A Scenario Approach to Optimization Subject to Robust and Average LMIs*

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Abstract

An optimization problem with robust and average constraints is introduced, where the constraints are described as parameter-dependent linear matrix inequalities. The difficulty of the parameter dependency is solved by using the scenario approach which employs random samples of the parameter. An explicit number of random samples such that the optimal solution of the scenario problem achieves prescribed accuracy and confidence is derived, which is the main result of this paper. It is then applied to an average pole placement problem with robust stability, and a numerical example is provided.

1 Introduction

Deterministic description of uncertainty is usually used in robust control theory [1], and linear matrix inequalities (LMIs) give an efficient tool for solving several robust control problems [2]. On the other hand, since the deterministic description often results in too conservative design, an average performance optimization has been proposed in a context of robust controller design which employs a probabilistic description of uncertainty [3].

In this paper, we consider a general LMI framework for robust controller design with both deterministic and probabilistic descriptions of uncertainty. Our problem is to minimize a linear function subject to robust and average LMIs:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad F^{(r)}(x, \Delta) \prec 0, \quad \forall \Delta \in \mathcal{D} \quad (1) \\
& \quad \mathcal{E}_P \{ F^{(a)}(x, \Delta) \} \prec 0 \quad (2)
\end{align*}
\]

where \( x \in \mathbb{R}^m \) is the decision variable, \( \Delta \) is the uncertain parameter, and \( \mathcal{D} \subseteq \mathbb{R}^p \) is the uncertainty set. Notice that \( \Delta \) is a deterministic parameter in the first constraint (1), while it is regarded as a random variable according to a probability distribution \( \mathcal{P}_D \) in the second constraint (2), where \( \mathcal{E}_D \{ \cdot \} \) denotes the expectation according to \( \mathcal{P}_D \). The constraints are defined by

\[
\begin{align*}
F^{(r)}(x, \Delta) &= F^{(r)}_0(\Delta) + \sum_{k=1}^m x_k F^{(r)}_k(\Delta), \\
F^{(a)}(x, \Delta) &= F^{(a)}_0(\Delta) + \sum_{k=1}^m x_k F^{(a)}_k(\Delta)
\end{align*}
\]

where \( F^{(r)}_k : \mathcal{D} \rightarrow \mathcal{S}_n, \quad k = 0, 1, 2, \ldots, m, \) and \( F^{(a)}_k : \mathcal{D} \rightarrow \mathcal{S}_n, \quad k = 0, 1, 2, \ldots, m, \) are symmetric matrix-valued measurable functions, \( x_i, i = 0, 1, 2, \ldots, m, \) are the elements of \( x, \) and \( \mathcal{S}_n \subseteq \mathbb{R}^{n \times n} \) is the set of symmetric matrices. Here, the inequality \( A \succ B \) for symmetric matrices \( A \) and \( B \) means that \( A - B \) is positive definite.

The constraints (1) (2) are affine with respect to the decision variables \( x_i, \) and thus this type of inequalities are called LMI. We assume that \( F^{(a)}_k(\Delta), i = 0, 1, 2, \ldots, m, \) are bounded for all \( \Delta \in \mathcal{D}. \)

Since this optimization problem has both robust and average constraints defined by parameter-dependent LMIs, it can describe average performance design as well as robust performance design in systems and control, which gives a unified framework for dealing with uncertain systems. In fact, the problem can deal even with a mixed average/robust performance design such as average pole placement for uncertain systems which is recently proposed by the authors [4].

However, it should be noted that solving the optimization is quite difficult. Indeed, we need to check the robust LMI constraint (1) for all possible values of \( \Delta \in \mathcal{D}, \) while the coefficient matrices \( F^{(r)}_k(\Delta) \) may be nonlinear functions of \( \Delta. \) Similarly, we have to compute an integral with respect to \( \Delta \) on the average LMI constraint (2), while the coefficient matrices \( F^{(a)}_k(\Delta) \) may also be nonlinear functions of \( \Delta. \)

In order to tackle these difficulties, in this paper, we take a probabilistic approach [5, 6, 7]. In particular, we here use a set of random samples of the uncertain parameter \( \Delta \) instead of the random variable \( \Delta \) itself, and we construct a scenario problem by using these random samples. Since the scenario problem is a convex optimization defied by a set of standard LMI constraints, it can be solved efficiently nowadays via well-developed semi-definite programming (SDP) solvers. Then, we here come up with a fundamental issue: How many random samples are needed for obtaining a “good” so-

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solution to the original optimization problem through solving the scenario problem? This important issue is actually solved in this paper.

## 2 Scenario Approach

Let us consider the scenario problem

\[
\begin{align*}
& \text{minimize} & c^T x \\
& \text{subject to} & F^{(r)}(x, \Delta_{0i}) < 0, & i = 1, 2, \ldots, N_r \\
& & 1 \frac{1}{N_a} \sum_{j=1}^{N_a} \{F^{(a)}(x, \Delta_{\ell j})\} < 0, & \ell = 1, 2, \ldots, L \\
& & |x_k| \leq d_k, & k = 1, 2, \ldots, m
\end{align*}
\]

(3) \hspace{1cm} (4)

where \(\Delta_{0i}, i = 1, 2, \ldots, N_r\), and \(\Delta_{\ell j}, j = 1, 2, \ldots, N_a, \ell = 1, 2, \ldots, L\), are random samples of \(\Delta\) extracted according to the probability distribution \(P_D\). That is, we prepare \(N_a + N_r L\) random samples of \(\Delta\) for describing the original problem approximately.

We assume that \(x\) can be taken from a bounded set

\[ \mathcal{X} = \{ x \in \mathbb{R}^m \mid |x_k| \leq d_k, \ k = 1, 2, \ldots, m \} \]

and the optimal solution of the scenario problem exists and it is unique. Suppose that coefficient matrices of the LMIs have bounds, i.e.,

\[ \|F_k^{(a)}(\Delta)\| \leq c_k, \quad k = 0, 1, 2, \ldots, m \]

for all \(\Delta \in \mathcal{D}\). Then, we see that

\[ \|F^{(a)}(x, \Delta)\| \leq b \]

holds for all \(x \in \mathcal{X}\) and \(\Delta \in \mathcal{D}\), where

\[ b = c_0 + \sum_{k=1}^{m} c_k d_k. \]

We assume that this \(b\) is available.

Employing a version of Hoeffding’s inequality for symmetric matrices [8] as well as an idea of the scenario approach to standard robust control design [6, 7], we obtain the following theorem.

**Theorem 1.** For given \(\epsilon_r \in (0, 1), \epsilon_a \in (0, \infty), \delta_r \in (0, 1), \) and \(\delta_a \in (0, 1)\), select \(N_r \in \mathbb{N}, N_a \in \mathbb{N}, \) and \(L \in \mathbb{N}\) which satisfy

\[ N_r \geq \frac{2}{\epsilon_r} \ln \frac{1}{2 \delta_r} + 2m + \frac{2m}{\epsilon_r} \ln 4 \]

(5)

\[ N_a \geq \frac{32b^2}{\epsilon_r^2} \ln \frac{n_a(m + 1)}{\delta_a} \]

(6)

\[ L = m + 1. \]

(7)

Then, the optimal solution \(\tilde{x}\) of the scenario problem satisfies

\[ P_D(\Delta \in \mathcal{D} \mid F^{(r)}(\tilde{x}, \Delta) < 0) > 1 - \epsilon_r \]

(8)

\[ E_{\mathcal{P}_D}[F^{(a)}(\tilde{x}, \Delta)] < \epsilon_a I \]

(9)

with probability at least \(1 - (\delta_r + \delta_a)\).

**Proof.** Since the scenario problem is defined by a set of random samples, its optimal solution \(\tilde{x}\) is also a random variable. Thus, sometimes it satisfies (8) and (9), and sometimes not. Then, if the risk such that \(\tilde{x}\) does not satisfy (8) is less than \(\delta_r\) and the risk such that \(\tilde{x}\) does not satisfy (9) is less than \(\delta_a\), we can conclude the result of the theorem. In the rest of this proof, we establish these bounds of the risks under the conditions (5), (6), and (7) regarding the number of samples.

Notice first that the dimension of the decision variable \(x\) is \(m\). Thus, the number of support constraints for the scenario problem is at most \(m\), where a support constraint is a constraint such that the optimal value changes when it is removed. Then, the probabilistic certificate (8) about the robust constraint (1) is guaranteed with probability at least \(1 - \delta_r\) by employing \(N_r\) random samples with (5), which can be shown by following Corollary 6 of [7]. That is, the risk regarding (8) is less than \(\delta_r\) under the condition (5).

Secondly, we show that the risk regarding (9) is less than \(\delta_a\). When we set \(L\) as (7), at the least one of the constraints (4) cannot be a support constraint, which gives a guarantee. To evaluate it, we use Jensen’s inequality and obtain

\[ \|E_{\mathcal{P}_D}[F^{(a)}(x, \Delta)] - F^{(a)}(x, \Delta_{t\ell})\| \]

\[ \leq E_{\mathcal{P}_D}[\|F^{(a)}(x, \Delta)\|] + \|F^{(a)}(x, \Delta_{t\ell})\| \]

\[ \leq 2b \]

for any \(\Delta_{t\ell}\). That is, we see that

\[ \mathcal{P}_D \{\mathcal{E}_D[F^{(a)}(x, \Delta)] - F^{(a)}(x, \Delta_{t\ell}) = 0, \}

\[ \left(\mathcal{E}_D[F^{(a)}(x, \Delta)] - F^{(a)}(x, \Delta_{t\ell})\right)^2 \leq (2b)^2 I \]

for all \(j = 1, 2, \ldots, N_a\) and \(\ell = 1, 2, \ldots, L\). From a Hoeffding’s inequality for symmetric matrices [8], we have

\[ \mathcal{P}_D\{\lambda_{\max}[\mathcal{E}_D[F^{(a)}(x, \Delta)]] - \frac{1}{N_a} \sum_{j=1}^{N_a} F^{(a)}(x, \Delta_{t\ell})] \geq \epsilon_a\} \]

\[ \leq n_a \exp(-\frac{N_a \epsilon_a}{32b^2}) \]

for arbitrary \(t \geq 0\), where \(\lambda_{\max}[A]\) represents the maximum eigenvalue of symmetric matrix \(A\). Here, it should be noted that we cannot know which one of the constraints (4) gives the guarantee above in advance. Let us define \(\delta_a\) as an upper bound of probability such that

\[ \mathcal{P}_D[F^{(a)}(x, \Delta)] - \frac{1}{N_a} \sum_{j=1}^{N_a} F^{(a)}(x, \Delta_{t\ell}) < \epsilon_a I \]

is not satisfied. Then, considering the number of \(m + 1\) cases, we can estimate \(\delta_a\) as follows:

\[ n_a (m + 1) \exp(-\frac{N_a \epsilon_a}{32b^2}) \leq \delta_a. \]

By solving the above with respect to \(N_a\), we obtain the inequality (6). \(\square\)
This theorem means that the optimal solution of the scenario problem becomes an “almost” feasible solution of the original problem with a high probability if we use an enough number of random samples. The required number \(N_r + N_a L\) of random samples can be known in advance of execution of the optimization, and it is determined by prescribed accuracy and confidence parameters \(\epsilon_r, \epsilon_a, \delta_r, \delta_a\) of the solution. This clarifies the sample complexity of the scenario approach to optimization subject to robust and average LMIs, which is indeed an important contribution to control theory.

3 Average Pole Placement with Robust Stability

In the previous section, a general scenario approach has been established for an optimization subject to robust and average LMIs. In this section, we apply this idea to a mixed average/robust performance design, i.e., average pole placement problem with robust stability.

Let us consider an uncertain system described by

\[
\dot{x}(t) = A(\Delta)x(t) + B(\Delta)u(t), \quad \Delta \in \mathcal{D}
\]

where \(A(\Delta) \in \mathbb{R}^{n \times n}, B(\Delta) \in \mathbb{R}^{n \times p}\). \(\Delta\) is the uncertain parameter which is time invariant, and \(\mathcal{D} \subseteq \mathbb{R}^q\) is the uncertainty set. Our control law is a state feedback

\[
u(t) = Kx(t)
\]

where \(K \in \mathbb{R}^{p \times n}\) is the gain that we have to design. Then, the closed-loop system is given as

\[
\dot{x}(t) = (A(\Delta) + B(\Delta)K)x(t), \quad \Delta \in \mathcal{D}
\]

The average pole placement considered in this paper is a problem to find a gain \(K\) of (11) which robustly stabilizes the uncertain system (10) and places the poles of the average closed-loop system (12) in a designated area \(\mathcal{R}\). That is, the problem is to find \(K\) such that

\[
\max \text{Re} \lambda_i[A(\Delta) + B(\Delta)K] < 0, \quad \forall \Delta \in \mathcal{D}
\]

\[
\lambda_i[\mathcal{E}_D\{A(\Delta) + B(\Delta)K\}] \in \mathbb{C}, \quad i = 1, 2, \ldots, n
\]

where \(\lambda_i[A], i = 1, 2, \ldots, n\) denote the eigenvalues of \(A \in \mathbb{R}^{n \times n}\) and \(\text{Re}\) denotes the real part of \(\lambda \in \mathbb{C}\). Here, we define the area \(\mathcal{R}\) as an open region of the complex plane described by

\[
\mathcal{R} = \left\{ s \in \mathbb{C} \mid \left[ \begin{array}{c} 1 \\ \bar{s} \end{array} \right]^T \left[ \begin{array}{cc} r_{11} & r_{12} \\ r_{12} & r_{22} \end{array} \right] \left[ \begin{array}{c} 1 \\ s \end{array} \right] < 0 \right\}
\]

where \(\bar{s}\) denotes the conjugate of \(s\). We assume that the symmetric matrix

\[
\left[ \begin{array}{cc} r_{11} & r_{12} \\ r_{12} & r_{22} \end{array} \right] \in \mathbb{R}^{2 \times 2}
\]

has one strictly negative eigenvalue and one strictly positive eigenvalue.

We remark that the average pole placement problem defined above is a generalized version of that proposed by the authors [4]. In fact, the above problem is reduced to that of [4] if we set \(r_{11} = 2\alpha, r_{12} = 1, r_{22} = 0\), where \(\alpha \in \mathbb{R}\) is a specified nonnegative number.

Now, let us recall that \(A \in \mathbb{R}^{n \times n}\) has all its eigenvalues in \(\mathcal{R}\). This fact follows [9] for example.

Employing the above fact, we can see that the conditions (13) and (14) can be rewritten as robust and average LMIs. To this end, let us introduce a standard parameterization of the gain

\[
K = GX^{-1}
\]

where \(X \in \mathbb{R}^{n \times n}\), \(X = X^T > 0\), and \(G \in \mathbb{R}^{p \times n}\) are the decision variables. Then, the robust stability condition (13) is guaranteed if the robust LMI (1) holds with

\[
F^{(r)}(x, \Delta) \doteq L(x, \Delta) + L^T(x, \Delta)
\]

and the elements of \(x\) are composed of the independent elements of \(X\) and \(G\). This is the case that we set \(r_{11} = 0, r_{12} = 1, r_{22} = 0\) in the fact above, where the open left half region of the complex plane is considered. Similarly, the average pole placement condition (14) is guaranteed if the average LMI (2) holds with

\[
F^{(a)}(x, \Delta) \doteq M_1(x, \Delta) + M_2(x)
\]

where

\[
M_1(x, \Delta) \doteq \left[ \begin{array}{cc} r_{12}(L(x, \Delta) + L^T(x, \Delta)) & r_{12}L(x, \Delta) \\ r_{22}L^T(x, \Delta) & 0 \end{array} \right]
\]

\[
M_2(x) \doteq \left[ \begin{array}{cc} r_{11}X & 0 \\ 0 & -r_{22}X \end{array} \right].
\]

Here, we set \(r_{11}, r_{12}, r_{22}\) according to \(\mathcal{R}\) which is specified by the designer. The above definition of \(F^{(a)}(x, \Delta)\) works well if \(r_{22} \neq 0\), while we have to define \(F^{(a)}(x, \Delta)\) as its \((1, 1)\) block if \(r_{22} = 0\).

In this way, the average pole placement problem with robust stability can be represented as the robust and average LMI constraints (1) and (2). We can therefore use the scenario approach for solving this problem if we introduce an appropriate objective function.

4 Numerical Example

Let us consider an uncertain system whose coefficient matrices are

\[
A(\Delta) = \left[ \begin{array}{cc} 0 & 1 \\ \delta_1 & -\delta_2 \end{array} \right], \quad B(\Delta) = \left[ \begin{array}{cc} 0 \\ \delta_3 \end{array} \right]
\]
where $\Delta = [ \delta_1 \delta_2 \delta_3 ]$. These uncertain parameters $\delta_1$, $\delta_2$, and $\delta_3$ are assumed to be random variables according to truncated Gaussian distributions. In this numerical example, they were generated as follows. The parameters were first generated as Gaussian distributions. Their averages were set as 4.1288, 1.4348, and 4.3478, and their standard deviations were set as 0.1376, 0.0478, and 0.1449, respectively. Then, if a generated value was out of $\pm 10\%$ of the corresponding average, it was rejected. In this way, we can prepare a set of random samples from the bounded intervals within $\pm 10\%$ of the averages. Notice here that these bounds are taken from a control problem of a vertical rotating arms system, where its nonlinearity is regarded as uncertainty.

The region $\mathcal{R}$ for average pole placement is given as

$$ \mathcal{R} = \{ s = \alpha + j\beta \mid (\alpha + 3.5)^2 + \beta^2 < 3^2 \} $$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. That is, the parameters $r_{11}$, $r_{12}$, and $r_{22}$ are 3.25, 3.5, and 1, respectively. Then, following the previous section, we have the robust and average LMIs for solving the average pole placement problem with robust stability.

In order to recast the problem as the scenario problem of Section 2, we further introduce a variable $\eta \in \mathbb{R}$ and a constraint

$$ X \succ \eta I \succ 0. $$

Then, we define the decision variable $x$ as

$$ x \doteq [ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 ]^T $$

where

$$ X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \quad G = \begin{bmatrix} x_4 & x_5 \end{bmatrix}, \quad \eta = x_6. $$

The objective function follows the definition of

$$ c \doteq [ 0 \ 0 \ 0 \ 0 \ 0 \ -1 ]^T. $$

This means maximization of $\eta$, which likely leads to a low gain $K$. If we choose the parameters related to the decision variable space as $d_1 = d_4 = d_5 = 0.3$, $d_2 = 0.35$, and $d_3 = 1.5$, we obtain $b = 62$ for example.

Now, let us consider the case that we want to solve the average pole placement problem with accuracy $\epsilon_r = 0.01$ and $\epsilon_a = 0.1$ and with confidence $\delta_1 = 0.001$ and $\delta_2 = 0.099$. That is, we here require that, with probability no smaller than 90%, the resultant $K$ can stabilize the uncertain system robustly with respect to more than 99% of the uncertainty set and can place the poles of the average system in $\mathcal{R}$. Then, Theorem 1 tells us the number of random samples for achieving these accuracy and confidence. Here we select $N_r = 3 \times 10^3$, $N_a = 6.1 \times 10^4$, and $L = 7$ following this theorem.

Solving a scenario problem defined by a set of random samples whose number satisfies the conditions of the theorem, we obtained

$$ K = \begin{bmatrix} -1.7702 & -0.6646 \end{bmatrix}. $$

Fig. 1: State behavior of the resultant closed-loop system with several uncertain values.

Fig. 1 shows the behavior of the closed-loop systems with several uncertain values, where we actually see that the system behaves appropriately. Furthermore, the poles of the average closed-loop system are $-1.1099$ and $-3.2144$, which are in fact in the specified region $\mathcal{R}$. We remark that more general distribution and uncertain system can be treated as is stated in the previous section, while here we used a simple example so that the exact poles can be computed.

References


