On Convergences of Estimates Concerned with Fuzzy Random Data

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Abstract
In this paper, the author investigates the convergence properties of estimators concerned with expectations for a class of fuzzy random sets, where the fuzzy random set is considered as a model of the capricious vague perception of a crisp phenomenon or a crisp random phenomenon.

First, the class of fuzzy sets, which has been proposed by the author, is refined from the practical point of view. Secondly, using the refined class of fuzzy sets, fuzzy random sets as vague perceptions of crisp phenomena and their extended one as vague perceptions of random phenomena are also refined. The expectations of fuzzy random sets are also reviewed. Finally applying the standard strong law of large numbers(SLLN) for the random elements in a separable Banach space, the convergence property of estimators for expectations of random fuzzy sets is examined.

Keywords: fuzzy random sets, capricious vague perception, expectation, strong law of large numbers

1 Metric Spaces of Fuzzy Sets
The class of fuzzy sets adopted in this paper is inspired by the one originally proposed by Kwakernaak[1], Kruse and Meyer[2], Kruse, Gebhardt and Klawonn[3], and the revised version of previously proposed by the author(see e.g., [4, 5, 6, 7, 8]).

Let $I$ be the open interval between 0 and 1, i.e., $I = (0, 1)$ and $\bar{I} = [0, 1]$. A fuzzy set $\bar{U}(u_0)$ as the vague perception of $u_0 \in \mathbb{R}^n$ is defined by the triple

$$\bar{U}(u_0) = \left(\mathbb{R}^n, [\bar{U}(u_0)], s_{\bar{U}}\right)$$

(1a)

with

$$[\bar{U}(u_0)] = \left\{ [\bar{U}(u_0)]_\alpha : \alpha \in \bar{I} \right\},$$

(1b)

where $\mathbb{R}^n$ is the $n$-dimensional Euclidean space called the basic space; $s_{\bar{U}}$ is the predicate, i.e., $s_{\bar{U}} : \mathbb{R}^n \to S$ with $S$ the “universe of discourse” defined by a set of statements, which assigns a proposition

$$s_{\bar{U}}(u) = \left\{ u \in \bar{U} \text{ coincides with } u_0 \right\}$$

(2)
to each element $u \in \mathbb{R}^n$; and $[\bar{U}(u_0)]$ is the family of subsets of $\mathbb{R}^n$ called the set representation of $\bar{U}(u_0)$, which satisfies

$$L_\alpha \bar{U}(u_0) \subseteq [\bar{U}(u_0)]_\alpha \subseteq L_{\bar{\alpha}} \bar{U}(u_0) \text{ for any } \alpha \in I,$$

(3)

where $L_\alpha \bar{U}(u_0)$ and $L_{\bar{\alpha}} \bar{U}(u_0)$ are the strong cut(strong level set) and the level set of $\bar{U}(u_0)$ at the level $\alpha$(see e.g., fukuda[7, 8]). The crisp point $u_0$ in $\mathbb{R}^n$, the vague perception of which gives the fuzzy set $\bar{U}(u_0)$, is called the original point of $\bar{U}(u_0)$.

Hereafter, a fuzzy set $\bar{U}(u_0)$ will be abbreviated as $\bar{U}$, when no confusion occurs or no special interest in the original point $u_0$ has been concerned.

Definition 1. The family of fuzzy sets is denoted by $F_{cc}(\mathbb{R}^n)$, whose element $\bar{U} = (\mathbb{R}^n, [\bar{U}], s_{\bar{U}})$ satisfies the following conditions:

(i) Any element $[\bar{U}]_\alpha$ of the set representation $[\bar{U}]$ of a fuzzy set $\bar{U}$ satisfies

$$[\bar{U}]_\alpha \in K_{cc}(\mathbb{R}^n),$$

(4)

where $K_{cc}(\mathbb{R}^n)$ is the family of nonempty compact and convex subsets of $\mathbb{R}^n$.

(ii) $[\bar{U}]_1$ defined by

$$[\bar{U}]_1 = \bigcap_{\alpha \in I}[\bar{U}]_\alpha$$

(5)

is the element of $K_{cc}(\mathbb{R}^n)$.

(iii) $[\bar{U}]_\alpha$ does not equal to $L_{\bar{\alpha}} \bar{U}$ only at the finite points of $\alpha$, i.e.,

$$[\bar{U}]_\alpha \neq L_{\bar{\alpha}} \bar{U} \text{ at most only at } \alpha \in \bar{I}_k = \{ \alpha_1, \alpha_2, \ldots, \alpha_k \}$$

(6)

The support of $\bar{U}$ is defined by

$$[\bar{U}]_0 = \text{cl.} \left( \bigcup_{\alpha \in I} [\bar{U}]_\alpha \right).$$

(7)

Then, the subclass $F_{cc}^R(\mathbb{R}^n)$ of $F_{cc}(\mathbb{R}^n)$ is defined as follows.
Definition 2. The subfamily of fuzzy sets $F^b_{cc}(\mathbb{R}^n)$ is the family of fuzzy sets in $F_{cc}(\mathbb{R}^n)$ whose each element $\tilde{U}$ satisfies

$$[\tilde{U}]_0 \in K_{cc}(\mathbb{R}^n).$$

Definition 3. The concept of support function $sp(x,A)$ of a nonempty compact convex subset of $A \subseteq \mathbb{R}^n$(see e.g. [10]) can be usefully generalized to that for the fuzzy sets in $F_{cc}(\mathbb{R}^n)$. The support function of the fuzzy set $\tilde{U}$ is defined by

$$\tilde{sp}(\tilde{U},\alpha,x) = \begin{cases} sp(x,[\tilde{U}]_\alpha) & \text{for } \alpha \in \mathbb{I}^1 = \mathbb{I} \cup \{1\} \\ 0 & \text{for } \alpha = 0 \end{cases} \quad (8)$$

for $(\tilde{U},\alpha,x) \in F_{cc}(\mathbb{R}^n) \times \mathbb{I} \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is a unit sphere in $\mathbb{R}^n$, and the support function $sp(x,[\tilde{U}]_\alpha)$ of $[\tilde{U}]_\alpha$ is given by

$$sp(x,[\tilde{U}]_\alpha) = \sup \{(x,u) \in [\tilde{U}]_\alpha\}$$

with the inner product $(x,u) \in \mathbb{R}^n$.

The definitions 1 and 3 are revised ones of those in [5].

Lemma 1. Let $\tilde{U} \in F_{cc}(\mathbb{R}^n)$. Then, the support function $\tilde{sp}(\tilde{U},\alpha,x)$ of $\tilde{U}$ has the following properties.

1. the support function $\tilde{sp}(\tilde{U},\alpha,x)$ is continuous on $x \in \mathbb{S}^{n-1}$ at every fixed $\alpha \in \mathbb{I}$;

2. the support function $\tilde{sp}(\tilde{U},\alpha,x)$ is nonincreasing on $\alpha \in \mathbb{I}^1 = \mathbb{I} \cup \{1\}$ at every fixed $x \in \mathbb{S}^{n-1}$ and left-continuous on $\alpha \in \mathbb{I}^1 \setminus \mathbb{I}_K$ at every fixed $x \in \mathbb{S}^{n-1}$.

Let $(R, B_0(R), \mu_R)$ be the complete measure space, where $B_0(R)$ is the completion of the Borel $\sigma$-algebra $\mathcal{B}(R)$ generated by the standard topology of $\mathbb{R}$, and $\mu_R$ is the Lebesgue measure. Then, $(\mathbb{I}, B_0(\mathbb{I}), \mu_\mathbb{I})$ is the relative measure space, where $B_0(\mathbb{I})$ is the relative $\sigma$-algebra given by the restriction of $B_0(R)$, i.e.

$$B_0(\mathbb{I}) = \{ \mathbb{I} \cap B \mid B \in B_0(R) \}$$

with $\mathbb{I} \subseteq B_0(R)$, and $\mu_\mathbb{I}$ is the relative Lebesgue measure on $\mathbb{I}$ ($\mu_\mathbb{I}(\mathbb{I}) = 1$). Then, it can be shown that $\tilde{sp}(\tilde{U},\alpha,x)$ is $B_0(\mathbb{I}) \otimes B_0(\mathbb{S}^{n-1})$-measurable.

The measurability of $\tilde{U} \in F_{cc}(\mathbb{R}^n)$ concerned with the product $\sigma$-algebra $B_0(\mathbb{I}) \otimes B_0(\mathbb{S}^{n-1})$ is the base of the integrability of $\tilde{U}$. Let $p \in [1, +\infty)$. Then, we can define the integrability of $\tilde{sp}(\tilde{U},\alpha,x)$ of the order $p$ with respect to $\mu_\mathbb{I} \otimes \mu_{\mathbb{S}^{n-1}}$ by

$$\int \tilde{sp}(\tilde{U},\alpha,x)^p d\mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}}(x) < +\infty \quad (9)$$

The family of integrable fuzzy sets of the order $p$ is denoted by $F^{(p)}_{cc}(\mathbb{R}^n)$. Furthermore, when $\tilde{U} \in F^{(p)}_{cc}(\mathbb{R}^n)$, we have

$$|\tilde{sp}(\tilde{U},\alpha,x)|^p \leq \left( \max_{u \in [\tilde{U}]_0} |u| \right)^p < +\infty \quad (10)$$

for any $\alpha \in \mathbb{I}$ and $x \in \mathbb{S}^{n-1}$. Hence, it follows

$$\int |\tilde{sp}(\tilde{U},\alpha,x)|^p d\mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}}(x) < +\infty$$

for $\tilde{U} \in F^{(p)}_{cc}(\mathbb{R}^n)$, which means

$$F^{(p)}_{cc}(\mathbb{R}^n) \subset F^{(p)}_{cc}(\mathbb{R}^n) \quad \text{for } p \in [1, +\infty). \quad (11)$$

Let $\tilde{U}_1$ and $\tilde{U}_2$ be the elements of $F^{(p)}_{cc}(\mathbb{R}^n)$. If

$$\int |\tilde{sp}(\tilde{U}_1,\alpha,x) - \tilde{sp}(\tilde{U}_2,\alpha,x)|^p d\mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}}(x) = 0$$

holds, we represent this binary relation by $\tilde{U}_1 \sim \tilde{U}_2$. Then, it can be shown that $\sim$ is the equivalence relation. Hence, we can define the quotient set of $F^{(p)}_{cc}(\mathbb{R}^n)$ as follows:

$$F^{(p)}_{cc}(\mathbb{R}^n) / \sim . \quad (12)$$

We can also consider the quotient set

$$F^{(p)}_{cc}(\mathbb{R}^n) = F^{(p)}_{cc}(\mathbb{R}^n) / \sim . \quad (13)$$

It may be clear that

$$F^{(p)}_{cc}(\mathbb{R}^n) \subset F^{(p)}_{cc}(\mathbb{R}^n) \quad \text{for } p \in [1, +\infty). \quad (14)$$

The mapping $\tilde{U} \mapsto \tilde{sp}(\tilde{U},\cdot,\cdot)$ is an isomorphism of $F^{(p)}_{cc}(\mathbb{R}^n)$ onto cone of $B_0(\mathbb{I}) \otimes B_0(\mathbb{S}^{n-1})$-measurable functions, preserving the semi-linear structure,

$$\tilde{sp}(\lambda \cdot \tilde{U} + \mu \cdot \tilde{V},\cdot,\cdot) = \lambda \cdot \tilde{sp}(\tilde{U},\cdot,\cdot) + \mu \cdot \tilde{sp}(\tilde{V},\cdot,\cdot) \quad (15)$$

for $\lambda, \mu \geq 0$ and $\tilde{U}, \tilde{V} \in F^{(p)}_{cc}(\mathbb{R}^n)$.

The metric $\rho_p(\tilde{U}, \tilde{V})$ for any $\tilde{U}, \tilde{V} \in F^{(p)}_{cc}(\mathbb{R}^n)$ is defined by

$$\rho_p(\tilde{U}, \tilde{V}) = \left( \int |\tilde{sp}(\tilde{U},\alpha,x) - \tilde{sp}(\tilde{V},\alpha,x)|^p d\mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}}(x) \right)^{\frac{1}{p}} . \quad (16)$$

Proposition 1. $(F^{(p)}_{cc}(\mathbb{R}^n), \rho_p)$ is a separable metric space and a dense subset of $(F^{(p)}_{cc}(\mathbb{R}^n), \rho_p)$ for any $p \in [1, +\infty)$.

For every $p \in [1, +\infty)$, $L^p(\mathbb{I} \times \mathbb{S}^{n-1}, || \cdot ||_p)$ is a separable Banach space with respect to the measure space $(\mathbb{I} \times \mathbb{S}^{n-1}, B_0(\mathbb{I}) \otimes B_0(\mathbb{S}^{n-1}), \mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}})$ (see e.g., [111]), where $\| f \|_p$ is the usual $L^p$-norm, i.e.,

$$\| f \|_p = \left( \int f(\alpha,x)^p d\mu_\mathbb{I}(\alpha) \otimes \mu_{\mathbb{S}^{n-1}}(x) \right)^{\frac{1}{p}} \quad (17)$$

for any $f \in L^p(\mathbb{I} \times \mathbb{S}^{n-1}, || \cdot ||_p)$.

For every $p \in [1, +\infty)$, we can embed $F^{(p)}_{cc}(\mathbb{R}^n)$ isomorphically into the separable Banach $L^p(\mathbb{I} \times \mathbb{S}^{n-1}, || \cdot ||_p)$-space by the mapping defined by

$$j_{F^{(p)}_{cc}(\mathbb{R}^n)} : F^{(p)}_{cc}(\mathbb{R}^n) \rightarrow L^p(\mathbb{I} \times \mathbb{S}^{n-1}, || \cdot ||_p), \quad (18)$$
property (15) of the support function for $F$ space, where
an elementary fuzzy random set as a vague perception of the
given by the subsets of $M$. Using the embedding defined by (18)
and (19), the following proposition is obtained.

**Proposition 2.** For every $p \in [1, +\infty)$, $(F_{cc}^p(R^n), \rho_p)$ is a
capacitance separable metric space.

### 2 Fuzzy Random Sets as Models of Capricious Vague Perceptions

#### 2.1 Vague Perception of Crisp Phenomena

First, we consider that the vague perception of a crisp pheno-
nomenon fluctuates slightly but randomly by the state of a
capricious person’s mind. Hence, the fuzzy set obtained as a
vague perception of a crisp phenomenon may be some kind of
‘fuzzy random set’, i.e., it is a function of the generating
point of some sample space. Using the class of fuzzy sets $F_{cc}^p(R^n)$, the elementary fuzzy random set proposed in [7]
is refined as follows:

**Definition 4.** Let $(\Omega, \mathcal{A}, P_{u_0})$ be an elementary probability
space, where $\Omega = \{\omega_1, \omega_2, \cdots, \omega_M\}$; $\mathcal{A}$ be a
$\sigma$-algebra given by the subsets of $\Omega$; and $P_{u_0}$ is a probability
measure such that $P_{u_0}(\omega_i) > 0$ for each $i = 1, 2, \cdots, M$. Then,
an elementary fuzzy random set as a vague perception of the
original point $u_0 \in R^n$ is defined as follows:

$$
\tilde{U}(u_0, \omega) = (R^n, [\tilde{U}(u_0, \omega)], s_{\tilde{U}(u_0, \omega)}) \in F_{cc}^p(R^n)
$$

with

$$
[\tilde{U}(u_0, \omega)] = \left\{ [\tilde{U}(u_0, \omega)]_\alpha | \alpha \in \mathbb{I} \right\},
$$

where

$$
s_{\tilde{U}(u_0, \omega)}(u) = \left\{ \begin{array}{ll}
\text{in } \tilde{U}(u_0, \omega) \text{ coincides with} \\
\text{the original point } u_0
\end{array} \right\}.
$$

Then, we can rewrite (21) by

$$
\tilde{U}(u_0, \omega) = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot \tilde{U}(u_0),
$$

where $\mathbf{1}_{\omega_i}(\omega)$ is the characteristic function of $\omega_i$ given by

$$
\mathbf{1}_{\omega_i}(\omega) = \left\{ \begin{array}{ll}
1 & \text{if } \omega = \omega_i \\
0 & \text{otherwise},
\end{array} \right\}
$$

and $\tilde{U}(u_0)$ is the fuzzy set given by the triple

$$
\tilde{U}(u_0) = \left( \mathbb{R}^n, [\tilde{U}(u_0)], s_{\tilde{U}(u_0)} \right) \in F_{cc}^p(R^n).
$$

Hence, we have the relation between $[\tilde{U}(u_0, \omega)]_\alpha$ and $[\tilde{U}(u_0)]_\alpha (i = 1, 2, \cdots, M)$ such that

$$
[\tilde{U}(u_0, \omega)]_\alpha = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot [\tilde{U}(u_0)]_\alpha \quad \text{for each } \alpha \in \mathbb{I}.
$$

The measurability of $\tilde{U}(u_0, \omega)$ is given through its $\mathcal{A}$-$\mathcal{B}(u_0)$
measurability, i.e.,

$$
\tilde{U}^{-1}(u_0, \cdot)(B) \in \mathcal{A} \quad \text{for all } B \in \mathcal{B},
$$

where $\mathcal{B}(u_0)$ is a $\sigma$-algebra generated by the subsets of
$\tilde{U}(u_0) = \{\tilde{U}(u_0), \tilde{U}(u_0, \omega_1), \cdots, \tilde{U}(u_0, \omega_M)\}$.

Since $\tilde{U}(u_0, \omega)$ is a some kind of the random quantity, it
should be possible to consider its statistical moments such as
its expectation, its variance and so on. Let restrict hereafter the
admissible class $\mathfrak{A}$ of the possible random original points
to integrable $\mathcal{A}$-measurable ones given as follows:

$$
\mathfrak{A} = \left\{ u | \mathfrak{I}(u) = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot u_i, \\
and \ u_i \in \mathbb{R}^n \text{ for each } i = 1, 2, \cdots, M \right\}.
$$

**Definition 5.** Let $\tilde{U}(u_0, \omega) \in F_{cc}^p(R^n)$ be an elementary
fuzzy random set. Then, the expectation of $\tilde{U}$ is given by

$$
E_{u_0}[\tilde{U}] = \sum_{i=1}^{M} \tilde{U}(u_0) \cdot P_{u_0}(\omega_i)
$$

with its set representation given by

$$
[E_{u_0}[\tilde{U}]] = \left\{ [E_{u_0}[\tilde{U}]]_\alpha | \alpha \in \mathbb{I} \right\}
$$

and

$$
[E_{u_0}[\tilde{U}]]_\alpha = E_{u_0}[\tilde{U}]]_\alpha = \sum_{i=1}^{M} [\tilde{U}(u_0)]_\alpha \cdot P_{u_0}(\omega_i).
$$

The predicate $s_{E_{u_0}[\tilde{U}]}$ of $E_{u_0}[\tilde{U}]$ is given through

$$
s_{E_{u_0}[\tilde{U}]}(x) = \left\{ \begin{array}{ll}
x = E_{u_0}(u) \\
and \ u = u_0 \text{ for some } u \in \mathfrak{A}
\end{array} \right\}
$$

with

$$
E_{u_0}(u) = \sum_{i=1}^{M} u_i \cdot P_{u_0}(\omega_i),
$$

where $u = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot u_i$ is an element of $\mathfrak{A}$, and it should
be noted that the probability $P_{u_0}(\omega_i)$ depends on the value of
the original point $u_0$. 


Then, we have the following proposition (see [6, 7]).

**Proposition 3.** Let \( \tilde{U}(u_0, \omega) \) be an n-dimensional elementary random set, i.e., \( \tilde{U} \in \mathcal{F}_c^p(\mathbb{R}^n) \). Then, there exists a fuzzy set such that

\[
E_{u_0}[\tilde{U}] = \sum_{i=1}^{M} \tilde{U}_i(u_0) \cdot P(u_0(\omega)) \in \mathcal{F}_c^p(\mathbb{R}^n).
\]

**Remark 1.** Let \( \mathcal{X}_\alpha \) be the selection set defined by

\[
\mathcal{X}_\alpha = \left\{ u \mid u(\omega) \in \mathcal{X} \text{ and } u(\omega) \in [\tilde{U}(u_0, \omega)]_\alpha \right\}.
\]

Then, the element of the set representation of \( E_{u_0}[\tilde{U}] \) given by (32) is rewritten as follows:

\[
[E_{u_0}[\tilde{U}]]_\alpha = E_{u_0}[\tilde{U}]_\alpha = \left\{ E_{u_0}(u) \mid u \in \mathcal{X}_\alpha \right\}.
\]

**2.2 Vague Perception of Random Phenomena**

There are many crisp phenomena that are perceived vaguely as mentioned in Sec. 2.1, and also there are many crisp phenomena which are themselves randomly changed. In order to consider the vague perceptions of random phenomena, two types of randomness should be considered, one of which is the randomness due to the capricious person’s feelings and another of which is the randomness of the phenomena themselves.

The class of fuzzy random sets reviewed and refined in this subsection is an extended one of that in Sec. 2.1. Let \((\Omega_1, A_1, P_1)\) be an elementary probability space describing the randomness of capricious persons’ minds defined as \((\Omega, A, P)\) in Def. 4, and let \((\Omega_2, A_2, P_2)\) be a probability space, on which an original random point \( u_0 \in \mathbb{R}^n \) as the model of a random phenomenon is defined. Then, the extended fuzzy random set as a capricious vague perception of the origina random point \( u_0 \) is defined on \((\Omega_2, A, P)\) as

\[
\tilde{U}(\omega) = \left[ \mathbb{R}^n, [\tilde{U}(\omega), \tilde{U}^c] \right] \in \mathcal{F}_c^p(\mathbb{R}^n)
\]

with

\[
[\tilde{U}(\omega)]_\alpha = \left\{ [\tilde{U}(\omega)]_\alpha \mid \alpha \in I \right\},
\]

where \( \tilde{U}(\omega) \) modeled by the fuzzy random set \( \tilde{U}(\omega) \). \( \tilde{U}(\omega) \)

**Definition 6.** An extended fuzzy random set \( \tilde{U}(\omega) \) on \((\Omega, A, P)\) obtained as the capricious vague perception of an original random point \( u_0(\omega^{(2)}) \) on \((\Omega_2, A_2, P_2)\) is defined by

\[
\tilde{U}(\omega) = \left[ \mathbb{R}^n, \tilde{U}(\omega), \tilde{U}^c(\omega) \right] \in \mathcal{F}_c^p(\mathbb{R}^n)
\]

with

\[
\tilde{U}(\omega) = \left\{ \tilde{U}(\omega) \mid \alpha \in I \right\},
\]

where \( \tilde{U}(\omega) \) is the predicate associated with the proposition such as

\[
s_{i}(u) = \left\{ u \text{ in } \tilde{U} \text{ coincides with } u_0 \right\}.
\]

Then, we can rewrite \( \tilde{U}(\omega) \) in (38) by

\[
\tilde{U}(\omega) = \sum_{i=1}^{M} \mathbf{1}_{a(i)}(\omega) \cdot \tilde{U}_i,
\]

where \( \{ \tilde{U}_i ; i = 1, 2, \ldots, M \} \) is a collection of fuzzy random sets similar to (26) given by

\[
\tilde{U}_i = \left[ \mathbb{R}^n, [\tilde{U}_i], s_{i}(\omega) \right] \in \mathcal{F}_c^p(\mathbb{R}^n),
\]

and

\[
1_{a(1)}(\omega) = \begin{cases} 1 & \text{if } \omega \in \{ \omega^{(1)} \} \times \Omega_2 \\ 0 & \text{otherwise.} \end{cases}
\]

The measurability of \( \tilde{U} \) is given through

\[
\tilde{U}^{-1}(B) \in A = A_1 \times A_2 \text{ for any } B \in \mathcal{B},
\]

where \( \mathcal{B} \) is a \( \sigma \)-algebra generated by the subsets of \( \tilde{U} = \{ U_1, U_2, \ldots, U_M \} \), and the admissible class of possible original random points \( \mathcal{X}_\epsilon \) is assumed to be given by

\[
\mathcal{X}_\epsilon = \left\{ u \mid \text{integrable random variables on } (\Omega, A, P) \right\}.
\]

As described above, a capricious vague perception of the original random point \( u_0(\omega^{(2)}) \) is modeled by the fuzzy random set \( \tilde{U}(\omega) \) given by (38).

Applying the similar procedure as that for \( \tilde{U}(u_0) \), we can show that the expectation of a fuzzy random set \( \tilde{U}(\omega) \) may be given as follows:

**Definition 7.** Let \( \tilde{U}(\omega) \) be an extended fuzzy random set given by (38). Then, the expectation of \( \tilde{U}(\omega) \) given by

\[
E[\tilde{U}] = \left[ \mathbb{R}^n, [E[\tilde{U}]], s_{E[\tilde{U}]} \right]
\]

with

\[
[E[\tilde{U}]]_\alpha = \left\{ E[\hat{U}] \mid \alpha \in I \right\},
\]

where \( s_{E[\tilde{U}]}(x) = \left\{ x \text{ coincides with the expectation of } u_0 \right\} \),

and \( [E[\tilde{U}]]_\alpha \) is the set representation of \( E[\tilde{U}] \) given through

\[
E[[\tilde{U}]]_\alpha = \sum_{i=1}^{M} [\tilde{U}_i]_\alpha \cdot P(\omega^{(1)}),
\]

where \( P(\omega^{(1)}) \) is the marginal probability, i.e., \( P(\omega^{(1)}) = P(\omega^{(1)}), \Omega_2) \).

Let here \( \mathcal{S} \) be the sub \( \sigma \)-algebra of \( A \) consisting all cylinder sets of the form \( A = \Omega_1 \times A_2^{(2)} \) with \( A^{(2)} \in A_2 \). Then,
the conditional expectation of $\tilde{U}$ concerned with $S$ should be given as follows:

$$E[\tilde{U} \mid S] = \left( \mathbb{R}^n, [E[\tilde{U} \mid S]], s_E[\tilde{U} \mid S] \right)$$

with

$$[E[\tilde{U} \mid S]] = \left\{ E[\tilde{U} \mid S] \mid \alpha \in I \right\}.$$  

**Proposition 4.** Let $\tilde{U} = (\mathbb{R}^n, [\tilde{U}]_r)$ be an extended fuzzy random set given by (38). Then, it follows

$$E[\tilde{U}] = E[E[\tilde{U} \mid S]]$$

where $E[\tilde{U} \mid S]$ defined by (49) is given by

$$E[\tilde{U} \mid S] = \sum_{i=1}^{M} \tilde{U}_i \cdot P(\omega_{i}^{(1)} \mid S).$$

**Remark 2.** Let $\mathcal{X}_{c, \alpha}$ be the selection set defined by

$$\mathcal{X}_{c, \alpha} = \left\{ u \mid u(\alpha) \in \mathcal{X}_{c} \text{ and } u(\alpha) \in [\tilde{U}(\alpha)] \alpha \right\}.$$  

Then, the element of the set representation of $E[\tilde{U}]$ given by (48) is rewritten as follows:

$$[E[\tilde{U}]_\alpha] = E[\tilde{U}_\alpha] = \left\{ E(u) \mid u \in \mathcal{X}_{c, \alpha} \right\},$$

where $E(u)$ is the expectation of $u$ given by $E(u) = \int u(\omega)dP(\omega)$.

### 3 Convergence for Fuzzy Random Sets

**3.1 Convergence for Capricious Vague Perceptions of Crisp Phenomena**

The distribution of an elementary fuzzy random set $\tilde{U}(u_{a}, \cdot) \in \tilde{U}(u_{a})$ is a probability measure on $\tilde{U}(u_{a})$ defined by

$$P_{\tilde{U}(u_{a}, \cdot)}(B) = \int_{\tilde{U}^{-1}(u_{a}, B)}$$

for any $B \in \mathcal{B}(u_{a})$. Let $A_{\tilde{U}(u_{a}, \cdot)}$ be the $\sigma$-algebra generated by $\tilde{U}^{-1}(u_{a}, B)$, i.e.,

$$A_{\tilde{U}(u_{a}, \cdot)} = \sigma\left\{ \tilde{U}^{-1}(u_{a}, B) \mid B \in \mathcal{B}(u_{a}) \right\}.$$  

Then, fuzzy random sets $\{\tilde{U}_i(u_{a}, \cdot); i = 1, 2, \cdots\}$ are said to be independent if $\{A_{\tilde{U}_i(u_{a}, \cdot)}; i = 1, 2, \cdots\}$ are independent, and identically distributed if all $\{P_{\tilde{U}_i(u_{a}, \cdot)}; i = 1, 2, \cdots\}$ are identical, and independent identically distributed (denoted by i.i.d. simply), if they are independent and identically distributed

As stated in Sec. 1, every fuzzy set $\tilde{U} \in \mathcal{F}^p_c(\mathbb{R}^n)$ is embedded into the separable Banach space $\mathbb{L}^p(\mathcal{I} \times S^{n-1}, \| \cdot \|_p)$ by the mapping (18) and (19), i.e.,

$$j^{\mathbb{F}^p_c}_{\tilde{U}}(\mathbb{R}^n) : \mathbb{F}^p_c(\mathbb{R}^n) \rightarrow \mathbb{L}^p(\mathcal{I} \times S^{n-1}, \| \cdot \|_p),$$

$$j^{\mathbb{F}^p_c}_{\tilde{U}}(\mathbb{R}^n)(\tilde{U}) = \tilde{s}(\tilde{U}, \alpha, x)$$

with the norm $\| \cdot \|_p$ defined by (20), i.e.,

$$\left\| j^{\mathbb{F}^p_c}_{\tilde{U}}(\mathbb{R}^n)(\tilde{U}) - j^{\mathbb{F}^p_c}_{\tilde{V}}(\mathbb{R}^n)(\tilde{V}) \right\|_p = P(\tilde{U}, \tilde{V}).$$  

Then, we can make use of the standard strong law of large numbers (SLLN) in a separable Banach space.[12]

**Proposition 5.** Let $(\mathbb{B}, \| \cdot \|)$ be a separable Banach space and let $\{X_i; i = 1, 2, \cdots\}$ be a sequence of i.i.d. Borel random variable distributed as $X$ with values in $\mathbb{B}$. Then,

$$\frac{1}{N} \sum_{i=1}^{N} X_i \rightarrow E(X) \quad a.s. \quad N \rightarrow \infty$$

if and only if $E(\|X\|) < +\infty$.

Substituting $\tilde{s}(\tilde{U}_i(u_{a}, \cdot), \alpha, x)$, $\tilde{s}(\tilde{U}(u_{a}, \cdot), \alpha, x)$ for $X_i$ and $X$, respectively, and using (8), it follows that

$$\frac{1}{N} \sum_{i=1}^{N} \tilde{s}(\tilde{U}_i(u_{a}, \cdot), \alpha, x) = \tilde{s}(\frac{1}{N} \sum_{i=1}^{N} \tilde{U}_i(u_{a}, \cdot), \alpha, x)$$

and

$$E(\tilde{s}(\tilde{U}(u_{a}, \cdot), \alpha, x)) = \tilde{s}(\sum_{i=1}^{M} \tilde{U}_i(u_{a}, \cdot), \alpha, x) = \tilde{s}(\tilde{E}_{u_{a}}(\tilde{U}(u_{a}, \cdot)), \alpha, x).$$

Furthermore, we have from (9) that

$$E(\| \tilde{s}(\tilde{U}(u_{a}, \cdot), \alpha, x) \|) = E(\rho_{P}(\tilde{U}(u_{a}, \cdot), \{0\}))$$

because of $\tilde{U}_i(u_{a}) \in \mathbb{F}^p_c(\mathbb{R}^n)$ for all $i = 1, 2, \cdots, M$. Then, using (58), (59) and (20), we have

$$\left\| \tilde{s}(\frac{1}{N} \sum_{i=1}^{N} \tilde{U}_i(u_{a}, \cdot), \alpha, x) - \tilde{s}(\tilde{E}_{u_{a}}(\tilde{U}(u_{a}, \cdot)), \alpha, x) \right\|_p = P(\tilde{U}_i(u_{a}), \tilde{E}_{u_{a}}(\tilde{U}(u_{a}, \cdot))) \rightarrow 0 \quad a.s. \quad N \rightarrow \infty$$

Then, the following Proposition is newly obtained.

**Proposition 6.** Let $\{\tilde{U}_i(u_{a}, \cdot); i = 1, 2, \cdots\}$ be a sequence of i.i.d. elementary fuzzy random sets distributed as $\tilde{U}(u_{a}, \cdot)$ with its expectation $\tilde{E}_{u_{a}}(\tilde{U}(u_{a}, \cdot))$. Then, the SLLN for $\{\tilde{U}_i(u_{a}, \cdot)\}$ is given by (61).

**3.2 Convergence for Capricious Vague Perceptions of Random Phenomena**

The distribution of an extended fuzzy random set $\tilde{U}(\omega) \in \tilde{U}$ is a probability measure on $\tilde{U}$ defined by

$$P_{\tilde{U}}(B) = P(\tilde{U}^{-1}(B)).$$  

Then, substituting $e$ as follows

$$
\mathcal{A}_U = \sigma \left\{ \widetilde{U}^{-1}(B) \in \mathcal{A}_1 \otimes \mathcal{A}_2; B \in \mathcal{B} \right\}.
$$

(63)

Then, fuzzy random sets $\{ \widetilde{U}_i; i = 1, 2, \cdots \}$ are said to be independent if $\{ \mathcal{A}_{U_i}; i = 1, 2, \cdots \}$ are independent, and identically distributed if all $\{ P_{U_i}; i = 1, 2, \cdots \}$ are identical, and independently distributed, if they are independent and identically distributed.

Since $\mathcal{E}[\widetilde{U}] \in \mathbb{P}_{cc}(\mathbb{R}^n)$, and using (48) it can be rewritten as follows

$$
\mathcal{E}[\widetilde{U}] = \sum_{i=1}^{M} \tilde{U}_i \cdot P(\omega_i^{(1)}).
$$

Then, substituting $\tilde{sp}(\widetilde{U}_i, \alpha, x), \tilde{sp}(\tilde{U}, \alpha, x)$ for $X_i$ and $X$ respectively in Proposition 5, and using (8), we have

$$
\frac{1}{N} \sum_{i=1}^{N} \tilde{sp}(\tilde{U}_i, \alpha, x) = \tilde{sp} \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{U}_i, \alpha, x \right)
$$

(64)

and

$$
E(\tilde{sp}(\tilde{U}, \alpha, x)) = \sum_{i=1}^{M} \tilde{sp}(\tilde{U}_i, \alpha, x) \cdot P(\omega_i)
$$

$$
= \tilde{sp} \left( \sum_{i=1}^{M} \tilde{U}_i \cdot P(\omega_i), \alpha, x \right)
$$

$$
= \tilde{sp}(\mathcal{E}[\tilde{U}], \alpha, x).
$$

(65)

Furthermore, we have from (9) that

$$
E(\|\tilde{sp}(\tilde{U}, \alpha, x)\|) = E(\rho_p(\tilde{U}, \{0\}))
$$

$$
= \sum_{i=1}^{M} \rho_p(\tilde{U}_i, \{0\}) \cdot P(\omega_i^{(1)}) < +\infty
$$

(66)

because of $\tilde{U}_i \in \mathbb{P}_{cc}(\mathbb{R}^n)$ for all $i = 1, 2, \cdots, M$. Then, using (64), (65) and (20), we have

$$
\left\| \tilde{sp} \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{U}_i, \alpha, x \right) - \tilde{sp}(\mathcal{E}[\tilde{U}], \alpha, x) \right\|_p
$$

$$
= \rho_p \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{U}_i, \mathcal{E}[\tilde{U}] \right) \rightarrow 0 \quad \text{a.s. as } N \rightarrow \infty
$$

(67)

Then, the following Proposition is newly obtained.

**Proposition 7.** Let $\{ \tilde{U}_i; i = 1, 2, \cdots \}$ be a sequence of i.i.d. extended fuzzy random sets distributed as $\tilde{U}$ with its expectation $\mathcal{E}[\tilde{U}]$. Then, the SLLN for $\{ \tilde{U}_i \}$ is given by (67).

### 4 Conclusions

In this paper, the author has investigated the convergence properties of estimators concerned with expectations for a class of fuzzy random sets, where the fuzzy random set is considered as a model of the capricious vague perception of a crisp phenomenon or a crisp random phenomenon.

First, the class of fuzzy sets, which has been proposed by author, has been refined from the practical point of view, and it has been shown that the refined class of fuzzy sets is a complete and separable metric space and hence it can be embedded into the separable Banach space. Secondly, using the refined class of fuzzy sets, fuzzy random sets as vague perceptions of crisp phenomena and their extended one as vague perceptions of random phenomena has been refined. Since the fuzzy random sets are embedded into the separable Banach space, the convergence property of estimators for expectations of random fuzzy sets has been finally examined by applying the standard strong law of large numbers for the random elements in a separable Banach space.

### References


