An Approach to Hybrid Smoothing for Linear Continuous-Time Systems with Non-Gaussian Noises

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Abstract

In this paper we study hybrid estimation for linear continuous-time systems with noises not to be restricted to be Gaussian. It is assumed that modes of the systems are not directly accessible. We consider optimal estimation problems to find both estimated states of the systems and a candidate of the distributions of the modes over the finite time interval. We adopt most probable trajectory (MPT) approach. Q. Zhang (2000) has presented hybrid filtering algorithm, i.e., causal estimation, by MPT approach. We consider both filtering and smoothing problems in this paper. We also consider the cases that mode transitions follow Markovian jump processes and present nearly optimal estimation algorithms for limiting mode distributions. We can expect better estimation performance by taking into consideration noncausal information of observations. The hybrid smoother is realized by two filters approach.

Key Words: Hybrid systems; Noncausal estimation; Non-Gaussian noise; Two filters approach; Smoothing

1 Introduction

It is very important to consider simultaneous estimation of both system states and inaccessible modes for hybrid systems with unknown modes ([5, 9]). This estimation is called hybrid estimation. By the hybrid estimation we often want to know current mode at each time through information of observation. However there exist cases that we want to know distributions of modes on long run time interval rather than each estimate of the modes themselves at each time to grasp global performance over long time intervals, for example, distributions of active modes in solar systems([6, 20]), distributions of active agents on formation or consensus via hybrid systems representation and so on.

Much work has been done for smoothing theory for both of continuous- and discrete-time systems ([7, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22] and so on). Various researchers have studied the smoothing problems by various approach, for example, maximum likelihood approach([11, 14, 18]), projection approach([15]) and so on. It is well known that smoothers (noncausal estimators) more effectively estimates the states than filters (causal estimators). It is well known that utilization of

accumulated information of observation improves estimation performance. Nevertheless, on research of estimation for hybrid systems, little work has been done from the point of view of the noncausal information of observation, i.e., smoothing. In [11] Helmick et al. have presented a fixed-interval smoothing algorithm for discrete-time Markovian jump systems by maximum likelihood (ML) approach. However they have considered the case with fully accessible modes and their approach is based on approximate approach to probability density functions (PDFs). While it is significant that optimality is guaranteed for estimation algorithms, in [5] and [9] Costa et al. have presented LMMSE filters to estimate both system states and inaccessible modes for continuous- and discrete-time Markovian jump systems affected by wide sense white noises, but in these LMMSE filters theory the optimality of estimation isn't guaranteed in the meaning that these filters aren't always MMSE. To the best of the author's knowledge the optimal smoothing problems in the cases with inaccessible modes have not yet fully investigated.

In this paper we study hybrid estimation for linear continuous-time systems with non-Gaussian noises. The concerned systems are general hybrid systems given below which aren't restricted to Markovian jump systems and where added noises aren't restricted to be Gaussian. It is assumed that modes of the systems are not directly accessible throughout this paper. We consider optimal estimation problems to find both estimated states of the systems and a candidate of the distributions of the modes over the finite time interval. We adopt most probable trajectory (MPT) approach to guarantee the optimality of estimation methods. On this approach, given information of observation, we consider optimal control problems where we seek optimal control by which averaged noises energies are minimized for averaged systems. In [23] Zhang has presented hybrid filtering algorithm by MPT approach. We consider both filtering and smoothing problems in this paper. We can expect better estimation performance by taking into consideration noncausal information of observations. The hybrid smoother is realized by two filters approach ([10, 13, 19, 21, 22]). We also consider cases that mode transitions follow Markovian jump processes and clarify relationship between estimators based on the Markovian mode transition probabilities and ones

based on the limiting probabilities, i.e., we show near optimality of the limiting estimators. Finally we give numerical examples and verify that we can obtain better estimation performance by smoothing than filtering.

Notations: Throughout this paper the superscript $\prime$ stands for the matrix transposition, $\|\cdot\|$ denotes the Euclidean vector norm and $\|v\|_R^2$ also denotes the weighted norm $v'Rv$. O denotes the matrix with all zero components.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and, on this space, we consider the following system with mode transitions and noises which aren't restricted to be Gaussian.

$$\begin{aligned} \dot{x}(t) &= A(t, \theta(t))x(t) + w(t, \theta(t)), \\ x(0) &= x_0, \theta(0) = i_0 \\ y(t) &= H(t, \theta(t))x(t) + v(t, \theta(t)) \end{aligned} \quad (1)$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^n$ is the exogenous random noise, $v \in \mathbf{R}^k$ is the measurement noise, and $y \in \mathbf{R}^k$ is the measured output. x_0 is an unknown initial state and it is assumed that a distribution of initial modes i_0 is given. The noises $w(\cdot)$ and $v(\cdot)$ aren't restricted to be Gaussian.

We assume that all these matrices are of compatible dimensions.

Let $\mathcal{M} = \{1, 2, \dots, m\}$ denote the state space of $\theta(t)$. In this paper it is assumed that the probability distribution of $\theta(\cdot)$ is unknown, but among a finite number of candidate distributions. Let $r \in \mathcal{N}_0 = \{1, 2, \dots, n_0\}$, and let $\mathcal{P} = \{\phi^{(1)}(\cdot), \dots, \phi^{(n_0)}(\cdot)\}$ denote the set of such candidate distributions on $\mathcal{M}$, i.e., for $r \in \mathcal{N}_0$ and $t \in [0, T]$, $\phi^{(r)}(t) = (\phi_1^{(r)}(t), \dots, \phi_m^{(r)}(t))$ with $\phi_i^{(r)}(t) \geq 0$ and $\sum_{i=1}^m \phi_i^{(r)}(t) = 1$.

The fixed-interval smoothing problem we address in this paper for the system (1) is to find the MPT (most probable trajectory) estimate of $x(t)$, $t \in [0, T]$, over the finite horizon $[0, T]$, using the information available on the known part of the observation $y(\cdot)$ for the given distributions of initial mode i_0 and initial state x_0 . We define the following performance indices for $r \in \mathcal{N}_0$ and $t \in [0, T]$:

$$\begin{aligned} J_{0t}^{(r)}(x_0, w, v) &:= \int_0^t \sum_{i=1}^m \phi_i^{(r)}(s) (w'(s, i)M(s, i)w(s, i) \\ &\quad + v'(s, i)N(s, i)v(s, i)ds \\ &\quad + (x_0 - \hat{x}_0)'D_0(x_0 - \hat{x}(0)) \\ J_{0T}^{(r)}(x_0, w, v) &:= \int_0^T \sum_{i=1}^m \phi_i^{(r)}(s) (w'(s, i)M(s, i)w(s, i) \end{aligned} \quad (2)$$

$$\begin{aligned} &+ v'(s, i)N(s, i)v(s, i)ds \\ &+ (x_0 - \hat{x}_0)'D_0(x_0 - \hat{x}(0)) \\ &+ (x(T) - \hat{x}_T)'D_T(x(T) - \hat{x}_T) \end{aligned} \quad (3)$$

where $\hat{x}_0$ is an initial estimate of x_0 and $\hat{x}_T$ is a terminal estimate of $x(T)$. $M(t, i) > O$, $N(t, i) \geq O$, $D_0 > O$ and $D_T > O$ are symmetric matrices which reflect the uncertainties on the estimate $\hat{x}(0)$, the noises $w(\cdot)$ and $v(\cdot)$. Thus these performance indices mean the energies of noises, initial and terminal estimates under some uncertainties averaged by the mode distributions for each $r \in \mathcal{N}_0$. We consider the optimization problems to decide $w(\cdot, i)$, $v(\cdot, i)$ and $r \in \mathcal{N}_0$ minimizing $J_{0t}^{(r)}$ and $J_{0T}^{(r)}$ utilizing the known parts of the observed information $\mathcal{Y}_T = \{y(s) | 0 \leq s \leq T\}$.

Since the mode $\theta(t)$ at each time is inaccessible, we cannot directly design estimators for the system (1) including the unknown modes. Also, even if the modes are accessible, the computational complexity can exponentially increase if we directly design the estimators for the system (1) including $\theta(t)$ explicitly. Hence we introduce the system averaged through the mode distributions for each $r \in \mathcal{N}_0$.

For notational simplicity, we adopt the following notation.

$$\overline{F}^{(r)}(t) = \sum_{i=1}^m \phi_i^{(r)}(t) F(t, i)$$

for a matrix function $F(t, i)$ and $r \in \mathcal{N}_0$. Similarly $\overline{F_1 F_2}^{(r)}(t) = \sum_{i=1}^m \phi_i^{(r)}(t) F_1(t, i) F_2(t, i)$ for matrix functions $F_1(t, i)$ and $F_2(t, i)$ and so on. Using these notations, we can shift the drift term in the system (1) to $\overline{A}^{(r)}(t)$ as follows:

$$dx(t) = \overline{A}^{(r)}(t)x(t)dt + w(t)dt$$

where

$$\begin{aligned} w(t) &= w^{(r)}(t) = (A(t, \theta(t)) - \overline{A}^{(r)}(t))x(t) + w(t, \theta(t)). \end{aligned}$$

By replacing the system noise $w(t, i)$ by $(\overline{A}^{(r)}(t) - A(t, i))x(t) + w(t)$ and the observation noise $v(t, i)$ by $y(t) - H(t, i)x(t)$ in the performance indices (2) and (3), we define

$$\begin{aligned} L^{(r)}(t, x, w, y) &:= \sum_{i=1}^m \phi_i^{(r)}(t) [(\overline{A}^{(r)}(t) - A(t, i))x + w]' \\ &\quad \times M(t, i) [(\overline{A}^{(r)}(t) - A(t, i))x + w] \\ &\quad + (y - H(t, i)x)' N(t, i) (y - H(t, i)x). \end{aligned}$$

Then we can define the following performance indices:

$$\begin{aligned} J_f^{(r)}(t, x, w(\cdot)) &:= \int_0^t L^{(r)}(s, x(s), w(s), y(s))ds + \Phi_0(x(0)) \end{aligned} \quad (4)$$

$$\begin{aligned}
& J_b^{(r)}(t, x, w(\cdot)) \\
& := \int_t^T L^{(r)}(s, x(s), w(s), y(s)) ds + \Phi_T(x(T)) \quad (5) \\
& J_s^{(r)}(t, x, w(\cdot)) := J_f^{(r)}(t, x, w(\cdot)) + J_b^{(r)}(t, x, w(\cdot))
\end{aligned}$$

where $\Phi_0(x(\cdot)) = (x(\cdot) - \hat{x}_0)' D_0(x(\cdot) - \hat{x}_0)$ and $\Phi_T(x(\cdot)) = (x(\cdot) - \hat{x}_T)' D_T(x(\cdot) - \hat{x}_T)$.

We consider the optimal control problems to minimize $J_f^{(r)}$ and $J_s^{(r)} = J_f^{(r)} + J_b^{(r)}$ for the given parts of $\mathcal{Y}_T$. Let $V_f^{(r)}(t, x)$ and $V_b^{(r)}(t, x)$ be the value functions of these control problems as follows:

$$\begin{aligned}
V_f^{(r)}(t, x) &:= \inf_{w(\cdot)} J_f^{(r)}(t, x, w(\cdot)) \\
V_b^{(r)}(t, x) &:= \inf_{w(\cdot)} J_b^{(r)}(t, x, w(\cdot)) \\
V_s^{(r)}(t, x) &:= V_f^{(r)}(t, x) + V_b^{(r)}(t, x) \\
w_f^{(r)*}(t) &:= \arg \min\{J_f^{(r)}(t, x, w(t)) : w \in \mathbf{R}^n\} \\
w_b^{(r)*}(t) &:= \arg \min\{J_b^{(r)}(t, x, w(t)) : w \in \mathbf{R}^n\} \\
w_s^{(r)*}(t) &:= \arg \min\{J_s^{(r)}(t, x, w(t)) : w \in \mathbf{R}^n\}
\end{aligned}$$

Then define

$$\begin{aligned}
\hat{x}_f^{(r)}(t) &:= \arg \min\{V_f^{(r)}(t, x) : x \in \mathbf{R}^n\}, \\
V_f^{(r)}(t) &:= V_f^{(r)}(t, \hat{x}_f^{(r)}(t))
\end{aligned}$$

and

$$\hat{r}_f(t) := \arg \min\{V_f^{(r)}(t) : r \in \mathcal{N}_0\}.$$

Then the most probable distribution is $\phi^{(\hat{r}_f(t))}(\cdot)$. Let $\hat{x}_f(t) = \hat{x}_f^{(\hat{r}_f(t))}(t)$ and we have

$$\begin{aligned}
& V_f^{(\hat{r}_f(t))}(t, \hat{x}_f(t)) \leq V_f^{(r)}(t, \hat{x}_f^{(r)}(t)) \\
& \leq V_f^{(r)}(t, x) = J_f^{(r)}(t, x, w_f^{(r)*}(t)) \leq J_f^{(r)}(t, x, w(t)).
\end{aligned}$$

Also define

$$\begin{aligned}
\hat{x}_s^{(r)}(t) &:= \arg \min\{V_s^{(r)}(t, x) : x \in \mathbf{R}^n\}, \\
V_s^{(r)}(t) &:= V_s^{(r)}(t, \hat{x}_s^{(r)}(t))
\end{aligned}$$

and

$$\hat{r}_s(t) := \arg \min\{V_s^{(r)}(t) : r \in \mathcal{N}_0\}.$$

Then the most probable distribution is $\phi^{(\hat{r}_s(t))}(\cdot)$. Let $\hat{x}_s(t) = \hat{x}_s^{(\hat{r}_s(t))}(t)$ and we have

$$\begin{aligned}
& V_s^{(\hat{r}_s(t))}(t, \hat{x}_s(t)) \leq V_s^{(r)}(t, \hat{x}_s^{(r)}(t)) \\
& \leq V_s^{(r)}(t, x) = J_s^{(r)}(t, x, w_s^{(r)*}(t)) \leq J_s^{(r)}(t, x, w(t)).
\end{aligned}$$

Now we define the following optimal estimators in the sense of most probable trajectories (MPTs).

Definition 2.1 Given the matrices M, N, D_0 and D_T , $(\hat{r}_f(t), \hat{x}_f(t))$, $t \geq 0$, is called an *optimal filter* (in the MPT sense) if it minimizes $V_f^{(r)}(t, x)$. $(\hat{r}_s(t), \hat{x}_s(t))$, $0 \leq t \leq T$, is called an *optimal smoother* (in the MPT sense) if it minimizes $V_s^{(r)}(t, x)$.

Then we formulate the following optimal hybrid estimation problems for the performance indices (4) and (5).

The Optimal Hybrid Filtering Problem for Linear Continuous-Time Systems:

Find the pair $(\hat{r}_f(t), \hat{x}_f^{(\hat{r}_f(t))}(t))$, $t \in [0, T]$ minimizing the performance index (4) based on the causal part $\mathcal{Y}_t = \{y(s) | 0 \leq s \leq t\}$ of the observed information $\mathcal{Y}_T$.

The Optimal Hybrid Smoothing Problem for Linear Continuous-Time Systems:

Find the pair $(\hat{r}_s(t), \hat{x}_s^{(\hat{r}_s(t))}(t))$, $t \in [0, T]$ minimizing the performance index (4)+(5) based on the whole observed information $\mathcal{Y}_T$.

3 Hybrid Estimation Algorithms

3.1 Optimal Hybrid Filtering

The Hamilton-Jacobi-Bellman (HJB) equations associated with the forward control problem to minimize $J_f^{(r)}$ with regard to $w(\cdot)$ are given as follows:

$$\begin{aligned}
\frac{\partial V_f^{(r)}(t, x)}{\partial t} &= \min_w \left\{ L^{(r)}(t, x, w, y(t)) \right. \\
&\quad \left. - (\bar{A}^{(r)}(t)x + w)' \frac{\partial V_f^{(r)}(t, x)}{\partial x} \right\}
\end{aligned}$$

$$V_f^{(r)}(0, x) = \Phi_0(x), \quad r \in \mathcal{N}_0$$

Then we obtain the following minimizing $w(\cdot)$.

$$\begin{aligned}
& w_f^{(r)*}(t, x) \\
& = (\bar{M}^{(r)}(t))^{-1} \left(\frac{1}{2} \frac{\partial V_f^{(r)}(t, x)}{\partial x} \right. \\
& \quad \left. - \bar{M}^{(r)}(t) \bar{A}^{(r)}(t)x + \bar{M} \bar{A}^{(r)}(t)x \right)
\end{aligned}$$

Let

$$V_f^{(r)}(t, x) = x' K_f^{(r)}(t)x + 2(p_f^{(r)}(t))'x + q_f^{(r)}(t) \quad (6)$$

for some functions $K_f^{(r)}$, $p_f^{(r)}$ and $q_f^{(r)}$ with appropriate dimensions. For any given t , by letting $\partial V_f^{(r)}/\partial x = 0$, we obtain

$$K_f^{(r)}(t)x + p_f^{(r)}(t) = 0.$$

Since it can be shown that the matrix $K_f^{(r)}(t)$ is positive-definite (see Lemma 2 in [23]), we obtain

$$\hat{x}_f^{(r)}(t) = -(K_f^{(r)}(t))^{-1} p_f^{(r)}(t)$$

as the minimizer of $V^{(r)}(t, x)$.

Then we obtain the following matrix differential equations, forward filter equations and scalar equations with initial conditions:

$$\begin{aligned} \dot{K}_f^{(r)}(t) &= -K_f^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_f^{(r)}(t) + \overline{A'MA}^{(r)}(t) \\ &\quad - \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t) + \overline{H'NH}^{(r)}(t) \\ &\quad - \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_f^{(r)}(t) \\ &\quad - K_f^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t), \quad K_f^{(r)}(0) = D_0 \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{\hat{x}}_f^{(r)}(t) &= -(K_f^{(r)}(t))^{-1}[\overline{A'MA}^{(r)}(t) \\ &\quad - \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t) \\ &\quad + \overline{H'NH}^{(r)}(t)]\hat{x}_f^{(r)}(t) \\ &\quad + (\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t)\hat{x}_f^{(r)}(t) \\ &\quad + (K_f^{(r)}(t))^{-1}\overline{H'N}^{(r)}(t)y(t), \quad \hat{x}_f^{(r)}(0) = \hat{x}_0 \end{aligned} \quad (8)$$

$$\begin{aligned} \dot{q}_f^{(r)}(t) &= -(\hat{x}_f^{(r)}(t))'K_f^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_f^{(r)}(t)\hat{x}_f^{(r)}(t) \\ &\quad + y'(t)\overline{N}^{(r)}(t)y(t), \quad q_f^{(r)}(0) = \hat{x}_0'D_0\hat{x}_0 \end{aligned} \quad (9)$$

We also obtain

$$V_f^{(r)}(t) = -(\hat{x}_f^{(r)}(t))'K_f^{(r)}(t)\hat{x}_f^{(r)}(t) + q_f^{(r)}(t).$$

Now we have the following filtering algorithm, which gives the solution of **the Optimal Hybrid Filtering Problem for Linear Continuous-Time Systems**.

Optimal hybrid filtering algorithm

Step 1) Obtain $K_f^{(r)}(t)$, $\hat{x}_f^{(r)}(t)$ and $q_f^{(r)}(t)$ for $r \in \mathcal{N}_0$ and $t \in [0, T]$ by solving (7), (8) and (9) with initial conditions.

Step 2) Choose $\hat{r}_f(t)$ that minimizes

$$V_f^{(r)}(t) = -(\hat{x}_f^{(r)}(t))'K_f^{(r)}(t)\hat{x}_f^{(r)}(t) + q_f^{(r)}(t).$$

Then the most probable distribution is $\phi^{(\hat{r}_f(t))}(\cdot)$ and the optimal filter is given by

$$(\hat{r}_f(t), \hat{x}_f(t)) = (\hat{r}_f(t), \hat{x}_f^{(\hat{r}_f(t))}(t)).$$

3.2 Optimal Hybrid Smoothing

The Hamilton-Jacobi-Bellman (HJB) equations associated with the backward control problem to minimize $J_b^{(r)}$ with regard to $w(\cdot)$ are given as follows:

$$\begin{aligned} -\frac{\partial V_b^{(r)}(t, x)}{\partial t} &= \min_w \left\{ L^{(r)}(t, x, w, y(t)) \right. \\ &\quad \left. + (\overline{A}^{(r)}(t)x + w)' \frac{\partial V_b^{(r)}(t, x)}{\partial x} \right\} \end{aligned}$$

$$V_b^{(r)}(T, x) = \Phi_T(x), \quad r \in \mathcal{N}_0$$

Then we obtain the following minimizing $w(\cdot)$.

$$\begin{aligned} w_b^{(r)*}(t, x) &= -(\overline{M}^{(r)}(t))^{-1} \left(\frac{1}{2} \frac{\partial V_b^{(r)}(t, x)}{\partial x} \right. \\ &\quad \left. + \overline{M}^{(r)}(t)\overline{A}^{(r)}(t)x - \overline{MA}^{(r)}(t)x \right) \end{aligned}$$

Let

$$V_b^{(r)}(t, x) = x'K_b^{(r)}(t)x + 2(p_b^{(r)}(t))'x + q_b^{(r)}(t) \quad (10)$$

for some functions $K_b^{(r)}$, $p_b^{(r)}$ and $q_b^{(r)}$ with appropriate dimensions. For any given t , by letting $\partial V_b^{(r)}/\partial x = 0$, we obtain

$$K_b^{(r)}(t)x + p_b^{(r)}(t) = 0.$$

The following lemma holds.

Lemma 3.1 *The matrix $K_b^{(r)}(t)$ is positive-definite.*

(Proof) See the appendix. (Q.E.D.)

Therefore we obtain

$$\hat{x}_b^{(r)}(t) = -(K_b^{(r)}(t))^{-1}p_b^{(r)}(t)$$

as the minimizer of $V_b^{(r)}(t, x)$.

Then we obtain the following matrix differential equations, backward filter equations and scalar equations with terminal conditions:

$$\begin{aligned} \dot{K}_b^{(r)}(t) &= K_b^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_b^{(r)}(t) - \overline{A'MA}^{(r)}(t) \\ &\quad + \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t) - \overline{H'NH}^{(r)}(t) \\ &\quad - \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_b^{(r)}(t) \\ &\quad - K_b^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t), \quad K_b^{(r)}(T) = D_T \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{\hat{x}}_b^{(r)}(t) &= (K_b^{(r)}(t))^{-1}[\overline{A'MA}^{(r)}(t) \\ &\quad - \overline{A'M}^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t) \\ &\quad + \overline{H'NH}^{(r)}(t)]\hat{x}_b^{(r)}(t) \\ &\quad + (\overline{M}^{(r)}(t))^{-1}\overline{MA}^{(r)}(t)\hat{x}_b^{(r)}(t) \\ &\quad - (K_b^{(r)}(t))^{-1}\overline{H'N}^{(r)}(t)y(t), \quad \hat{x}_b^{(r)}(T) = \hat{x}_T \end{aligned} \quad (12)$$

$$\begin{aligned} \dot{q}_b^{(r)}(t) &= \hat{x}_b^{(r)'}(t)K_b^{(r)}(t)(\overline{M}^{(r)}(t))^{-1}K_b^{(r)}(t)\hat{x}_b^{(r)}(t) \\ &\quad - y'(t)\overline{N}^{(r)}(t)y(t), \quad q_b^{(r)}(T) = \hat{x}_T'D_T\hat{x}_T \end{aligned} \quad (13)$$

We also obtain

$$V_b^{(r)}(t) = -(\hat{x}_b^{(r)}(t))'K_b^{(r)}(t)\hat{x}_b^{(r)}(t) + q_b^{(r)}(t).$$

Using (6) and (10), we can express $V_s^{(r)}(t, x)$ as

$$V_s^{(r)}(t, x) = x'[K_f^{(r)}(t) + K_b^{(r)}(t)]x + 2[p_f^{(r)}(t) + p_b^{(r)}(t)]'x + q_f^{(r)}(t) + q_b^{(r)}(t)$$

Let

$$\partial V_s^{(r)}/\partial x = 0$$

and we obtain the following form.

$$\hat{x}_s^{(r)}(t) = -[K_f^{(r)}(t) + K_b^{(r)}(t)]^{-1}(p_f^{(r)}(t) + p_b^{(r)}(t))$$

Since $p_f^{(r)}(t) = -K_f^{(r)}(t)\hat{x}_f^{(r)}(t)$ and $p_b^{(r)}(t) = -K_b^{(r)}(t)\hat{x}_b^{(r)}(t)$, for each candidate r of given distributions, we can obtain the following form of smoothed estimate at time t by the forward and backward filtered estimates.

$$\hat{x}_s^{(r)}(t) = K_s^{(r)}(t)[K_f^{(r)}(t)\hat{x}_f^{(r)}(t) + K_b^{(r)}(t)\hat{x}_b^{(r)}(t)] \quad (14)$$

where $K_s^{(r)}(t) = [K_f^{(r)}(t) + K_b^{(r)}(t)]^{-1}$.

Now we have the following smoothing algorithm, which gives the solution of the **Optimal Hybrid Smoothing Problem for Linear Continuous-Time Systems**.

Optimal hybrid smoothing algorithm

Step 1) Obtain $K_b^{(r)}(t)$, $\hat{x}_b^{(r)}(t)$ and $q_b^{(r)}(t)$ for $r \in \mathcal{N}_0$ and $t \in [0, T]$ by solving (11), (12) and (13) with terminal conditions.

Step 2) Choose $\hat{r}_s(t)$ that minimizes

$$V_s^{(r)}(t) = V_f^{(r)}(t) + V_b^{(r)}(t)$$

where

$$V_b^{(r)}(t) = -\hat{x}_b^{(r)'}(t)K_b^{(r)}(t)\hat{x}_b^{(r)}(t) + q_b^{(r)}(t).$$

Then the most probable distribution is $\phi^{(\hat{r}_s(t))}(\cdot)$ and the optimal smoother is given by

$$\begin{aligned} (\hat{r}_s(t), \hat{x}_s(t)) &= (\hat{r}_s(t), \hat{x}_s^{(\hat{r}_s(t))}(t)) \\ &= (\hat{r}_s(t), K_s^{(\hat{r}_s(t))}(t)[K_f^{(\hat{r}_s(t))}(t)\hat{x}_f^{(\hat{r}_s(t))}(t) \\ &\quad + K_b^{(\hat{r}_s(t))}(t)\hat{x}_b^{(\hat{r}_s(t))}(t)]) \end{aligned} \quad (15)$$

where $K_s^{(\hat{r}_s(t))}(t) = [K_f^{(\hat{r}_s(t))}(t) + K_b^{(\hat{r}_s(t))}(t)]^{-1}$.

4 Markovian Mode Transitions and Limiting Estimation

In this section it is assumed that the mode transitions follow some Markovian jump processes.

For each $r \in \mathcal{N}_0$ $\{\theta^{(r)}(t)\}$ is a homogeneous Markov process with right continuous trajectories and taking values on the finite set $\mathcal{M} = \{1, 2, \dots, m\}$ with the following stationary transition probabilities:

$$\begin{aligned} \mathcal{P}\{\theta^{(r)}(t + \Delta) = j | \theta^{(r)}(t) = i\} \\ = \begin{cases} \pi_{ij}^{(r)}\Delta + o(\Delta) & i \neq j \\ 1 + \pi_{ii}^{(r)}\Delta + o(\Delta) & i = j \end{cases} \end{aligned}$$

where $\pi_{ij}^{(r)} \geq 0$ is the transition rate from the state i to j , $i \neq j$, and $\pi_{ii}^{(r)} = -\sum_{j=1, j \neq i}^N \pi_{ij}^{(r)} < 0$. Let $\Pi^{(r)} = [(\pi_{ij}^{(r)})]$ be the transition matrix of $\theta^{(r)}(t)$ and define $\phi_i^{(r)}(t) := \mathcal{P}\{\theta^{(r)}(t) = i\} > 0$ for any $i \in \mathcal{M}$ ([8, 9]).

Let $\phi^{(r)}(t)$ denote the probability distribution of $\theta^{(r)}(t)$. Then $\phi^{(r)}(t)$ satisfies the Kolmogorov forward differential equation

$$\dot{\phi}^{(r)}(t) = \phi^{(r)}(t)\Pi^{(r)}(t) \quad (16)$$

for the given initial distribution $\phi^{(r)}(0)$ ([8, 9]).

Definition 4.1 For each $r \in \mathcal{N}_0$ and t , $\Pi^{(r)}(t) = [(\pi_{ij}^{(r)}(t))]$ is weakly irreducible if the equations

$$\nu^{(r)}(t)\Pi^{(r)}(t) = 0 \text{ and } \sum_{i=1}^m \nu_i^{(r)}(t) = 1$$

have a unique nonnegative solution

$\nu^{(r)}(t) = (\nu_1^{(r)}(t), \dots, \nu_m^{(r)}(t))$. Then the solution $\nu(t)$ is called the quasi-stationary distribution corresponding to $\Pi^{(r)}(t)$.

We consider the generator of the Markov chain with fast and slow motions as follows:

For sufficiently small $\epsilon > 0$,

$$\Pi^{(r), \epsilon}(t) = \frac{1}{\epsilon} \tilde{\Pi}^{(r)}(t) + \hat{\Pi}^{(r)}(t).$$

Let $\phi^{(r), \epsilon}(\cdot)$ denote the solution of (16) with $\Pi^{(r)}(t)$ replaced by $\Pi^{(r), \epsilon}(t)$.

For simplicity, we assume that $\tilde{\Pi}^{(r)}(t)$ consists of a single block of irreducible matrix, i.e., $\tilde{\Pi}^{(r)}(t)$ itself is weakly irreducible. Then define $\tilde{\phi}^{(r)}(t) := \lim_{\epsilon \rightarrow 0} \tilde{\phi}_i^{(r), \epsilon}(t)$. Let $\tilde{\nu}^{(r)}(t)$ denote the quasi-stationary distribution corresponding to $\tilde{\Pi}^{(r)}(t)$ and $\tilde{\nu}^{(r)}(t) = \tilde{\phi}^{(r)}(t)$ holds.

We adopt the notation

$$\overline{F}_\lambda^{(r)}(t) = \sum_{i=1}^m \tilde{\phi}_i^{(r)}(t) F(t, i)$$

for any matrix function $F(t, i)$ and $r \in \mathcal{N}_0$.

Then, for some $\tilde{K}_f^{(r)}(t)$, $\tilde{x}_f^{(r)}(t)$, $\tilde{q}_f^{(r)}(t)$, $\tilde{K}_b^{(r)}(t)$, $\tilde{x}_b^{(r)}(t)$ and $\tilde{q}_b^{(r)}(t)$, consider the following differential equations with the initial or terminal conditions:

$$\begin{aligned} &\dot{\tilde{K}}_f^{(r)}(t) \\ &= -\tilde{K}_f^{(r)}(t)(\overline{M}_\lambda^{(r)}(t))^{-1}\tilde{K}_f^{(r)}(t) + \overline{A'MA}_\lambda^{(r)}(t) \\ &\quad - \overline{A'M}_\lambda^{(r)}(t)(\overline{M}_\lambda^{(r)}(t))^{-1}\overline{MA}_\lambda^{(r)}(t) + \overline{H'NH}_\lambda^{(r)}(t) \\ &\quad - \overline{A'M}_\lambda^{(r)}(t)(\overline{M}_\lambda^{(r)}(t))^{-1}\tilde{K}_f^{(r)}(t) \\ &\quad - \tilde{K}_f^{(r)}(t)(\overline{M}_\lambda^{(r)}(t))^{-1}\overline{MA}_\lambda^{(r)}(t), \quad \tilde{K}_f^{(r)}(0) = D_0 \quad (17) \end{aligned}$$

$$\begin{aligned}
\dot{\hat{x}}_f^{(r)}(t) = & -(\tilde{K}_f^{(r)}(t))^{-1}[\overline{A'MA_\lambda}^{(r)}(t) \\
& -\overline{A'M_\lambda}^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1}\overline{MA_\lambda}^{(r)}(t) \\
& +\overline{H'NH_\lambda}^{(r)}(t)]\tilde{x}_f^{(r)}(t) \\
& +(\overline{M_\lambda}^{(r)}(t))^{-1}\overline{MA_\lambda}^{(r)}(t)\tilde{x}_f^{(r)}(t) \\
& +(\tilde{K}_f^{(r)}(t))^{-1}\overline{H'N}^{(r)}(t)y(t), \quad \tilde{x}_f^{(r)}(0) = \hat{x}_0 \quad (18)
\end{aligned}$$

$$\begin{aligned}
& \dot{\hat{q}}_f^{(r)}(t) \\
= & -(\tilde{x}_f^{(r)}(t))' \tilde{K}_f^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \tilde{K}_f^{(r)}(t) \tilde{x}_f^{(r)}(t) \\
& +y'(t) \overline{N}^{(r)}(t)y(t), \quad \tilde{q}_f^{(r)}(0) = \hat{x}_0' D_0 \hat{x}_0 \quad (19)
\end{aligned}$$

$$\begin{aligned}
& \dot{\hat{K}}_b^{(r)}(t) \\
= & \tilde{K}_b^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \tilde{K}_b^{(r)}(t) - \overline{A'MA_\lambda}^{(r)}(t) \\
& + \overline{A'M_\lambda}^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \overline{MA_\lambda}^{(r)}(t) - \overline{H'NH_\lambda}^{(r)}(t) \\
& - \overline{A'M_\lambda}^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \tilde{K}_b^{(r)}(t) \\
& - \tilde{K}_b^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \overline{MA_\lambda}^{(r)}(t), \quad \tilde{K}_b^{(r)}(T) = D_T \quad (20)
\end{aligned}$$

$$\begin{aligned}
\dot{\hat{x}}_b^{(r)}(t) = & (\tilde{K}_b^{(r)}(t))^{-1}[\overline{A'MA_\lambda}^{(r)}(t) \\
& -\overline{A'M_\lambda}^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1}\overline{MA_\lambda}^{(r)}(t) \\
& +\overline{H'NH_\lambda}^{(r)}(t)]\tilde{x}_b^{(r)}(t) \\
& +(\overline{M_\lambda}^{(r)}(t))^{-1}\overline{MA_\lambda}^{(r)}(t)\tilde{x}_b^{(r)}(t) \\
& -(\tilde{K}_b^{(r)}(t))^{-1}\overline{H'N}^{(r)}(t)y(t), \quad \tilde{x}_b^{(r)}(T) = \hat{x}_T \quad (21)
\end{aligned}$$

$$\begin{aligned}
& \dot{\hat{q}}_b^{(r)}(t) \\
= & \tilde{x}_b^{(r)'}(t) \tilde{K}_b^{(r)}(t)(\overline{M_\lambda}^{(r)}(t))^{-1} \tilde{K}_b^{(r)}(t) \tilde{x}_b^{(r)}(t) \\
& -y'(t) \overline{N}^{(r)}(t)y(t), \quad \tilde{q}_b^{(r)}(T) = \hat{x}_T' D_T \hat{x}_T \quad (22)
\end{aligned}$$

Then the following proposition holds.

Proposition 4.1 For $r \in \mathcal{N}_0$, let $K_f^{(r),\epsilon}(t)$, $x_f^{(r),\epsilon}(t)$, $q_f^{(r),\epsilon}(t)$, $K_b^{(r),\epsilon}(t)$, $x_b^{(r),\epsilon}(t)$ and $q_b^{(r),\epsilon}(t)$ denote the solutions of the differential equations (7)-(9) and (11)-(13) respectively with $\phi^{(r)}(\cdot)$ replaced by $\phi^{(r),\epsilon}(\cdot)$. Then

$$\begin{aligned}
K_f^{(r),\epsilon}(t) & \rightarrow \tilde{K}_f^{(r)}(t) \quad K_b^{(r),\epsilon}(t) \rightarrow \tilde{K}_b^{(r)}(t) \\
\hat{x}_f^{(r),\epsilon}(t) & \rightarrow \tilde{x}_f^{(r)}(t) \quad \hat{x}_b^{(r),\epsilon}(t) \rightarrow \tilde{x}_b^{(r)}(t) \\
q_f^{(r),\epsilon}(t) & \rightarrow \tilde{q}_f^{(r)}(t) \quad q_b^{(r),\epsilon}(t) \rightarrow \tilde{q}_b^{(r)}(t)
\end{aligned}$$

uniformly hold on $[0, T]$ as $\epsilon \rightarrow 0$ where $\tilde{K}_f^{(r)}(t)$, $\tilde{x}_f^{(r)}(t)$, $\tilde{q}_f^{(r)}(t)$, $\tilde{K}_b^{(r)}(t)$, $\tilde{x}_b^{(r)}(t)$ and $\tilde{q}_b^{(r)}(t)$ are the unique solutions of (17)-(19) and (20)-(22) respectively.

(Proof) See the appendix. (Q.E.D.)

The limiting smoother $(\tilde{r}_s(t), \tilde{x}_s(t))$ can be obtained as follows: Solve (17)-(19) with the initial conditions

and (20)-(22) with the terminal conditions and obtain $\tilde{K}_f^{(r)}(t)$, $\tilde{x}_f^{(r)}(t)$, $\tilde{q}_f^{(r)}(t)$, $\tilde{K}_b^{(r)}(t)$, $\tilde{x}_b^{(r)}(t)$ and $\tilde{q}_b^{(r)}(t)$, $t \in [0, T]$. Then, for each t , choose $\tilde{r}_s(t)$ that minimizes

$$\tilde{V}_s^{(r)}(t) = \tilde{V}_f^{(r)}(t) + \tilde{V}_b^{(r)}(t)$$

where

$$\tilde{V}_f^{(r)}(t) = -\tilde{x}_f^{(r)'}(t) \tilde{K}_f^{(r)}(t) \tilde{x}_f^{(r)}(t) + \tilde{q}_f^{(r)}(t)$$

and

$$\tilde{V}_b^{(r)}(t) = -\tilde{x}_b^{(r)'}(t) \tilde{K}_b^{(r)}(t) \tilde{x}_b^{(r)}(t) + \tilde{q}_b^{(r)}(t).$$

Then the most probable distribution is $\tilde{\phi}^{(\tilde{r}_s(t))}(\cdot)$ and the optimal smoother is given by

$$\begin{aligned}
(\tilde{r}_s(t), \tilde{x}_s(t)) & = (\tilde{r}_s(t), \tilde{x}_s^{(\tilde{r}_s(t))}(t)) \\
& = (\tilde{r}_s(t), \tilde{K}_s^{(\tilde{r}_s(t))}(t) [\tilde{K}_f^{(\tilde{r}_s(t))}(t) \tilde{x}_f^{(\tilde{r}_s(t))}(t) \\
& \quad + \tilde{K}_b^{(\tilde{r}_s(t))}(t) \tilde{x}_b^{(\tilde{r}_s(t))}(t)])
\end{aligned}$$

where $\tilde{K}_s^{(\tilde{r}_s(t))}(t) = [\tilde{K}_f^{(\tilde{r}_s(t))}(t) + \tilde{K}_b^{(\tilde{r}_s(t))}(t)]^{-1}$.

The next theorem gives the near optimality of the limiting smoother.

Theorem 4.1 Let $\hat{x}_s^{(r),\epsilon}(t)$ and $\hat{x}_s^\epsilon(t)$ denote the smoothers defined by (14) and (15) respectively with $\phi^{(r)}(\cdot)$ replaced by $\phi^{(r),\epsilon}(\cdot)$. Then, for each $r \in \mathcal{N}_0$ and $t \in [0, T]$, we have

i) $\hat{x}_s^{(r),\epsilon}(t) \rightarrow \hat{x}_s^{(r)}(t)$ as $\epsilon \rightarrow 0$.

ii) $\hat{x}_s^\epsilon(t) \rightarrow \tilde{x}_s(t)$ as $\epsilon \rightarrow 0$ if $\tilde{r}_s(t)$ is the unique minimizer of $\tilde{V}_s^{(r)}(t)$.

(Proof) See the appendix. (Q.E.D.)

5 Numerical Examples

In this section, we study numerical examples to demonstrate the effectiveness of the presented design algorithms.

We consider the following two mode systems and assume that the system parameters are as follows:

$$\begin{aligned}
\dot{x}(t) & = A(\theta(t))x(t) + w(t, \theta(t)), \\
x(0) & = x_0, \theta(0) = i_0 \\
y(t) & = Hx(t) + v(t, \theta(t))
\end{aligned} \quad (23)$$

where

$$\begin{aligned}
& \cdot \text{ Mode 1:} \quad \cdot \text{ Mode 2:} \\
A_1 & = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 1 \\ 0.8 & -0.2 \end{bmatrix},
\end{aligned}$$

$$H = [1, 0]$$

and

$$M(t, i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N(t, i) = 1, \quad D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $i = 1, 2$. We set $\hat{x}_0 = \text{col}(-0.1, 0)$ and the distribution of the initial mode i_0 as $(1/2, 1/2)$. $w(\cdot)$ and $v(\cdot)$ are stochastic noises which aren't restricted to be Gaussian white. The candidates of mode distributions are given as follows:

$$\phi^{(1)} = \left(\frac{2}{5}, \frac{3}{5}\right), \phi^{(2)} = \left(\frac{1}{10}, \frac{9}{10}\right), \phi^{(3)} = \left(\frac{7}{10}, \frac{3}{10}\right)$$

Note that these distributions are limiting distributions if the mode transition law follows the Markovian jump processes. Actually the corresponding Markovian transition matrices are

$$\Pi^{(1)} = \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}, \Pi^{(2)} = \begin{pmatrix} -9 & 9 \\ 1 & -1 \end{pmatrix}$$

and

$$\Pi^{(3)} = \begin{pmatrix} -3 & 3 \\ 7 & -7 \end{pmatrix}$$

respectively. The paths of $\theta(t)$ are generated randomly, and the performances are compared under the same circumstance, that is, the same set of the paths so that the performances can be easily compared.

We consider the whole system (23) with the true mode distribution $\phi^{(3)}$ over the time interval $t \in [0, 15]$. We verify the effectiveness of the presented hybrid estimation algorithms and compare the estimation performances for the optimal filtering and smoothing algorithms. In order to carry out these algorithms we solve the forward or backward triplet of the differential equations (7)-(9) or (11)-(13) with the initial or terminal conditions for given observation $y(\cdot)$ and each candidate $r = 1, 2, 3$ of given distributions, and obtain the pair $(\hat{r}(t), \hat{x}_f(t))$ minimizing $V_f^{(r)}(t)$ in the filtering case or the pair $(\hat{r}_s(t), \hat{x}_s(t))$ minimizing $V_s^{(r)}(t)$ in the smoothing case for $t \in [0, T]$. (If it is assumed that the mode transitions follow some Markovian jump processes, we consider that we solve the differential equations (17)-(19) or (20)-(22) and obtain the nearly optimal limiting estimators for $t \in [0, T]$.)

Filtered and smoothed values of the first components of the whole system states are given by **Fig. 1** and **Fig. 2** respectively. **Fig. 3** and **Fig. 4** show the square errors between the states and filtered values, and the states and smoothed values respectively. The mean square errors over the time interval $[0, 15]$ are 0.0025 in the filtering case, and 0.0019 in the smoothing case respectively. From these figures and calculation results it is shown that the smoother gives better estimation than the filter. Filtered and smoothed mode distributions are given by **Fig. 5** and **Fig. 6**. Notice that the vertical axes show the candidates of the mode distributions not the modes themselves. In **Fig. 5** the filtered values of the mode distributions rapidly change to be left undecided. To the contrary in **Fig. 6** the smoothed values of the mode distributions are firmly decided. Through these four figures it is shown that the

Fig. 1: The state of the system and filtered values

Fig. 2: The state of the system and smoothed values

optimal smoother presented in this paper gives better estimate performance than the optimal filter presented in the previous work ([23]) from the point of view of both state and modes estimation.

6 Concluding Remarks

In this paper we have studied the state and mode estimation problems for linear continuous-time hybrid systems over the fixed time interval. The systems aren't restricted to the Markovian jump systems except for Section 4 and the added noises aren't restricted to be Gaussian. We have adopted the MPT approach. The state and mode estimation approach adopted in this paper guarantees the optimality of estimation performance in the meaning of MPT different from the previous work ([5, 9]).

In this paper we have considered the problems that both the system state and modes are estimated. However we have considered the problems that the distributions of the modes over the fixed time interval not the modes themselves are estimated to grasp the global behavior of the hybrid systems over the long time intervals. In order to estimate both the system state and distributions of the modes we have introduced the averaged performance indices with respect to the candidates

Fig. 3: The square errors between the state and filtered values

Fig. 4: The square errors between the state and smoothed values

of the mode distributions for the averaged systems. For these performance indices we have formulated the optimal filtering and smoothing problems for the observed information. The estimation problems have been reduced to the optimal control problems to find the noises minimizing the introduced performance indices. We have derived the forward and backward Riccati type differential equations and the forward and backward filters, which give the necessary conditions for the solvability of the optimal control problems. Then we have presented the optimal hybrid smoothing algorithm by the two filters approach. We have also considered the cases that the mode transitions follow the Markovian jump processes and shown the near optimality of the limiting estimators. Finally we have studied the numerical examples to compare the estimation performances by filtering and smoothing. We have obtained the better estimation performance by the smoothing algorithm than the filtering algorithm from the point of view of both state and modes estimation.

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Fig. 5: The filtered mode distributions

Fig. 6: The smoothed mode distributions

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Appendix 1: Proof of Lemma 3.1

In this appendix we show the positive-definiteness of the matrix $K_b^{(r)}(t)$.

(Proof of Lemma 3.1) For each $r \in \mathcal{N}_0$,

$$\begin{aligned} V_b^{(r)}(t, x) &= J_b^{(r)}(t, x, w_b^{(r)*}(\cdot)) \\ &= \int_t^T L^{(r)}(s, x^*(s), w_b^{(r)*}(s), y(s)) ds \end{aligned} \quad (24)$$

where $w_b^{(r)*}(s) = w_b^{(r)*}(s, x^*(s))$ is the optimal control with

$$\begin{aligned} w_b^{(r)*}(s, x) &= -(\overline{M}^{(r)}(s))^{-1} \left(K_b^{(r)}(s)x + p_b^{(r)}(s) \right. \\ &\quad \left. + \overline{M}^{(r)}(s)\overline{A}^{(r)}(s)x - \overline{M}\overline{A}^{(r)}(s)x \right) \end{aligned}$$

and

$$\begin{aligned} \dot{x}^*(s) &= -(\overline{M}^{(r)}(s))^{-1} \left(K_b^{(r)}(s) - \overline{M}\overline{A}^{(r)}(s) \right) x^*(s) \\ &\quad - (\overline{M}^{(r)}(s))^{-1} p_b^{(r)}(s), \quad x^*(t) = x. \end{aligned}$$

Let $\Psi(s, t)$ be the state transition matrix of the linear time-varying system

$$\dot{z}(s) = -(\overline{M}^{(r)}(s))^{-1} \left(K_b^{(r)}(s) - \overline{M}\overline{A}^{(r)}(s) \right) z(s)$$

with $\Psi(t, t) = I$. Then we can show

$$\frac{\partial x^*}{\partial x} = \Psi(s, t) \text{ and } \frac{\partial^2 x^*}{\partial x^2} = 0.$$

Differentiating both sides of (24) twice with respect to x and using the quadratic form of $V_b^{(r)}(t, x)$, we can obtain

$$K_b^{(r)}(t) \geq \int_t^T [\Psi'(s, t) \overline{H}^T \overline{N} \overline{H}^{(r)}(s) \Psi(s, t)] ds > 0$$

because of $M(t, i) > 0$, $N(t, i) \geq 0$ and $D > 0$ and

$$\begin{aligned} &L^{(r)}(s, x^*(s), w_b^{(r)*}(s), y(s)) \\ &:= \sum_{i=1}^m \phi_i^{(r)}(s) [(\overline{A}^{(r)}(s) - A(s, i))x^*(s) + w_b^{(r)*}(s)]' \\ &\quad \times M(s, i) [(\overline{A}^{(r)}(s) - A(s, i))x^*(s) + w_b^{(r)*}(s)] \\ &\quad + (y(s) - H(s, i)x^*(s))' N(s, i) (y(s) - H(s, i)x^*(s)) \end{aligned}$$

which concludes the proof. (Q.E.D.)

Appendix 2: Proof of Proposition 4.1 and Theorem 4.1

First the following lemmas hold ([23]).

Lemma 6.1 For each $r \in \mathcal{N}_0$ and $t \in [0, T]$, the inequalities

$$\begin{aligned} \|K_f^{(r)}(t)\| &\leq C_0(T+1), \quad \|p_f^{(r)}(t)\| \leq C_0(T+1), \\ \|q_f^{(r)}(t)\| &\leq C_0(T+1), \\ \|K_b^{(r)}(t)\| &\leq C_T(T+1), \quad \|p_b^{(r)}(t)\| \leq C_T(T+1) \\ \text{and } \|q_b^{(r)}(t)\| &\leq C_T(T+1) \end{aligned}$$

hold where $K_f^{(r)}(t)$, $p_f^{(r)}(t)$, $q_f^{(r)}(t)$, $K_b^{(r)}(t)$, $p_b^{(r)}(t)$ and $q_b^{(r)}(t)$ are the solutions of (7)-(9) and (11)-(13) respectively and

$$\begin{aligned} C_0 &= \max \left\{ \sup_{t,i} \{ \|M(t,i)\| \|A(t,i)\|^2 \right. \\ &\quad \left. + 2\|N(t,i)\| \|H(t,i)\|^2 \}, 2 \sup_{t,i} \{ \|N(t,i)\| \|y(t)\|^2 \}, \right. \\ &\quad \left. 2\|D_0\|, 2\|D_0\| \|\hat{x}_0\|^2 \right\}, \\ C_T &= \max \left\{ \sup_{t,i} \{ \|M(t,i)\| \|A(t,i)\|^2 \right. \\ &\quad \left. + 2\|N(t,i)\| \|H(t,i)\|^2 \}, 2 \sup_{t,i} \{ \|N(t,i)\| \|y(t)\|^2 \}, \right. \\ &\quad \left. 2\|D_T\|, 2\|D_T\| \|\hat{x}_T\|^2 \right\}. \end{aligned}$$

Lemma 6.2 Let F^ϵ and F be invertible matrices such that $F^\epsilon \rightarrow F$ as $\epsilon \rightarrow 0$. Then $(F^\epsilon)^{-1} \rightarrow F^{-1}$ holds.

(Proof of Proposition 4.1)

We only show the uniform convergence of $K_b^{(r),\epsilon}(\cdot)$. The others can be shown by the similar arguments.

By Lemma 6.1 $K_b^{(r),\epsilon}(t)$ is uniformly bounded on $[0, T]$ for each $r \in \mathcal{N}_0$. Therefore, from (11), $\dot{K}_b^{(r),\epsilon}(t)$ is uniformly bounded on $[0, T]$ for each $r \in \mathcal{N}_0$, which implies the sequence $\{K_b^{(r),\epsilon}(\cdot)\}$ is equicontinuous on $[0, T]$. From Arzela-Ascoli theorem it follows that, for each sequence of $\{\epsilon \rightarrow 0\}$, there exists a subsequence (still indexed by ϵ) such that $\{K_b^{(r),\epsilon}(t)\}$ uniformly converges to a continuous matrix-valued function $K^{(r),0}(t)$. It suffices to show $K_b^{(r),0}(t) = \tilde{K}_b^{(r)}$. In order to show this equality we write $K_b^{(r),\epsilon}(\cdot)$ as the integral form

$$\begin{aligned} &K_b^{(r),\epsilon}(t) \\ &= D_T - \int_t^T \left\{ K_b^{(r),\epsilon}(s) (\overline{M}^{(r),\epsilon}(s))^{-1} K_b^{(r),\epsilon}(s) \right. \\ &\quad - \overline{A'MA}^{(r),\epsilon}(s) \\ &\quad + \overline{A'M}^{(r),\epsilon}(s) (\overline{M}^{(r),\epsilon}(s))^{-1} \overline{MA}^{(r),\epsilon}(s) \\ &\quad - \overline{H'NH}^{(r),\epsilon}(s) \\ &\quad - \overline{A'M}^{(r),\epsilon}(s) (\overline{M}^{(r),\epsilon}(s))^{-1} K_b^{(r),\epsilon}(s) \\ &\quad \left. - K_b^{(r),\epsilon}(s) (\overline{M}^{(r),\epsilon}(s))^{-1} \overline{MA}^{(r),\epsilon}(s) \right\} ds \end{aligned}$$

where $\overline{F}^{(r),\epsilon}(t) = \sum_{i=1}^m \phi_i^{(r),\epsilon}(t) F(t, i)$ for any function $F(t, i)$.

The convergence $\phi_i^{(r),\epsilon}(t) \rightarrow \tilde{\phi}_i^{(r)}(t)$, $\epsilon \rightarrow 0$ implies

$$\begin{aligned} \overline{M}^{(r),\epsilon}(t) &\rightarrow \overline{M}^{(r)}(t), \quad \overline{A'M}^{(r),\epsilon}(t) \rightarrow \overline{A'M}^{(r),\epsilon}(t), \\ \overline{MA}^{(r),\epsilon}(t) &\rightarrow \overline{MA}^{(r)}(t) \end{aligned}$$

and

$$\overline{A'MA}^{(r),\epsilon}(t) \rightarrow \overline{A'MA}^{(r)}(t) \text{ as } \epsilon \rightarrow 0.$$

From Lemma 6.2 we have

$$(\overline{M}^{(r),\epsilon}(t))^{-1} \rightarrow (\overline{M}^{(r)}(t))^{-1} \text{ as } \epsilon \rightarrow 0.$$

Letting $\epsilon \rightarrow 0$ and noting the uniform convergence $K_b^{(r),\epsilon}(t) \rightarrow K_b^{(r),0}(t)$, we can show $K_b^{(r),0}(t)$ corresponds to the integral form of the unique solution of (20). Therefore $K_b^{(r),0}(t) = \tilde{K}_b^{(r)}(t)$. (Q.E.D.)

(Proof of Theorem 4.1)

By Proposition 4.1 and Lemma 6.2, we have

$$\begin{aligned} \hat{x}_f^{(r),\epsilon}(t) &\rightarrow \tilde{x}_f^{(r)}(t), \quad \epsilon \rightarrow 0 \\ \hat{x}_b^{(r),\epsilon}(t) &\rightarrow \tilde{x}_b^{(r)}(t) \quad \epsilon \rightarrow 0 \end{aligned}$$

and

$$\hat{x}_s^{(r),\epsilon}(t) \rightarrow \tilde{x}_s^{(r)}(t), \quad \epsilon \rightarrow 0$$

which imply $V_f^{(r),\epsilon}(t) \rightarrow \tilde{V}_f^{(r)}(t)$, $V_b^{(r),\epsilon}(t) \rightarrow \tilde{V}_b^{(r)}(t)$ and $V_s^{(r),\epsilon}(t) \rightarrow \tilde{V}_s^{(r)}(t)$ $\epsilon \rightarrow 0$. In order to show (ii) of Theorem 4.1 let denote r_0 denote the only minimizer of $\tilde{V}_s^{(r)}(t)$, i.e., $\tilde{V}_s^{(r_0)}(t) < \tilde{V}_s^{(r)}(t)$ for all $r \neq r_0$. Then $\tilde{r}_s(t) = r_0$ by definition. Thus, for sufficiently small $\epsilon > 0$,

$$V_s^{(\tilde{r}_s(t)),\epsilon}(t) = V_s^{(r_0),\epsilon}(t) < V_s^{(r),\epsilon}(t) \text{ for all } r \neq r_0.$$

Hence let $\hat{r}_s^\epsilon(t)$ denote the only minimizer of $V_s^{(r),\epsilon}(t)$ and then $\hat{r}_s^\epsilon(t) = \hat{r}_s(t)$ holds for sufficiently small $\epsilon > 0$. Therefore

$$\begin{aligned} |\hat{x}_s^\epsilon(t) - \tilde{x}_s(t)| &= \left| \sum_{r=1}^{n_0} \mathbf{1}_{\{\tilde{r}_s(t)=r\}} (\hat{x}_s^{(r),\epsilon}(t) - \tilde{x}_s^{(r)}(t)) \right| \\ &\leq \sum_{r=1}^{n_0} \mathbf{1}_{\{\tilde{r}_s(t)=r\}} |\hat{x}_s^{(r),\epsilon}(t) - \tilde{x}_s^{(r)}(t)| \end{aligned}$$

holds as $\epsilon \rightarrow 0$ where the indicator function $\mathbf{1}_{\{\tilde{r}_s(t)=r\}} := 1$ if $\tilde{r}_s(t) = r$, and $\mathbf{1}_{\{\tilde{r}_s(t)=r\}} := 0$ if $\tilde{r}_s(t) \neq r$. (Q.E.D.)