Optimization of the Observation Gain Matrix for Stationary LQG Control Systems

Yoshiki TAKEUCHI
Dept. of Information Science, Osaka Univ. of Education
4-698-1 Asahigaoka, Kashiwara City, Osaka 582-8582, Japan
Tel. +81-729-78-3669, Fax. +81-729-78-3554
e-mail: takeuti@cc.osaka-kyoiku.ac.jp

Abstract

In this paper, we are concerned with the optimization of observations in stationary LQG stochastic control systems which employ the stationary Kalman filter. The performance of the LQG stochastic control system is dependent on the gain matrix in the linear observation. From the view point of the performance of the LQG regulator, it is better to take the dimension and the value of this gain matrix as large as possible. However, it is usually the case that we cannot take these values so large but there exist certain physical restrictions. By taking a performance criterion for the selection of the gain matrix as a quadratic function on the estimation error and the gain matrix and by introducing the eigenvalues-eigenvectors representation of a nonnegative definite symmetric matrix, the condition of optimality is derived under weaker assumptions than already known. Also, numerical calculations are easily carried out by introducing an $n$-dimensional polar coordinates system.

1 Introduction

In this paper, we consider an optimization problem of observations for the LQG optimal control systems. This type of problem is rather classical, and there were a number of studies [1]-[4] under an performance criterion related with the estimation errors. A typical criterion is a quadratic form on the estimation error and the observation gain matrix. Since, as it is well-known, the separation theorem holds for this class of systems, the optimal control input is a linear function of the filtered state estimate which is generated by the Kalman filter.

We have already considered the optimization of the observation for the LQG optimal control systems by applying the information theoretic criterion as follows [12]:

(i) Information theoretic optimization to maximize the mutual information between the signal and the observation subject to a power constraint concerned with the innovations process;

(ii) Optimization of the performance of the LQG regulator, i.e., minimization of performance criterion of the control.

In this paper, we apply the method of optimization of the observation for the stationary Kalman filter developed in [13] to the optimization of the observation for the LQG optimal control systems.

There does not exist much difference between the mathematical features of the problem discussed in this paper and that of [13]. However, a new proof of the condition of optimality is given in this paper under weaker assumptions than those in [13]. As we will see, the problem is formulated as an optimization problem with respect to variables which are components of an orthogonal matrix. In order to construct an easily calculable optimization algorithm, this problem is converted to one without constraint by introducing an $n$-dimensional polar coordinates system. Also, for this unconstrained problem, we can use the connection rule of the angular parameters [12], [13] which makes us possible to find an interior point with the same value as any exterior point near the boundary of the domain. A numerical example shows that we can easily get the solution.

Mathematical symbols, in this paper, are used in the following way. $R$ is the space of all real numbers, i.e., $R \triangleq (-\infty, \infty)$. For positive integers $m$ and $n$, $R^*$ and $R^{m \times n}$ denote the spaces of $n$-dimensional vectors and $m \times n$-dimensional matrices whose components take values in $R$. The prime denotes the transpose of a vector or a matrix and the Euclidean norm is $| \cdot |$. Thus, for $x \in R^*$, $|x| = \sqrt{x^t x}$. The identity matrix of any dimension is denoted by $I$. The components of a matrix are denoted by using subscripts. Thus, $[A]_{ij}$ is the $(i,j)$-component of $A$. In the case where no confusion may arise, we denote $[A]_0$ simply by $a_0$. If $A$ is a square matrix, $\det(A)$ and $\text{tr}(A)$ respectively denote the determinant and the trace of $A$. We use $A > 0$ and $A \geq 0$ to denote that $A$ is positive definite and nonnegative definite, respectively. For any pair of matrices $A$ and $B$, $A \otimes B$ denotes the Kronecker product of $A$ and $B$, and $\text{vec}(A)$ is the vector formed by stacking the columns of $A$ into a single column vector. The triplet $(\Omega, \mathcal{F}, P)$ is a complete probability space where $\Omega$ is a sample space with elementary events $\omega$, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$, and $P$ is a probability measure. $E\{\cdot\}$ denotes the expectation and $E\{\cdot|\mathcal{G}\}$, $\mathcal{G} \subset \mathcal{F}$ the conditional expectation, given $\mathcal{G}$, with respect.
to $P$. $\sigma\{\cdot\}$ is the minimal sub-$\sigma$-field of $\mathcal{F}$ with respect to which the family of $\mathcal{F}$-measurable sets or random variables $\{\cdot\}$ is measurable.

2 Problem Formulation

2.1 Stationary Optimal LQG Regulator System

Let $x = \{x_i(\omega); t = 0, 1, \cdots\}$ denote an $n$-dimensional Gaussian stochastic process described by

$$x_{t+1}(\omega) = A x_t(\omega) + C u_t(\omega) + G w_t(\omega),$$

where $A \in \mathbb{R}^{nxn}$, $C \in \mathbb{R}^{nxr}$, $G \in \mathbb{R}^{rxd}$, $x^0(\omega)$ is a Gaussian random vector with mean $\hat{x}$ and covariance $Q^0$, $u = \{u(t); t = 0, 1, \cdots\}$ is an $r$-dimensional control input, and $w = \{w_t(\omega); t = 0, 1, \cdots\}$ is a $d_w$-dimensional white Gaussian noise sequence. Suppose that the value of $x$ is not available but we have $m$-dimensional observations described by

$$y_i(\omega) = H x_i(\omega) + R v_i(\omega),$$

where $y = \{y_i(\omega); t = 1, 2, \cdots\}$ is an $m$-dimensional observation process, $H \in \mathbb{R}^{mxm}$, $R \in \mathbb{R}^{mxr}$, and $v = \{v_t(\omega); t = 1, 2, \cdots\}$ is a $d_v$-dimensional standard white Gaussian noise sequence. We will assume the following two conditions throughout this paper.

(C1) $R \triangleq R'R > 0$.

(C2) $x^0(\omega), w$ and $v$ are mutually independent.

It is well-known that the least-squares estimate $\hat{x}_{ij}(\omega) \triangleq E\{x_i(\omega)|\mathcal{F}_t\}$ of $x_i(\omega)$ based on $\mathcal{F}_t \triangleq \sigma\{y_i(\omega); s = 1, 2, \cdots, t\}$ is given by the Kalman filter:

$$\hat{x}_{ij}(t|\omega) = A \hat{x}_{i-1}(t|\omega) + C u(t - 1)$$

and

$$\hat{x}_{ij}(t|\omega) = \hat{x}_{ij}(t - 1|\omega) + Q' H' (HQ' + R)'^{-1} Q' + R_{ij}(\omega),$$

where

$$Q' \triangleq E\{[x_i(\omega) - \hat{x}_{ij}(\omega)][x_i(\omega) - \hat{x}_{ij}(\omega)]'\},$$

$$Q \triangleq E\{[x_i(\omega) - \hat{x}_{ij}(\omega)][x_i(\omega) - \hat{x}_{ij}(\omega)]\}.$$ (5)

Also, $\tilde{y}_i(\omega) = \{\tilde{y}_i(\omega); t = 1, 2, \cdots\}$ in (3) is the innovations process:

$$\tilde{y}_i(\omega) = y_i(\omega) - H \hat{x}_{ij}(\omega),$$

$$= H \{x_i(\omega) - \hat{x}_{ij}(\omega)\} + R v_i(\omega).$$ (8)

The stationary optimal input $u = \{u(t); t = 0, 1, \cdots\}$ is determined based on the well-known solution of the LQG regulator problem with the performance criterion:

$$\mathcal{J} \triangleq \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sum_{t=1}^{T} [x_i(\omega) M x_i(\omega) + u'(t - 1) N u(t - 1)] \right\},$$ (9)

where $M \in \mathbb{R}^{nxn}$ and $N \in \mathbb{R}^{rxd}$ are, respectively, non-negative definite and positive definite symmetric matrices. As it is well-known, for (1), (2) and (9), the optimal control is given by

$$u^*(t) \triangleq -(C' W C + N)^{-1} C' W A \tilde{y}_i(\omega),$$

where $W \in \mathbb{R}^{nxn}$ is given by the positive definite solution of the matrix Riccati equation:

$$\begin{align*}
W &= A' Y A + M \\
Y &= W - W C' (C' W C + N)^{-1} C'
end{align*}$$

From (9)-(11), the minimal value of $\mathcal{J}$ is given by

$$\mathcal{J} = tr[A'(W - Y) AQ] + tr[W GG']$$

2.2 Optimization of the Observations

From the view point of the LQG regulator performance, the observation is better when we have a smaller value of $\mathcal{J}$. In fact, $\mathcal{J}$ is dependent on $H$ because $Q$ in the first term in the right-hand side of (12) is dependent on $H$ by (4). Since we must take $H$ larger in order to decrease $Q$ and the second term in $\mathcal{J}$ is independent of $H$, it may be reasonable to take the performance criterion for $H \in \mathbb{R}^{mxm}$ as the quadratic form:

$$\hat{J} \triangleq tr[A'(W - Y) AQ] + tr[HN'H'],$$

where $N \in \mathbb{R}^{nxn}$ is a positive definite symmetric matrix. Here, the second term, for example, denotes the cost of the observation.

[Problem 1] Find $H \in \mathbb{R}^{mxm}$ such that (13) is minimized subject to (4).

The formulation of Problem 1 based on (13) as the optimization problem with respect to the observation gain matrix $H$ is well-known classical one, and a number of literatures have been devoted to the problem in this formulation [1]-[4]. The main difference between the present and those previous works is that we do not consider obtaining the optimal condition for $H$ itself but for the eigenvectors and eigenvalues of the nonnegative definite symmetric matrix $\tilde{N}^{1/2} H R_{ij}^{-1} H \tilde{N}^{1/2}$. In this approach, Theorem 1 below plays an important role.

First, let us denote

$$\tilde{N}^{1/2} H R_{ij}^{-1} H \tilde{N}^{1/2} = \tilde{U} \tilde{\Xi} \tilde{U}',$$

where

$$\tilde{\Xi} \triangleq \text{diag}(\xi_1, \xi_2, \cdots, \xi_m), \quad \xi_i \geq 0, i = 1, 2, \cdots, m,$$

$$\tilde{m} \triangleq \text{rank}(H) \leq m,$$

and $\tilde{U} = [u_1 u_2 \cdots u_\tilde{m}]$ is the set of eigenvectors of the
matrix given by the left-hand side of (14) corresponding to the positive eigenvalues \( \xi_i, i = 1, 2, \cdots, \hat{m} \). Note that we have \( \hat{U}^*\hat{U} = I \). Without loss of generality, we can assume that
\[
\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\hat{m}} > 0. 
\] (17)

Since (14) implies that
\[
H^\dagger R_0^{-1}H = \hat{N}^{-1/2}\hat{U}^*\hat{U}\hat{N}^{-1/2}, 
\] (18)
we have the following theorem which guarantees that we can take \( H \) in the form:
\[
H = R_0^{-1/2}\hat{U}^*\hat{U}\hat{N}^{-1/2}, 
\] (19)
where \( \hat{\Gamma} \) represents the first \( \hat{m} \) columns of an orthogonal matrix \( \Gamma \in R^{\hat{m}\times \hat{m}} \) such that \( \Gamma^\dagger = \Gamma\Gamma = I \).

«Theorem 1» Assume (C-1) and (C-2). Then, any \( H \in R^{\hat{m}\times \hat{m}} \) which satisfies (14) for a fixed set of values \( \hat{U} \), \( \hat{\Xi} \) and \( \hat{N} \) yields the same value of \( J \).

(Proof) We have the assertion immediately by noting the fact that the second relation of (4) can be rewritten as
\[
J = \hat{\Xi}^* = (Q^-)^{-1} + H^\dagger R_0^{-1}H. 
\] (20)

Thus, without loss of generality, we can take \( H \) as the form given by (19) which is an expression of \( H \) with property (14) and/or (18). Thus, the problem has been converted to the optimization with respect to \( \hat{\Gamma} \in R^{\hat{m}\times \hat{m}} \), \( \hat{U} \in R^{\hat{m}\times n} \) and \( \hat{\Xi} \equiv \text{diag}(\xi_1, \xi_2, \cdots, \xi_{\hat{m}}) \).

3 The condition of Optimality

As we see from (18), (19) and (20), the value of \( J \) is independent of that of \( \hat{\Gamma} \) and is determined by (18), (20) and the first part of (4). Hence, the optimal value of \( \hat{\Gamma} \) should be determined as in such a way that \( \text{tr}[HNH'] \), the second term in (13), is minimized.

«Theorem 2» Assume (C-1)-(C-2). Then the optimal value of \( \hat{\Gamma} \) given by the left-hand side of (14) corresponding to the eigenvectors of \( R_0 \) given by (22) is quite similar to the result of the optimization by an information theoretic criterion [12, Theorem 3.2] which has the following form.
\[
H = \hat{\Gamma}^{-\dagger/2}\hat{U}(Q^-)^{-1/2} 
\] (22)

In both cases, \( \hat{\Gamma} \) is the set of eigenvectors of \( R_0 \). The main difference between these two expressions of \( H \) exists in the points:

(i) \( \hat{N}^{-1/2} \) is a constant matrix in the present problem whereas it is replaced by \( (Q^-)^{-1/2} \) which is a variable to be determined in [12].

(ii) \( \hat{\Xi} \) is, here, determined to achieve the minimal value of \( \hat{J} \) given by (13) whereas it is determined by a Generalized Water Filling Theorem in [12], i.e., \( \hat{\Xi} = \alpha I - \hat{\Psi} \) for some positive constant \( \alpha \).

Now, we have converted Problem 1 to the following form.

[Problem 2] Find \( \hat{U} \in R^{\hat{m}\times \hat{n}} \) and \( \hat{\Xi} \equiv \text{diag}(\xi_1, \xi_2, \cdots, \xi_{\hat{m}}) \) such that
\[
\hat{J} = \text{tr}[A'(W - Y)AQ] + \text{tr}[\hat{\Psi}\hat{\Xi}] \to \min, 
\] (24)
subject to (4), (17) and
\[
\hat{U}^*\hat{U} = I. 
\] (25)

For Problem 2, let us define the Lagrangean by
\[
L(\hat{\Xi}, \hat{U}, \lambda) 
\] (26)
\[
\triangleq \text{tr}[A'(W - Y)AQ] + \text{tr}[\hat{\Psi}\hat{\Xi}] + \text{tr}[\lambda(\hat{U}^*\hat{U} - I)], 
\] (26)
where \( \lambda \in R^{\hat{m}\times \hat{m}} \) is a symmetric matrix whose (i, j)-component is a Lagrange multiplier for the (i, j)-component of (25), i.e.,
\[
\text{tr}[\lambda(\hat{U}^*\hat{U} - I)] = \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} \lambda_{ij} (\hat{U}^*\hat{U} - I)_{ij}, 
\] (27)

For Problem 2 and (26), we have the following result.

«Theorem 3 (Condition of Optimality)» Assume (C-1), (C-2) and
\[
\mathcal{M} \triangleq \{ (\hat{U}, \hat{\Xi}); \text{det}[F \otimes \hat{U} - I] \neq 0 \}. 
\] is not empty.

Then, the condition of optimality of \( \hat{U} \) and \( \hat{\Xi} \) is given by
\[
\hat{N}^{-1/2}QXQ\hat{N}^{-1/2} = \hat{U}\hat{\Xi}, 
\] (28)
where \( X \in R^{\hat{m}\times n} \) is a solution of...
\[ X = F'XF + A'(W - Y)A. \quad (29) \]

Since \( \Psi \) is a diagonal matrix given by (23), (28) implies that

« Corollary 1 » Assume (C-1)-(C-3). The optimal \((\bar{U}, \bar{Z})\) is such that

(i) Each column vector of \( \bar{U} \) is an eigenvector of \( \tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2} \) which is a symmetric matrix.

(ii) The order of column vectors in \( \bar{U} \) is the one that the corresponding eigenvalues are in ascending order.

(iii) The first \( m \) eigenvalues of \( \tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2} \) coincide with those of \( R_0 \).

[Remark 2] Clearly, the condition \( \text{det} \{ F \otimes F - I \} \neq 0 \) is equivalent to the one that no eigenvalue of \( F \otimes F \) is equal to 1. Let \( \mu_i, i=1,2,\ldots,n \) denote the eigenvalues of \( F = Q(Q')^{-1}A \). Then, the condition holds if and only if the following two conditions are fully satisfied.

(i) \( \mu_i \neq 1, i=1,2,\ldots,n \).

(ii) \( \mu_i \mu_j \neq 1, i < j, i, j=1,2,\ldots,n \). ■

[Remark 3] As we see, the set of optimal values \((Q, Q^*, \bar{U}, \bar{Z})\) is given by a solution of the set of equations (4), (19), (25), (28) and (29). Although it is not easy to get an analytical solution of these equations, these relations together with the properties given in Corollary 1 are applicable in constructing a recursive numerical algorithms.

4 Proofs of Theorems

In this section, we will give proofs of the theorems presented in the previous section.

(Proof of Theorem 2) By (19), it is seen that

\[ \text{tr} \left[ HNH' \right] = \text{tr} \left[ R_0^{1/2} \tilde{\Gamma} \tilde{\Gamma}' R_0^{1/2} \tilde{Z} \tilde{U} \tilde{N}^{-1/2} \tilde{N}^{-1/2} \tilde{N} \tilde{N}^{-1/2} \tilde{U} \tilde{Z} \tilde{\Gamma} \tilde{\Gamma}' \right] \]

\[ = \text{tr} \left[ R_0^{1/2} \tilde{\Gamma} \tilde{\Gamma}' R_0^{1/2} \right] = \text{tr} \left[ \tilde{\Gamma} R_0 \tilde{\Gamma} \right], \quad (30) \]

where we have used (25) and the relation \( \text{tr} \left[ X'Y \right] = \text{tr} \left[ XY \right] \). By (17) and the last expression in (30), we see that \( \text{tr} \left[ HNH' \right] \) is minimized when the diagonal components of \( \tilde{\Gamma} R_0 \tilde{\Gamma} \) are minimized and in ascending order. Since \( R_0 > 0 \), we can conclude that \( \tilde{\Gamma} \) is optimal when the diagonal components of \( \tilde{\Gamma} R_0 \tilde{\Gamma} \) are the first \( m \) eigenvalues of \( R_0 \) which satisfy (21). Hence, the column vectors of \( \tilde{\Gamma} \) are the corresponding eigenvectors. This completes the proof. ■

For the proof of Theorem 3, we need the following lemma.

«Lemma 1» Assume (C-1)-(C-3). Then, the solutions of the matrix Lyapunov equations:

\[ X = F'XF + Z, \quad (31) \]

and

\[ \hat{X} = F'\hat{X}F' + Z, \quad (32) \]

are given, respectively, by

\[ \text{vec}(X) = (I - F' \otimes F')^{-1} \text{vec}(Z), \quad (33) \]

and

\[ \text{vec}(\hat{X}) = (I - F \otimes F)^{-1} \text{vec}(Z). \quad (34) \]

(Proof) Noting the well-known relation \( \text{vec}(XYZ) = (X \otimes Z) \text{vec}(Y) \), (31) implies

\[ (I - F' \otimes F') \text{vec}(X) = \text{vec}(Z). \quad (35) \]

Thus, we have (33). We also have (34) from (32) extremely the same way. This completes the proof. ■

Now, let us proceed to the proof of Theorem 3.

(Proof of Theorem 3) Substituting (18) into (20), we have

\[ Q^{-1} = (Q^{-1}) + \tilde{N}^{-1/2} \tilde{U} \tilde{Z} \tilde{U} \tilde{N}^{-1/2}. \quad (36) \]

First, let us note that

\[ \frac{\partial Q}{\partial u_{ij}} = \frac{\partial (Q^{-1})^{-1}}{\partial u_{ij}} = -Q^{-1} \frac{\partial Q^{-1}}{\partial u_{ij}} Q. \quad (37) \]

Then, it follows from (36) and (37) that

\[ \frac{\partial Q}{\partial u_{ij}} = Q(Q^{-1}) \frac{\partial Q}{\partial u_{ij}} Q^{-1} Q \]

\[ -Q \tilde{N}^{-1/2} \{ E_{ij} \tilde{U} + U \tilde{Z} E_{ij} \} \tilde{N}^{-1/2} Q, \quad (38) \]

where

\[ E_{ij} \triangleq \begin{bmatrix} j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (39) \]

Note that from the first relation of (4), we have

\[ \frac{\partial Q}{\partial u_{ij}} = A \frac{\partial Q}{\partial u_{ij}} A'. \quad (40) \]

Substitution of (35) into (33) yields

\[ \frac{\partial Q}{\partial u_{ij}} = F \frac{\partial Q}{\partial u_{ij}} F' - Q \tilde{N}^{-1/2} \{ E_{ij} \tilde{U} + U \tilde{Z} E_{ij} \} \tilde{N}^{-1/2} Q. \quad (41) \]

Hence, by using (32) and (34) in Lemma 1, we have
For simplicity, let
\[ \tilde{M} \triangleq A'(W - Y)A, \]
and
\[ S \triangleq Q\tilde{N}^{-1/2} \{ E_{ij} \tilde{X}_U' + \tilde{U}_E E_{ij} \} \tilde{N}^{-1/2} Q, \]
Then, it follows that
\[
\frac{\partial}{\partial u_y} \text{tr}[\tilde{M}Q] = \text{vec}(\tilde{M})^\top \text{vec} \left( \frac{\partial Q}{\partial u_y} \right) = -\text{vec}(\tilde{M})(I - F \otimes F)^{-1} \text{vec}(S) \\
= -\left[ (I - F' \otimes F')^{-1} \text{vec}(\tilde{M}) \right]^\top \text{vec}(S) \\
= -\text{vec}(X)' \text{vec}(S) \\
= -\text{tr}[XS],
\]
where we have used the fact \((F \otimes F') = F' \otimes F\) in the third equality, and applied (31) and (33) with \( Z = M \) in the forth equality. Thus, we see that
\[
\frac{\partial}{\partial u_y} \text{tr}[\tilde{M}Q] = -\text{tr}[XS].
\]
Using (45) and (26), we have the condition
\[
\frac{\partial}{\partial u} \text{tr}[\tilde{M}Q] = -2\tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U} \tilde{\Xi}.
\]
and which implies
\[
\tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U} \tilde{\Xi} = \tilde{U} \tilde{\Lambda}.
\]
(47)
Here, note that we can take \( \tilde{\Lambda} \) as a diagonal matrix because \( \tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \) is symmetric and \( \tilde{\Xi} \) is diagonal [10; Lemma 4.2]. Similarly, for the derivatives with respect to \( \tilde{\Xi} \) are computed as
\[
\begin{align*}
\text{vec} \left( \frac{\partial Q}{\partial \tilde{\Xi}} \right) &= -(I - F \otimes F)^{-1} \\
\times \text{vec} \left( Q\tilde{N}^{-1/2} \{ E_{ij} \tilde{X}_U' + \tilde{U}_E E_{ij} \} \tilde{N}^{-1/2} Q \right), \\
\frac{\partial}{\partial \tilde{\Xi}} \text{tr}[\tilde{M}Q] &= -[\tilde{U}' \tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U}],
\end{align*}
\]
and
\[
\frac{\partial}{\partial \tilde{\Xi}} \text{tr}[\tilde{M}Q] = -[\tilde{U}' \tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U}],
\]
where (49) implies (50) because of (47) and the fact that \( \tilde{\Lambda} \) is diagonal. Now, using (50) and (51), we have the condition
\[
\frac{\partial}{\partial \tilde{\Xi}} \text{tr}[\tilde{M}Q] = -[\tilde{U}' \tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U}] = 0,
\]
and, hence, we have
\[
\tilde{U}' \tilde{N}^{-1/2} QXQ \tilde{N}^{-1/2} \tilde{U} = \tilde{\Psi}.
\]
(53)
By applying (53) and (25) to (47), we have
\[
\tilde{\Lambda} = \tilde{\Psi} = \tilde{\Xi},
\]
and which, together with (47), implies (28). This completes the proof.

\section{5 Representation of \( \tilde{U} \) by an \( n \)-dimensional Polar Coordinates System}

In this section, we convert the constrained problem given by (24) and (25) to an unconstrained one by introducing an \( n \)-dimensional polar coordinates system in \( \mathbb{R}^n \). Let us denote
\[
\tilde{U} = T(n) \left[ \begin{array}{cc} I_{n-\ell} & 0 \\
0 & T(n-\ell) \end{array} \right] \left[ \begin{array}{c} I_{\ell \times \ell} \\
0 \end{array} \right],
\]
where
\[
\ell = \min(\tilde{m} - 1, n - 2),
\]
and, for \( k = n, n-1, \cdots, n-\ell \),
\[
T(k) \triangleq T(k; \theta_{1,1}, \theta_{1,2}, \cdots, \theta_{k+1})
= [z_1 z_2 \cdots z_k],
\]
and
\[
\begin{bmatrix}
\prod \cos \theta_{1,1} \\
\sin \theta_{1,1} \prod \cos \theta_{1,1} \\
\vdots \\
\sin \theta_{1,1} \prod \cos \theta_{1,1} \\
\vdots \\
\sin \theta_{k-1,1} \cos \theta_{k-1,1} \\
\sin \theta_{k-1,1} \cos \theta_{k-1,1} \\
\sin \theta_{k-1,1} \cos \theta_{k-1,1} \\
\sin \theta_{k-1,1} \cos \theta_{k-1,1} \\
\cos \theta_{k-1,1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\ell = 2, 3, \cdots, k,
\end{bmatrix}
\]
and
0 \leq \theta_{i1} \leq 2\pi, \quad -\frac{\pi}{2} \leq \theta_{ik} \leq \frac{\pi}{2}, \quad i = 2, 3, \ldots, k - 1. \quad (59)

Then, it can be easily seen that \( T'(k)T(k) = T(k)T'(k) = I_{k \times k} \) and, hence, we have (25) for \( \bar{U} \) given by (55). The fact that all values in \( \mathbb{R}^{\text{new}} \) of \( \bar{U} \) with property (25) is given by (55)-(59) is shown in [12], [13].

Thus, we have converted Problem 2 with constraint (25) to the one with the unconstrained angular variables given by (59) for \( k = n, n - 1, \ldots, n - \bar{k} \).

6 A method of Connections of Angular Parameters at the Boundary of the Domain

For simplicity, let
\[
\Theta \leftarrow \left\{ \theta_{i1}, \theta_{ik}, i = 2, 3, \ldots, k - 1, k = n, n - 1, \ldots, n - \bar{k} \right\}. \quad (60)
\]

Clearly, \( \bar{U} = \bar{U}(\Theta) \) is a periodic function of \( \Theta \). Hence, for an exterior point of the domain given by (59), there always exists an interior point (of the domain) for which \( \bar{U} \) has the same value as the exterior point. If we can find such a pair of values of \( \Theta \), the implementation of the searching algorithm of finding the minimal point of \( \bar{U} \) becomes much simpler by replacing the exterior point generated by the algorithm with the corresponding interior point. This is because, in usual cases of optimization, we must stop searching when the algorithm generates an exterior point. In such a case, we usually take a nearest boundary point and memorize the value of the objective function at that point as a local minimal point. However, by using the replacement of the exterior point with the interior one, we can continue searching until we really reach a minimal point. By the following theorem, we show that it suffices to search over the set with \( 1/2^{r+1} \) size:
\[
0 \leq \theta_{i1} \leq \pi, \quad -\frac{\pi}{2} \leq \theta_{ik} \leq \frac{\pi}{2}, \quad i = 2, 3, \ldots, k - 1, k = n, n - 1, \ldots, n - \bar{k}, \quad (61)
\]
and obtain the rule of replacement of exterior points of (61) with the corresponding interior points.

«Theorem 4 (Connections of Angular Parameters)»[12]

For \( \bar{U} \) given by (55), we have the following rules for the case when only one parameter violates (61). For \( k = n, n - 1, \ldots, \bar{k} \), let
\[
\Theta_{n} \leftarrow \left\{ \theta_{11}, \theta_{12}, \ldots, \theta_{1k-1} \right\}, \quad q \in \left\{ n, n - 1, \ldots, n - \bar{k} \right\}, \quad (62)
\]
and assume that all \( \theta_{j}, k \neq q \) satisfy (61). Then,

(i) When \( \theta_{q1} \in [-\pi, 0) \) or \( \theta_{q1} \in (\pi, 2\pi] \) and \( \theta_{q1} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( i = 2, 3, \ldots, q - 1 \), the value of \( \bar{U} \) is equal to the one at the point by the following substitutions.

(a) \( q : \) odd,
\[
\theta_{q1} \rightarrow \theta_{q1} \pm \pi, \quad \theta_{q1} \rightarrow -\theta_{q1}, i = 2, 3, \ldots, q - 1,
\]

\[
\left\{ \begin{array}{l}
0_{(2j)(2j-1)} \rightarrow -0_{(2j)(2j-1)}, \\
0_{(2j)(2j-1)} \rightarrow \pi - 0_{(2j)(2j-1)}, \quad j = 2, 3, \ldots, (q - 1)/2,
\end{array} \right.
\]

\[
\theta_{31} \rightarrow \pi - \theta_{31},
\]

(b) \( q : \) even,
\[
\theta_{q1} \rightarrow \theta_{q1} \pm \pi, \quad \theta_{q1} \rightarrow -\theta_{q1}, i = 2, 3, \ldots, q - 1,
\]

\[
\left\{ \begin{array}{l}
0_{(2j)(2j-1)} \rightarrow -0_{(2j)(2j-1)}, \\
0_{(2j)(2j-1)} \rightarrow \pi - 0_{(2j)(2j-1)}, \quad j = 1, 2, \ldots, (q - 2)/2,
\end{array} \right.
\]

\[
\theta_{31} \rightarrow \pi - \theta_{31},
\]

(ii) When \( \theta_{q1} \in \left[ -\frac{\pi}{2}, -\frac{\pi}{4} \right) \) or \( \theta_{q1} \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right] \), \( \tau = 2, 3, \ldots, q - 2 \) and \( \theta_{q1} \in [0, \pi] \) and \( \theta_{q1} \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), \( i \neq \tau, i = 2, 3, \ldots, q - 1 \), the value of \( \bar{U} \) is equal to the one at the point by the following substitutions.

(a) \( 2\tau < q, \)
\[
\theta_{q1} \rightarrow \theta_{q1} \pm \pi, \quad \theta_{q1} \rightarrow -\theta_{q1}, i = \tau + 1, \tau + 2, \ldots, q - 1,
\]

\[
\left\{ \begin{array}{l}
0_{(q-2)(q-1)} \rightarrow -0_{(q-2)(q-1)}, \\
0_{(q-2)(q-1)} \rightarrow \pi - 0_{(q-2)(q-1)}, \quad j = 1, 2, \ldots, \tau - 1,
\end{array} \right.
\]

\[
\theta_{(q-2)(q-1)} \rightarrow \pi - 0_{(q-2)(q-1)} , \quad \theta_{(q-2)(q-1)} = -0_{(q-2)(q-1)},
\]

\[
i = 2, 3, \ldots, q - 2\tau + 1,
\]

(b) \( 2\tau \geq q, \)
\[
\theta_{q1} \rightarrow \theta_{q1} \pm \pi, \quad \theta_{q1} \rightarrow -\theta_{q1}, i = \tau + 1, \tau + 2, \ldots, q - 1,
\]

\[
\left\{ \begin{array}{l}
0_{(q-2)(q-1)} \rightarrow -0_{(q-2)(q-1)}, \\
0_{(q-2)(q-1)} \rightarrow \pi - 0_{(q-2)(q-1)}, \quad j = 1, 2, \ldots, \tau - 2,
\end{array} \right.
\]

\[
\theta_{(q-2)(q-1)} \rightarrow \pi - 0_{(q-2)(q-1)} , \quad \theta_{(q-2)(q-1)} = -0_{(q-2)(q-1)},
\]

\[
i = 2, 3, \ldots, q + 2\tau + 2.
\]

(Proof) The outline of the proof is given in [12].

7 A Numerical Example

In this section, we will give an illustrative example for a 3-dimensional LQG system for signal \( x \) and observations \( y \) (\( n = m = 3 \)).

Example 1. Let us consider a 3-dimensional system with
\[ A = \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \]
\[ C = \begin{bmatrix} 1.2 & 0.1 & 0.2 \\ 0.2 & 1.4 & 0.2 \\ 0.1 & 0.5 & 1.1 \end{bmatrix}, \]
\[ M = \text{diag}(40.0, 100.0, 55.0), \quad N = I, \]
\[ R_0 = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1.44 \end{bmatrix}, \]
\[ W = \begin{bmatrix} 0.0937 & 0.9097 & 0.0117 \\ 0.0256 & 0.1872 & 0.26248 \end{bmatrix}, \]
\[ Y = \begin{bmatrix} 0.7184 & -0.0857 & -0.1342 \\ -0.0857 & 0.6969 & -0.4178 \\ -0.1342 & -0.4178 & 0.9710 \end{bmatrix}. \]

For the above system and observations, we have
\[ \Psi = \text{diag}(0.49, 1.0, 1.44), \quad \Gamma = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1.44 \end{bmatrix}. \]

By taking \( \bar{N} = (1/150.0) I \), we made numerical computations for three cases: \( \bar{m} = 3 \), \( \bar{m} = 2 \) and \( \bar{m} = 1 \). We have carried out the optimization by a simple alternate search algorithm with respect to the angular parameters: \( \theta_{11}, \theta_{12} \) and \( \theta_{21} \), and \( \Xi \triangleq \text{diag}(\xi_1, \cdots, \xi_3) \) by making use of the connection rule shown in Section 5. The result are summarized in Table 1.

Table 1. The optimal values of \( \Theta = \{ \theta_{11}, \theta_{12}, \theta_{21} \} \), \( \Xi \), and \( \hat{J} \) for \( \bar{m} = 3, 2 \) and 1.

<table>
<thead>
<tr>
<th>( \bar{m} )</th>
<th>( \theta_{11} )</th>
<th>( \theta_{12} )</th>
<th>( \theta_{21} )</th>
<th>( \xi_1 )</th>
<th>( \xi_2 )</th>
<th>( \xi_3 )</th>
<th>( \hat{J} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.1019</td>
<td>0.6179</td>
<td>0.7854</td>
<td>0.8951</td>
<td>0.2162</td>
<td>0.0354</td>
<td>1.4873</td>
</tr>
<tr>
<td>2</td>
<td>1.2463</td>
<td>0.7227</td>
<td>0.8754</td>
<td>0.5906</td>
<td>0.5031</td>
<td>0.1015</td>
<td>1.7886</td>
</tr>
<tr>
<td>1</td>
<td>1.1014</td>
<td>0.6177</td>
<td>0.6177</td>
<td>0.8957</td>
<td>0.1872</td>
<td>0.8231</td>
<td>3.8220</td>
</tr>
</tbody>
</table>

In Fig. 1, the results of the optimization of \( \xi_1 \), \( \xi_2 \), and \( \xi_3 \) by the alternate search for \( \bar{m} = 3 \) with the initial values \( \xi_1 = \xi_2 = \xi_3 = 1.0 \) is shown. Fig. 2 shows the corresponding change of the value of \( \hat{J} \). As we see from Figs. 1 and 2, we have good convergences. Thus, we got good convergences to the optimal value shown in Table 1.

References


