A Stopping Rule for Finite-Difference Stochastic Approximation*

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Abstract

A stopping rule is developed for finite-difference stochastic approximation (FDSA) which is to minimize an unknown objective function based on random noise corrupted observation of the function. When it is assumed that the function is convex quadratic, the necessary number of iterations for achieving a given probabilistic accuracy of the resultant solution is derived, which gives a rigorous stopping rule for FDSA. This number is polynomial in the problem size.

1 Introduction

Optimization problems with an *unknown* objective function have many applications in several control synthesis, signal processing, and pattern recognition such as on-line identification, adaptive control, and nonlinear filtering. *The Finite-Difference Stochastic Approximation* (FDSA) [1,2] is a recursive procedure which has ability to find an optimal solution of an unknown objective function via noisy measurements. Due to the unknown objective function, we have to estimate a gradient of the objective function based on the measurements. The algorithm employs difference quotient to estimate it. Then, the stochastic approximation [3–6] is applied to guarantee validity of this approximation. In fact, convergence to the optimal solution in a probabilistic sense is established in [1, 2]. In addition to this, it is important to consider stopping rules for stochastic approximation. For example, in [7–9], it has been shown that the candidate of the solution is close enough to the optimal solution in a *practical* sense when it satisfies a stopping rule.

In this paper, we develop a stopping rule for FDSA. We give a theoretical guarantee on the stopping rule. We show that the corresponding candidate of the solution is close to the optimal solution within specified estimation error with high probability when the number of iterations approaches to the derived number. Note that once the parameters are specified, the necessary number of the measurements can readily be computed in advance of execution of FDSA. Although the authors have already proposed a stopping rule of linear stochastic approximation for solving an unknown linear equation [10], construction of a stopping rule of FDSA is more difficult than that of the stochastic approximation.

This is because the number of parameters of FDSA is greater than that of the stochastic approximation.

This paper is organized as follows. In Section 2, we formulate our problem and show the main result. Then in Section 3, we prove the main result. Finally, we make some concluding remarks in Section 5.

2 Problem Settings and Main Results

We consider an unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad (1)$$

where $x \in \mathbb{R}^n$ is a decision variable and the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is unknown. FDSA is a kind of gradient methods [11] according to the update rule

$$x_{k+1} = x_k - a_k \tilde{g}_k. \quad (2)$$

The gain a_k is monotone decreasing sequence which converges to zero as k goes to ∞ . The vector $\tilde{g}_k$ is an estimation of the gradient of f with respect to x at x_k based on the difference quotient

$$\tilde{g}_k = \begin{bmatrix} \tilde{g}_{k,1} \\ \tilde{g}_{k,2} \\ \vdots \\ \tilde{g}_{k,n} \end{bmatrix}, \quad \tilde{g}_{k,i} = \frac{y_{k,i}^+ - y_{k,i}^-}{2c_k}, \quad i = 1, 2, \dots, n, \\ y_{k,i}^\pm = f(x_k \pm c_k \psi_i) + \epsilon_{k,i}^\pm,$$

where $y_{k,i}^\pm \in \mathbb{R}$ are the measurements of $f(x)$ at $x_k \pm c_k \psi_i$, $\psi_i \in \mathbb{R}^n$ denotes the i -th unit vector, and $\epsilon_{k,i}^\pm \in \mathbb{R}$ are the measurement noise. The step size c_k is monotone decreasing sequence which converges to zero as k goes to ∞ .

In this paper, we assume that the objective function is convex quadratic. This is given by

$$f(x) \doteq \frac{1}{2} x^T H x + h^T x + d$$

where coefficients $H = H^T$, $I/2 \preceq H \preceq I$, $h \in \mathbb{R}^n$, and $d \in \mathbb{R}$ are unknown. It is supposed that the measurement noise $\epsilon_k^\pm$ are random variables according to the independent and identically distribution with zero mean and the variance $\sigma^2 < \infty$.

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We select a_k and c_k as k^{-1} and $k^{-1/6}$ respectively, which are the best parameter settings in a theoretical sense [12].

Under these settings, we can construct a stopping rule of FDSA, which is the main result in this paper.

Theorem 1 For given $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, and $\gamma \in (0, 1)$, select $\bar{k} \in \mathbb{N}$ which satisfies

$$\bar{k} \geq \max \{ \tau_1, \tau_2 \}, \quad (3)$$

$$\tau_1 \doteq \left(\frac{1}{\alpha} \right)^2, \quad (4)$$

$$\tau_2 \doteq \left(\frac{3n\sigma^2}{2\beta^2\gamma} \right)^{3/2}. \quad (5)$$

Then, for any initial candidate $x_1 \in \mathbb{R}^n$ of the solution, the $\bar{k}$ -th candidate $x_{\bar{k}}$ of the solution satisfies

$$\mathbb{P}^\infty (\|x_{\bar{k}} - x^*\|_2 \leq \alpha \|x_1 - x^*\|_2 + \beta) \geq 1 - \gamma, \quad (6)$$

where $\mathbb{P}^\infty$ is a probability measure on a noise sequence and $\|\cdot\|_2$ denotes the Euclidean norm.

This theorem says that $x_{\bar{k}}$ is enough close to x^* at the $\bar{k}$ -th iteration. The distance between $x_{\bar{k}}$ and x^* is determined by α , β , and γ . These parameters correspond to the initial error, the error caused by noise, and probabilistic guarantee, respectively.

Since α , β , and γ can be chosen as arbitrarily small numbers in this theorem, we have immediately the following corollary which is actually given in a previous work [1].

Corollary 1 For any initial candidate $x_1 \in \mathbb{R}^n$, sequences $x_1, x_2, \dots$ of estimations of the solution converges in probability towards x^* .

3 Proof of Theorem 1

We first derive the dynamics of the estimation error $e_k \doteq x_k - x^*$. The gradient estimation $g_{k,i}$ with respect to the i -th element of x is

$$\begin{aligned} \tilde{g}_{k,i} &= \frac{y_{k,i}^+ - y_{k,i}^-}{2c_k} \\ &= \frac{f(x_k + c_k \psi_i) - f(x_k - c_k \psi_i)}{2c_k} + \frac{\epsilon_{k,i}^+ - \epsilon_{k,i}^-}{2c_k} \\ &= x_k^T H \psi_i + h^T \psi_i + \frac{\epsilon_{k,i}^+ - \epsilon_{k,i}^-}{2c_k}. \end{aligned}$$

The first-order optimality condition [11]

$$\frac{df}{dx}(x^*) = Hx^* + h = 0$$

leads to $x^* = -H^{-1}h$, then we have an expression of the gradient estimation:

$$\begin{aligned} \tilde{g}_k &= (x_k - x^*)^T H \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_n \end{bmatrix} \\ &\quad + \frac{1}{2c_k} \begin{bmatrix} \epsilon_{k,1}^+ - \epsilon_{k,1}^- \\ \epsilon_{k,2}^+ - \epsilon_{k,2}^- \\ \vdots \\ \epsilon_{k,n}^+ - \epsilon_{k,n}^- \end{bmatrix} \\ &= He_k + \frac{\epsilon_k}{2c_k}, \end{aligned}$$

where we introduce

$$\epsilon_k \doteq \begin{bmatrix} \epsilon_{k,1}^+ - \epsilon_{k,1}^- \\ \epsilon_{k,2}^+ - \epsilon_{k,2}^- \\ \vdots \\ \epsilon_{k,n}^+ - \epsilon_{k,n}^- \end{bmatrix}$$

for readability. The update rule (2) is rewritten as

$$x_{k+1} = x_k - \frac{1}{k} \left(He_k + \frac{\epsilon_k}{2c_k} \right).$$

Thus, we immediately see that

$$e_{k+1} = \left(I - \frac{1}{k} H \right) e_k - \frac{1}{2c_k k} \epsilon_k$$

when we subtract x^* from the both sides. Then, the dynamics of the expectation $\mathbb{E}[e_k]$ and the variance V_k on e_k are

$$\mathbb{E}[e_{k+1}] = \left(I - \frac{1}{k} H \right) \mathbb{E}[e_k], \quad (7)$$

$$\begin{aligned} V_{k+1} &= \mathbb{E} [(e_{k+1} - \mathbb{E}[e_{k+1}])(e_{k+1} - \mathbb{E}[e_{k+1}])^T] \\ &= \left(I - \frac{1}{k} H \right) (e_k - \mathbb{E}[e_k])(e_k - \mathbb{E}[e_k])^T \left(I - \frac{1}{k} H \right)^T \\ &\quad + \frac{1}{2c_k^2 k^2} \sigma^2 I \\ &= \left(I - \frac{1}{k} H \right) V_k \left(I - \frac{1}{k} H \right)^T + \frac{1}{2c_k^2 k^2} \sigma^2 I. \end{aligned} \quad (8)$$

The solutions of (7) and (8) are given by

$$\begin{aligned} \mathbb{E}[e_k] &= \Phi(k, 1) e_1, \\ V_k &= \sum_{j=1}^{k-1} \frac{\sigma^2}{2c_j^2 j^2} \Phi(k, j) (\Phi(k, j))^T, \end{aligned}$$

where $\Phi(k, j)$ is a transition matrix which is defined by

$$\Phi(k, j) \doteq \begin{cases} \left(I - \frac{1}{k-1} H \right) \left(I - \frac{1}{k-2} H \right) \\ \quad \times \dots \times \left(I - \frac{1}{j} H \right) & \text{if } k > j \\ I & \text{if } k \leq j. \end{cases}$$

Finally, norm of the k -th estimation error is bounded by

$$\begin{aligned} \|e_k\|_2 &= \|\mathbb{E}[e_k] + e_k - \mathbb{E}[e_k]\|_2 \\ &\leq \|\mathbb{E}[e_k]\|_2 + \|e_k - \mathbb{E}[e_k]\|_2. \end{aligned}$$

That is, all we have to do is prove

$$\|E[e_{\bar{k}}]\|_2 \leq \alpha \|e_1\|_2, \quad (9)$$

$$P^\infty(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2 \leq \beta) \geq 1 - \gamma, \quad (10)$$

when the number of iterations reaches $\bar{k}$ which is defined by (3). The following lemmas show that these inequalities actually hold.

Lemma 1 For a given constant $\alpha \in (0, \infty)$, selecting $\bar{k} \in \mathbb{N}$ which is greater than τ_1 defined by (4),

$$\|\Phi(\bar{k}, 1)\|_2 \leq \alpha$$

holds, where $\|\cdot\|_2$ is the matrix norm induced by the Euclidean norm.

Proof: From the definition of the transition matrix and sub-multiplicativity of the matrix norm, we obtain

$$\|\Phi(\bar{k}, 1)\|_2 \leq \prod_{\ell=1}^{\bar{k}-1} \left\| I - \frac{1}{\ell} H \right\|_2.$$

The assumption $I/2 \preceq H \preceq I$ leads to

$$\left\| I - \frac{1}{\ell} H \right\|_2 \leq 1 - \frac{1}{2\ell} \quad \ell = 1, 2, \dots, \bar{k} - 1.$$

Employing

$$1 - x \leq \exp(-x),$$

$$\sum_{\ell=1}^{\bar{k}-1} \frac{1}{\ell} \geq \int_1^{\bar{k}} \frac{1}{x} dx,$$

we see that

$$\begin{aligned} \|\Phi(\bar{k}, 1)\|_2 &\leq \prod_{\ell=1}^{\bar{k}-1} \left(1 - \frac{1}{2\ell} \right) \\ &\leq \prod_{\ell=1}^{\bar{k}-1} \exp\left(-\frac{1}{2\ell}\right) = \exp\left(-\frac{1}{2} \sum_{\ell=1}^{\bar{k}-1} \frac{1}{\ell}\right) \\ &\leq \exp\left(-\frac{1}{2} \int_1^{\bar{k}} \frac{1}{x} dx\right) = \sqrt{\frac{1}{\bar{k}}} \\ &\leq \sqrt{\frac{1}{\tau_1}} = \alpha. \end{aligned}$$

That is, we can conclude that the inequality (9) holds. $\square$

Lemma 2 For given constants $\beta \in (0, \infty)$ and $\gamma \in (0, 1)$, selecting $\bar{k} \in \mathbb{N}$ which is greater than τ_2 defined by (5),

$$P^\infty(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2 \leq \beta) \geq 1 - \gamma$$

holds.

Proof: Markov inequality [13] leads to

$$P^\infty\left(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2 \leq \frac{E[\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2]}{\gamma}\right) \geq 1 - \gamma.$$

Noting that

$$\begin{aligned} P^\infty\left(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2 \leq \frac{E[\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2]}{\gamma}\right) \\ = P^\infty\left(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2 \leq \sqrt{\frac{E[\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2]}{\gamma}}\right) \end{aligned}$$

and

$$\begin{aligned} E[\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2^2] &= E[(e_{\bar{k}} - E[e_{\bar{k}}])^T (e_{\bar{k}} - E[e_{\bar{k}}])] \\ &= E[\text{trace}((e_{\bar{k}} - E[e_{\bar{k}}])(e_{\bar{k}} - E[e_{\bar{k}}])^T)] \\ &= \text{trace} E[(e_{\bar{k}} - E[e_{\bar{k}}])(e_{\bar{k}} - E[e_{\bar{k}}])^T] \\ &= \text{trace} V_{\bar{k}}, \end{aligned}$$

the above probabilistic inequality implies

$$P^\infty\left(\|e_{\bar{k}} - E[e_{\bar{k}}]\|_2 \leq \sqrt{\frac{\text{trace} V_{\bar{k}}}{\gamma}}\right) \geq 1 - \gamma.$$

From the solution of $V_{\bar{k}}$, $\sqrt{\text{trace} V_{\bar{k}}/\gamma}$ is rewritten by

$$\begin{aligned} \sqrt{\frac{\text{trace} V_{\bar{k}}}{\gamma}} &= \sqrt{\frac{1}{\gamma} \text{trace} \left(\sum_{j=1}^{\bar{k}-1} \frac{\sigma^2}{2c_j^2 j^2} (\Phi(\bar{k}, j))^T \Phi(\bar{k}, j) \right)} \\ &= \sqrt{\frac{\sigma^2}{2\gamma} \sum_{j=1}^{\bar{k}-1} \frac{1}{c_j^2 j^2} \text{trace}((\Phi(\bar{k}, j))^T \Phi(\bar{k}, j))}. \end{aligned}$$

Similar discussion to the proof of Lemma 1 gives us

$$\text{trace}((\Phi(\bar{k}, j))^T \Phi(\bar{k}, j)) \leq \frac{jn}{\bar{k}}.$$

This estimation leads to

$$\sqrt{\frac{\text{trace} V_{\bar{k}}}{\gamma}} \leq \sqrt{\frac{\sigma^2}{2\gamma} \sum_{j=1}^{\bar{k}-1} \frac{n}{c_j^2 j \bar{k}}} = \sqrt{\frac{n\sigma^2}{2\gamma \bar{k}} \sum_{j=1}^{\bar{k}-1} \frac{1}{j^{2/3}}}.$$

We therefore conclude that the statement of the lemma as

$$\begin{aligned} \sqrt{\frac{\text{trace} V_{\bar{k}}}{\gamma}} &\leq \sqrt{\frac{n\sigma^2}{2\gamma \bar{k}} \int_1^{\bar{k}} x^{-2/3} dx} \\ &\leq \sqrt{\frac{3n\sigma^2}{2\gamma \bar{k}^{2/3}}} \\ &\leq \sqrt{\frac{3n\sigma^2}{2\gamma \tau_2^{2/3}}} = \beta. \end{aligned}$$

$\square$

Theorem 1 is immediately established by these lemmas. That is, from Lemmas 1 and 2 we can see that

$$\begin{aligned} \|x_{\bar{k}} - x^*\|_2 &= \|e_{\bar{k}}\|_2 \leq \|E[e_{\bar{k}}]\|_2 + \|e_{\bar{k}} - E[e_{\bar{k}}]\|_2 \\ &\leq \sqrt{\frac{1}{\bar{k}}} \|e_1\|_2 + \sqrt{\frac{3n\sigma^2}{2\gamma \bar{k}^{2/3}}} \\ &\leq \alpha \|e_1\|_2 + \beta = \alpha \|x_1 - x^*\|_2 + \beta \end{aligned}$$

with probability $1 - \gamma$.

Finally, we hasten to compare difficulty between FDSA and the stochastic approximation (SA). SA is a method for finding a solution of an unknown equation $f(x) = 0$ from noisy measurements of its residual. Its update rule is

$$x_{k+1} = x_k - a_k y_k,$$

where measurement y_k of residual is

$$y_k = f(x_k) + \varepsilon_k,$$

and $\varepsilon_k \in \mathbb{R}^n$ is a measurement noise. Now, when we assume that ε_k is a random variable with mean 0 and variance $\sigma^2 I$, this implies that the mean of y_k and its variance are also 0 and $\sigma^2 I$, respectively. On the other hand, we can interpret solving the optimization (1) as finding x^* which satisfies the equation

$$\frac{df}{dx}(x) = Hx + h = 0.$$

due to the first order optimality condition. That is, FDSA is a SA for finding the root of the above equation whose measurement corresponds to the gradient estimation

$$\tilde{g}_k = He_k + \frac{\varepsilon_k}{2c_k} = (Hx_k + h) + \frac{\varepsilon_k}{2c_k}.$$

Since its variance is $\sigma^2/(2c_k^2)$ and c_k is monotone decreasing sequence, the variance of the measurement is monotone increasing. We therefore see that construction of a stopping rule of FDSA is more difficult than that of SA.

4 Numerical Example

In this section, we investigate the necessary number $\bar{k}$ of iterations of the stochastic approximation. We first prepare some parameters of this algorithm. We give the dimension of x as $n = 4$ and we set the variance $\sigma^2 = 0.01$ of $\varepsilon_k^\pm$. Setting $\gamma = 0.01$, we plot τ_1 and τ_2 for each α and β on Fig. 1. The solid and the dashed lines denote τ_1 and τ_2 , respectively. From this figure, we see that τ_1 and τ_2 is a polynomial in $1/\alpha$ and $1/\beta$. Furthermore τ_1 does not become too large for each α . On the other hand, it seems that τ_2 becomes large for small β . Since we do not have to set too small β when we utilize FDSA, this issue is not important. For example, if we set $\beta = 0.1$, $\tau_2 \approx 14,697$ and it is not large.

5 Concluding Remarks

In this paper, we have developed a stopping rule for FDSA. Here we have assumed that the objective function is convex quadratic. Then, we have obtained the number of iterations for guaranteeing prespecified quality of the resultant solution. Furthermore, we have clarified that its number is bounded by a polynomial in the problem size. Even if the convex objective function is not quadratic, we may construct a stopping rule based on a similar idea, which is now under investigation.

Fig. 1: The necessary number of iterations of FDSA with $\gamma = 0.01$ and $\sigma^2 = 0.01$.

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