1 Introduction

In this paper, we work on the pricing and hedging of look-back power options with the payoff given by

\[ Z := S_T^\alpha \left( \sup_{0 \leq t \leq T} S_t \right)^\beta \quad (\alpha, \beta > 0), \]

where \( S_t \) is the underlying stock price. When \( \beta = 0 \), \( Z \) is a usual power option, which was studied in [1]. We adopt the Black-Scholes model and describe the behavior of \( S_t \) under the risk neutral measure \( Q \) as follows.

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = S \]

\[ \Rightarrow S_t = S e^{r t + \sigma W_t}, \]

where \( r, \sigma \) and \( S \) are positive constants, \( r \) is the risk-free rate and \( \{W_t\} \) is a one-dimensional standard Brownian motion under \( Q \). Following the familiar procedure for the pricing and hedging under the Black-Scholes model, we begin with the calculation of \( E_t := E_Q[e^{-rt} Z | F_t] \), where \( F_t = \sigma(W_t; 0 \leq s \leq t) \).

2 Pricing and hedging

We define

\[ \tilde{r} := r - \frac{1}{2} \sigma^2, \quad X_t := W_t + \tilde{r} t, \quad M_t := \sup_{0 \leq s \leq T} X_s, \]

\[ H_t := e^{-\tilde{r} t} E_t = E_Q[e^{\alpha X_t + \beta M_T} | F_t], \]

\[ U_t := e^{\tilde{r} W_t - \frac{1}{2} \tilde{r}^2 t} = e^{-\tilde{r} X_t + \frac{1}{2} \tilde{r}^2 t}, \]

\[ \tilde{Q}[A] := \int_A U_T dQ \quad \text{for} \quad A \in \mathcal{F}_T, \]

\[ \tilde{\alpha} := \alpha \sigma + \tilde{r}, \quad \tilde{\beta} := \beta \sigma. \]

Remark that \( \{X_t; 0 \leq t \leq T\} \) under \( \tilde{Q} \) is a standard Brownian motion, we have the following.

\[ H_t = e^{\tilde{\alpha} X_t} E_{\tilde{Q}}[e^{\alpha (X_T - X_t)} + \beta \{M_t \vee (X_t + \sup_{s \leq t} X_s - X_t)\}] | F_t \]

\[ = e^{\tilde{\alpha} X_t} E_{\tilde{Q}}[e^{\alpha (X_T - X_t)} + \beta \{M_t \vee (X_t + \sup_{s \leq t} X_s - X_t)\}] | F_t \]

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Here for \( x \leq m, \)

\[ E_{\tilde{Q}}[e^{\tilde{\alpha} (X_T - x)} + \beta \{m \vee (x + M_T - m)\}] \]

\[ = \int_0^{2(2v-u)} e^{\tilde{\alpha} u + \beta \{2(2v-u) - (2u-v)^2\}} du \]

\[ = \int_0^{2(2v-u)} e^{\tilde{\alpha} u + \beta \{2(2v-u) - (2u-v)^2\}} du \]

\[ = I + J. \]

For the first integral \( I \),

\[ I e^{-m} \]

\[ = \int_0^{2(2v-u)} e^{\tilde{\alpha} u - \frac{u^2}{2(2v-u)}} du \]

\[ = \int_0^{2(2v-u)} e^{\tilde{\alpha} u - \frac{u^2}{2(2v-u)}} du \]

\[ = E[e^{\tilde{\alpha} N(0,T-t)}] \]

\[ = e^{\tilde{\alpha} N(2(m-x),T-t)} - e^{2(m-x)\tilde{\alpha}^2(T-t)} + \tilde{\alpha}^2(T-t) \]

where \( \Phi(z) := \int_{-\infty}^{z} \frac{-1}{\sqrt{2\pi x}} e^{-\frac{x^2}{2}} dx \) and the last equality comes form the Esscher transform immediately. On the
other hand,
\[
J e^{-\hat{\beta}z} = \int_{-\infty}^{m-x} du \int_0^\infty \frac{2(2\pi)^{1/2}}{\sqrt{2\pi(\beta - t)^2}} e^{\hat{\alpha}u + \hat{\beta}v - \frac{(u-v)^2}{2(\beta - t)}} dv
\]
\[
+ \int_{m-x}^\infty du \int_0^\infty \frac{2(2\pi)^{1/2}}{\sqrt{2\pi(\beta - t)^2}} e^{\hat{\alpha}u + \hat{\beta}v - \frac{(u-v)^2}{2(\beta - t)}} dv
\]
\[
=: \int_{-\infty}^\infty K du + \int_{m-x}^\infty L du .
\]

Here,
\[
K = \int_{m-x}^\infty \frac{1}{\sqrt{2\pi(\beta - t)}} e^{-\frac{(u-\frac{1}{2}m)^2}{2(\beta - t)}} e^{\hat{\alpha}u + \hat{\beta}v} dv
\]
\[
= \frac{1}{\sqrt{2\pi(\beta - t)}} e^{\frac{\hat{\alpha}u + \hat{\beta}v}{\beta - t}} \exp\left(\frac{u-2(m-x)+\frac{\hat{\beta}}{2}(T-t)}{\sqrt{\beta - t}}\right)
\]
\[
=: K_1 + K_2 .
\]

Replacing \(m-x\) with \(u\) in the above equation,
\[
L = \frac{1}{\sqrt{\pi(T-t)}} e^{-\frac{u^2}{2T-t}} + (\hat{\alpha} + \hat{\beta})u
\]
\[
+ \frac{\beta}{\sqrt{\pi}} e^{(\hat{\alpha} + \frac{\beta}{2})u + \frac{\beta}{2}(T-t)} \Phi\left(-u + \frac{\beta}{2}(T-t)\right)
\]
\[
=: L_1 + L_2 .
\]

When \(\hat{\alpha} + \frac{\beta}{2} = \frac{\hat{e}}{2} + \sigma(\alpha - \frac{1}{2} + \frac{\beta}{2}) = 0\) we especially denote \(K, K_1, K_2, L, L_1\) and \(L_2\) as \(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{L}, \bar{L}_1\) and \(\bar{L}_2\), respectively. Then,
\[
\int_{-\infty}^{m-x} K_1 du = e^{(2\hat{\alpha} + \hat{\beta})(m-x) + \frac{\beta}{2}(T-t)} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right) ,
\]
\[
\int_{m-x}^\infty \bar{K}_1 du = e^{\frac{\beta}{2}(T-t)} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right) ,
\]
\[
\frac{2}{\beta} e^{-\frac{\beta}{2}(T-t)} \int_{-\infty}^{m-x} K_2 du
\]
\[
= \int_{-\infty}^{m-x} \left(\frac{2(m-x)}{\alpha + \frac{\beta}{2}}\right)^{\frac{1}{2}} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right) du
\]
\[
= \frac{\beta}{\sqrt{\alpha + \frac{\beta}{2}}} \left\{ e^{\frac{\beta}{2}(T-t)} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right)
\right. 
\]
\[
- \left. e^{(\alpha + \frac{\beta}{2})(m-x)} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right) \right\}
\]
\[
\Rightarrow \int_{m-x}^\infty L_2 du
\]
\[
= \frac{2}{\beta} e^{\frac{\beta}{2}(T-t)} \int_{-\infty}^{m-x} \bar{L}_2 du = \int_{-\infty}^{m-x} \Phi\left(-m-x - \frac{\beta}{2}(T-t)\right) du
\]
\[
\Rightarrow \int_{m-x}^\infty \bar{L}_2 du = \int_{-\infty}^{m-x} \bar{K}_2 du
\]
Putting all of these together, \(E_t\) is given as follows when \(\hat{\alpha} + \frac{\beta}{2} \neq 0\).
\[ E_t = e^{-rT}S^{\alpha + \beta}H_t \]
\[ = S^{\alpha + \beta}e^{-rT - \frac{1}{2}\sigma^2(T-t)}(\alpha + \beta-X_t(I + J)) \big|_{x=X_t, m=M_t} \]
\[ \times \left\{ e^{\frac{1}{2}\sigma^2(T-t)} \Phi \left( \frac{M_t-X_t+\alpha(T-t)}{\sqrt{T-t}} \right) \right. \]
\[ - e^{2(m-z)\alpha + \frac{1}{2}\sigma^2(T-t)} \Phi \left( \frac{(M_t-z-X_t+\alpha(T-t))}{\sqrt{T-t}} \right) \]
\[ + e^{\frac{1}{2}\sigma^2} \left( \int_{m-x}^{m-z} K(x) du + \int_{m-z}^{\infty} L(x) du \right) \bigg|_{x=X_t, m=M_t}, \]
\[ g_1(t) = \frac{S^{\alpha + \beta}e^{-rT + \frac{1}{2}\sigma^2(T-t)}(\alpha + \beta)^2}{2(\alpha + \beta)(\alpha + \beta)^2}, \quad g_2(t) = \frac{S^{\alpha + \beta}e^{-rT + \frac{1}{2}\sigma^2(T-t)}(\alpha + \beta)^2}{2(\alpha + \beta)(\alpha + \beta)^2}. \]

\[ E_0 = E^Q \left[ e^{-rT}S_T \left( \sup_{0 \leq t \leq T} S_t \right)^\beta \right] \]
\[ = \frac{2\alpha + \beta}{2(\alpha + \beta)} g_1(0) \Phi(-\alpha \sqrt{T}) + \frac{2(\alpha + \beta)^2}{2(\alpha + \beta)} g_2(0) \Phi((\alpha + \beta)\sqrt{T}). \]

On the other hand, let \( \tilde{E}_t \) for \( \alpha + \beta = 0 \), \( \tilde{E}_t \) is given as follows.
\[ \tilde{E}_t = g_1(t) \frac{S^{\alpha + \beta}e^{-rT - \frac{1}{2}\sigma^2(T-t)}}{2\alpha + \beta} \Phi \left( \frac{M_t-X_t-\alpha(T-t)}{\sqrt{T-t}} \right) \]
\[ + 2g_1(t) \left( \frac{1}{2} - \alpha(X_t - M_t) + \alpha^2(T-t) \right) e^{-(\alpha + \beta)X_t} \]
\[ \times \Phi \left( \frac{X_t-\alpha(T-t)}{\sqrt{T-t}} \right) \]
\[ - 2g_2(t) \tilde{\alpha} \sqrt{\frac{T}{2\pi}} e^{-\tilde{r}X_t-\tilde{\alpha}M_t} \left( \frac{X_t-M_t^2}{\tilde{\alpha}^2} \right). \]

Remark: \( \tilde{E}_t \) and \( \tilde{E}_t \) being \( Q \)-martingales, (1) and (3) give us examples of Azéma-Yor martingales, or rather Kennedy martingales.

To order to find the hedging strategy, we calulate \( dE_t \) and \( d\tilde{E}_t \).

\[ \tilde{E}_0 = 2g_2(0)(1 + \tilde{\alpha}^2T) \Phi(-\tilde{\alpha} \sqrt{T}) - 2g_2(0)\tilde{\alpha} \sqrt{\frac{T}{2\pi}}. \]
3 Conclusions

In this paper, we dealt with pricing and hedging of lookback power options. We obtained (2) or (4) as the price formula and (5) or (6) as the hedging strategy.

References