delta formula for this option, we can consider the result as a benchmark.
2. Asian option (average option): the pay-off is given by $(\frac{1}{T} \int_0^T S_s ds - K)^+$ (where $A^+ = \max(A, 0)$), where K is a strike price. It is approximated by $(\frac{S_0}{T} \{\int_0^{t_1} e^{\sigma \frac{n}{T} s \Delta W_1 - \frac{1}{2} \sigma^2 s} ds + \sum_{l=2}^n \int_{t_{l-1}}^{t_l} e^{\sum_{k=1}^{l-1} \sigma \Delta W_k + \sigma \frac{n}{T} (s - t_{l-1}) \Delta W_l - \frac{1}{2} \sigma^2 s} ds\} - K)^+$. If we used Yor’s formula, it would be possible to obtain a closed form of the delta, but we will not work on this since it is too complicated.

Other than these, as a benchmark of our framework, we will present simulation results for the plain call option, pay-off of which is $F = (S_T - K)^+$. In this case we do not have to approximate the pay-off since

$$F^n = E[F | \mathcal{G}_n^n] = (S_0 e^{\sigma \sum_{i=1}^n \Delta W_i - \frac{\sigma^2}{2} T} - K)^+ = F.$$

Remark 1.4. In [3], Leitz-Martini introduced another discrete version of (1) in the Bernoulli random walk setting. The random walk has a martingale representation property and therefore a perfect hedging strategy will be given by the discrete Clark-Ocone formula. On the other hand, embedding the Bernoulli setting to the standard continuous time framework is quite tedious. The priority of our methods lies in the tractability of the embedding, its error estimate, and specific computations.

2 Numerical Results

In this paper, we will give the simulation results of the approximating values of the hedging portfolio of the three path-dependent options together with the benchmark result of the plain call option, as described above.

2.1 The plain vanilla case: a benchmark

First, we will give the simulation results of the plain option, and draw a comparison between them and a theoretical value obtained by Black-Scholes formula (4).

$$\theta_0 := \text{BS hedging} = N(d), \quad (4)$$

where $d = \frac{\log \frac{S_0}{K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}$ (see[8]). Let S_0 denote the initial stock price and K , σ and T denote the strike, the volatility and the maturity respectively. We will consider plane call option with the initial stock price. We set $S_0 = 100$, strike $K = 120$, maturity $T = 0.5$ and volatility $\sigma = 0.3$. The theoretical value obtained by Black-Scholes formula (4) is 0.2256030.

On the other hand, the hedging portfolio of $X^{(n)}(\Delta W_1, \dots, \Delta W_n) = (S_0 \exp(\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n) - K)^+$ at time t_i obtained by Theorem 1.2 is the following:

$$\begin{aligned} \theta_{t_{i-1}}^{(n)} &:= E[\partial_i X^{(n)} | \mathcal{G}_{i-1}^n] / \sigma S_{t_{i-1}} \\ &= E[1_{[K, \infty)}(S_0 \exp(\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n)) \\ &\quad \sigma S_0 \exp(\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n) | \mathcal{G}_{i-1}^n] / \sigma S_{t_{i-1}}. \end{aligned} \quad (5)$$

In particular, if $i = 1$, then

$$\begin{aligned} \theta_0^{(n)} &= E[\partial_1 X^{(n)}] / \sigma S_0 \\ &= E[1_{[K, \infty)}(S_0 \exp(\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n)) \\ &\quad \exp(\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n)]. \end{aligned} \quad (6)$$

Though we can calculate (6) directly using Gauss kernel just as the theoretical value (4), we use a Monte Carlo method aiming that it gives, as a benchmark, information of the accuracy and the speed (and the mixture) of our framework. To accomplish the task, we generate n (the number of the time step discretization $\times N$ (the number of the sample paths for the Monte-Carlo simulation) Gaussian (quasi-)random numbers with mean 0 and variance $\frac{T}{n}$.

We set $N = 1000000$ and $n = 256$ in the simulation of Fig. 1. The values and the errors are indicated in Table 1. It is very accurate and they took just 43 seconds respectively. This accuracy is almost always attained.

value	error(%)
0.225844	0.106824821
0.225435	-0.074467095
0.22504	-0.249553419
0.225616	0.005762335
0.225743	0.062055912

Table 1: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}] / \sigma S_0$ in the case of the plain option.

2.2 Look-back option

As we have written, we use

$$\begin{aligned} X^{(n)}(\Delta W_1, \dots, \Delta W_n) &= S_0 \max(1, \max_{1 \leq l \leq n} e^{\sum_{k=1}^l \sigma \Delta W_k - \frac{1}{2} \sigma^2 t_l}) \\ &\quad - S_0 e^{\sum_{k=1}^n \sigma \Delta W_k - \frac{1}{2} \sigma^2 t_n}. \end{aligned}$$

Fig. 1: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}] / \sigma S_0$ in the case of the plain option.

A closed form of the delta is known in this case (see e.g.[8]) as

$$\begin{aligned} &\frac{\sigma \sqrt{T-t}}{\sqrt{2\pi}} \frac{Y(t)}{S(t)} e^{-\frac{\delta_-(T-t, \frac{S(t)}{Y(t)})^2}{2}} \\ &+ (\log \frac{S(t)}{Y(t)} + \frac{\sigma(T-t)}{2}) N(-\delta_-(T-t, \frac{Y(t)}{S(t)})) \\ &+ 2N(\delta_+(T-t, \frac{S(t)}{Y(t)})) - 1, \end{aligned} \quad (7)$$

where $S(t) = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$, $Y(t) = \max_{0 \leq u \leq t} S(u)$, $\delta_{\pm}(\tau, z) = \frac{1}{\sigma \sqrt{\tau}} (\log z \pm \frac{\sigma^2 \tau}{2})$. At $t=0$,

$$\begin{aligned} &\frac{\sigma \sqrt{T}}{\sqrt{2\pi}} e^{-\frac{\delta_-(T, 1)^2}{2}} + \frac{\sigma^2 T}{2} N(-\delta_-(T, 1)) \\ &+ 2N(\delta_+(T, 1)) - 1, \end{aligned} \quad (8)$$

where $\delta_{\pm}(T, 1) = \pm \frac{\sigma \sqrt{T}}{2}$.

We will calculate the price of a look back call option with the initial stock price $S_0 = 100$, maturity $T = 0.5$ and volatility $\sigma = 0.3$. The theoretical value obtained by (8) is 0.1808241.

On the other hand, the hedging portfolio of $X^{(n)}(\Delta W_1, \dots, \Delta W_n) = S_0 \max(1, \max_{1 \leq l \leq n} e^{\sum_{k=1}^l \sigma \Delta W_k - \frac{1}{2} \sigma^2 t_l}) - S_0 e^{\sum_{k=1}^n \sigma \Delta W_k - \frac{1}{2} \sigma^2 t_n}$ at time 0 obtained by our

value	error(%)
0.131884	-27.06503171
0.131886	-27.06392566
0.1316	-27.22209042
0.131784	-27.12033407
0.131715	-27.1584927

Table 2: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}]/\sigma S_0$ in the case of look back option.

DCO formula is the following:

$$\begin{aligned}\theta_0^{(n)} &:= E[\partial_1 X^{(n)}]/S_0\sigma \\ &= E[1_{[1,\infty)}(\max_{1 \leq l \leq n} e^{\sum_{k=1}^l \sigma \Delta W_k - \frac{1}{2}\sigma^2 t_l}) \\ &\quad \max_{1 \leq l \leq n} e^{\sum_{k=1}^l \sigma \Delta W_k - \frac{1}{2}\sigma^2 t_l} - e^{\sigma \sum_{k=1}^n \Delta W_k - \frac{\sigma^2}{2} t_n}]. \quad (9)\end{aligned}$$

We set $N = 1000000$ and $n = 256$ in the simulation of Fig. 2. It looks the simulation value converges to a wrong one. Equation (9) implies that if we increase the number of partitions, we will get a value closer to the theoretical value. How closer the values are (as the number increases) are indicated in Fig.3 and Table 3, by which we may conclude that the simulation with 131072 partitions is fine. We set $N = 1000$ and $n = 131072$ and the results of five simulations are given in Fig. 4 and Table 4. Each simulation took 26 seconds.

Fig. 2: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}]/\sigma S_0$ in the case of look back option.

Fig. 3: Simulation results of $\theta_1^{(n)} = E[\partial_1 X^{(n)}]/\sigma S_0$, $n = 512, 2048, 8192, 32768, 131072$, in the case of look back option.

partitions	value	error(%)
512	0.13698	-24.24682329
2048	0.166283	-8.041571892
8192	0.165227	-8.625564845
32768	0.173249	-4.189209292
131072	0.1792656	-0.86221914

Table 3: Simulation results of $\theta_1^{(n)} = E[\partial_1 X^{(n)}]/\sigma S_0$, $n = 512, 2048, 8192, 32768, 131072$, with $N = 1\text{million}$ in the case of look back option.

value	error(%)
0.178158	-1.474416297
0.181798	0.538589712
0.179054	-0.978907126
0.175995	-2.670606407
0.177141	-2.036841328

Table 4: Simulation results of $\theta_1^{(131072)} = E[\partial_1 X^{(131072)}]/\sigma S_0$ in the case of look back option.

Fig. 4: Simulation results of $\theta_1^{(131072)} = E[\partial_1 X^{(131072)}]/\sigma S_0$ in the case of look back option.

2.3 Asian option

As we have described, we use

$$\begin{aligned}
X^{(n)}(\Delta W_1, \dots, \Delta W_n) &= \left(\frac{S_0}{T} \left\{ \int_0^{t_1} e^{\sigma \frac{n}{T} s \Delta W_1 - \frac{1}{2} \sigma^2 s} ds \right. \right. \\
&+ \sum_{l=2}^n \int_{t_{l-1}}^{t_l} e^{\sum_{k=1}^{l-1} \sigma \Delta W_k + \sigma \frac{n}{T} (s-t_{l-1}) \Delta W_l - \frac{1}{2} \sigma^2 s} ds \Big\} - K \Big)^+ \\
&= \left(\frac{S_0}{T} \left\{ \int_0^{t_1} e^{\sigma \frac{n}{T} s \Delta W_1 - \frac{1}{2} \sigma^2 s} ds \right. \right. \\
&+ \sum_{l=2}^n e^{\sum_{k=1}^{l-1} \sigma \Delta W_k} \int_{t_{l-1}}^{t_l} e^{\sigma \frac{n}{T} (s-t_{l-1}) \Delta W_l - \frac{1}{2} \sigma^2 s} ds \Big\} - K \Big)^+.
\end{aligned}$$

We can calculate the differentiation in this case as

$$\begin{aligned}
\theta_1^{(n)} &= E[\partial_1 X^{(n)}]/\sigma S_0 \\
&= E[1_{[K, \infty)} \left(\frac{S_0}{T} \left\{ \int_0^{t_1} e^{\sigma \frac{n}{T} s \Delta W_1 - \frac{1}{2} \sigma^2 s} ds \right. \right. \\
&+ \sum_{l=2}^n e^{\sum_{k=1}^{l-1} \sigma \Delta W_k} \int_{t_{l-1}}^{t_l} e^{\sigma \frac{n}{T} (s-t_{l-1}) \Delta W_l - \frac{1}{2} \sigma^2 s} ds \Big\} \Big) \\
&\quad \times \frac{1}{T} \left\{ \int_0^{t_1} \frac{n}{T} s e^{\sigma \frac{n}{T} s \Delta W_1 - \frac{1}{2} \sigma^2 s} ds \right. \\
&+ \sum_{l=2}^n e^{\sum_{k=1}^{l-1} \sigma \Delta W_k} \int_{t_{l-1}}^{t_l} e^{\sigma \frac{n}{T} (s-t_{l-1}) \Delta W_l - \frac{1}{2} \sigma^2 s} ds \Big\}].
\end{aligned}$$

We will consider the price of an Asian call option with the initial stock price $S_0 = 100$, strike $K = 120$, maturity $T = 0.5$ and volatility $\sigma = 0.3$. We rely again on the Monte Carlo simulation as we have done for the previous cases. Fig.5 and Table 5 show how the simulated

partitions	value
16	0.076709
64	0.078467
256	0.079045
1024	0.078402
4096	0.078957

Table 5: Simulation results of $\theta_1^{(n)} = E[\partial_1 X^{(n)}]/\sigma S_0$, $n = 16, 64, 256, 1024, 4096$, with $N = 1\text{million}$

values converge as the number of partitions increases. They imply that a simulation with 64 partitions suffice. We set $N = 1000000$ and $n = 64$ in the simulations of Fig. 6 and Table 6. They took 199 seconds.

Fig. 5: Simulation results of $\theta_1^{(n)} = E[\partial_1 X^{(n)}]/\sigma S_0$, $n = 16, 64, 256, 1024, 4096$, with $N = 1\text{million}$

3 Conclusion

We have used a discrete version of Clark-Ocone formula as a tool for computing the delta, hedging portfolio, of path-dependent options under the Black-Scholes economy. The simulation results imply that

- For the look back case: the stability of Monte Carlo simulation is not bad, but the convergence in respect of time discretization does not seem nice.
- For the Asian: the stability of Monte Carlo simulation is not so good, but the convergence in respect of time discretization is not bad.

We have only considered the Geometric Brownian case. If the underlying price process is given by a solution

Fig. 6: the simulation results of $\theta_1^{(64)} = E[\partial_1 X^{(64)}]/\sigma S_0$, with $N = 1$ million

value
0.078259
0.078397
0.077716
0.078366
0.077904

Table 6: Simulation results of $\theta_1^{(64)} = E[\partial_1 X^{(64)}]/\sigma S_0$, with $N = 1$ million

Fig. 7: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}]/\sigma S_0$, with $N = 1$ million

value
0.078836
0.078730
0.078845
0.078774
0.078443

Table 7: Simulation results of $\theta_1^{(256)} = E[\partial_1 X^{(256)}]/\sigma S_0$, with $N = 1$ million

to a more complicated stochastic differential equation, we need another tractable scheme to obtain X^n . Also, some error estimated should be established.

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