Simulation Studies for a Class of Fuzzy Random Data

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Abstract

In this paper, the author numerically investigates the convergence properties of estimators concerned with expectations for a class of fuzzy random sets, where the fuzzy random set is considered as a model of the capricious vague perception of a crisp phenomenon or a crisp random phenomenon.

First, fuzzy random sets as vague perceptions of crisp phenomena and their extended one as vague perceptions of random phenomena are reviewed. Secondly applying the standard strong law of large numbers (SLLN) for the random elements in a separable Banach space, the convergence property of estimators for expectations of random fuzzy sets is examined, and finally they are confirmed numerically by simulation studies.

The proposed fuzzy random sets are expected to apply to the social system modelings including the mathematical finance, where many data have both the properties of vagueness and randomness.

Keywords: simulation studies, fuzzy random sets, capricious vague perception, expectation, strong law of large numbers (SLLN)

1 Vague Perception of Crisp Phenomena

First, we consider that the vague perception of a crisp phenomenon fluctuates slightly but randomly due to the state of a capricious person’s mind. Hence, the fuzzy set obtained as a vague perception of a crisp phenomenon may be some kind of ‘fuzzy random set’, i.e., it is a function of the generating point of some sample space. Using the class of fuzzy sets $\mathbb{F}_{cc}(\mathbb{R}^n)$, the elementary fuzzy random set is described as follows[1, 2]:

Definition 1. Let $(\Omega, \mathcal{A}, P_{\omega})$ be an elementary probability space, where $\Omega = \{\omega_1, \omega_2, \cdots, \omega_M\}; \mathcal{A}$ be a $\sigma$-algebra given by the subsets of $\Omega$; and $P_{\omega}$ is a probability measure such that $P_{\omega}(\omega_i) > 0$ for each $i = 1, 2, \cdots, M$. Then, an elementary fuzzy random set as a vague perception of the original point $u_0 \in \mathbb{R}^n$ is defined by

$$\overline{U}(u_0, \omega) = (\mathbb{R}^n, \{\overline{U}(u_0, \omega)\}, s_{\overline{U}(u_0, \omega)}^\delta) \in \mathbb{F}_{cc}(\mathbb{R}^n)$$

with

$$\{\overline{U}(u_0, \omega)\} = \{\overline{U}(u_0, \omega)|_{\alpha \in \mathbb{I}}\},$$

where

$$s_{\overline{U}(u_0, \omega)}^\delta(u) = \begin{cases} u \text{ in } \overline{U}(u_0, \omega) \text{ coincides with} \\ \text{the original point } u_0 \end{cases}.$$

Then, we can rewrite (1) by

$$\overline{U}(u_0, \omega) = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot \overline{U}(u_0),$$

where $\mathbf{1}_{\omega_i}(\omega)$ is the characteristic function of $\omega_i$ given by

$$\mathbf{1}_{\omega_i}(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_i \\ 0 & \text{otherwise}, \end{cases}$$

and $\overline{U}(u_0)$ is the fuzzy set given by the triple

$$\overline{U}(u_0) = (\mathbb{R}^n, \{\overline{U}(u_0)\}, s_{\overline{U}(u_0)}^\delta) \in \mathbb{F}_{cc}(\mathbb{R}^n).$$

Hence, we have the relation between $[\overline{U}(u_0, \omega)]_\alpha$ and $[\overline{U}(u_0)]_\alpha(i = 1, 2, \cdots, M)$ such that

$$[\overline{U}(u_0, \omega)]_\alpha = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot [\overline{U}(u_0)]_\alpha \text{ for each } \alpha \in \mathbb{I}.$$ (7)

The measurability of $\overline{U}(u_0, \omega)$ is given through its $\mathcal{A}$-$\mathcal{B}(u_0)$ measurability, i.e.,

$$\overline{U}^{-1}(u_0, \cdot)(\mathcal{B}) \in \mathcal{A} \text{ for all } \mathcal{B} \in \mathcal{B}(u_0),$$

where $\mathcal{B}(u_0)$ is a $\sigma$-algebra generated by the subsets of $\overline{U}(u_0) = \{\overline{U}(u_0), \overline{U}(u_0), \cdots, \overline{U}(u_0)\}$.

Since $\overline{U}(u_0, \omega)$ is a some kind of the random quantity, it should be possible to consider its statistical moments such as its expectation, its variance and so on. Let restrict hereafter the admissible class $\mathcal{F}$ of the possible random original points to integrable $\mathcal{A}$-measurable ones given as follows:

$$\mathcal{F} = \left\{ u \in \mathbb{R}^n | u(\omega) = \sum_{i=1}^{M} \mathbf{1}_{\omega_i}(\omega) \cdot u_i, \text{ and } u_i \in \mathbb{R}^n \text{ for each } i = 1, 2, \cdots, M \right\}.$$ (9)
Definition 2. Let \( \tilde{U}(u_0, \omega) \in \mathbb{F}_c^{(p)}(\mathbb{R}^n) \) be an elementary fuzzy random set. Then, the expectation of \( \tilde{U} \) is given by

\[
E_{\omega}\{\tilde{U}\} = \sum_{i=1}^{M} \tilde{U}_i(u_0) \cdot P_u(\omega_i) \tag{10}
\]

with its set representation given by

\[
\{E_{\omega}\{\tilde{U}\}\alpha \in \mathbb{R}\} = \left\{E_{\omega}\{\tilde{U}\}_\alpha \right\} \tag{11}
\]

and

\[
\{E_{\omega}\{\tilde{U}\}_{\alpha} = E_{\omega}\{\tilde{U}\}_{\alpha} = \sum_{i=1}^{M} \tilde{U}_i(u_0) \cdot P_u(\omega_i). \tag{12}
\]

The predicate \( s_{E_{\omega}\{\tilde{U}\}} \) of \( E_{\omega}\{\tilde{U}\} \) is given through

\[
s_{E_{\omega}\{\tilde{U}\}}(x) = \left\{ \begin{array}{ll}
(x = E_{\omega}(u)) \\
\text{and} (u = u_0 \text{ for some } u \in \mathbb{R})
\end{array} \right. \tag{13}
\]

with

\[
E_{\omega}(u) = \sum_{i=1}^{M} U_i \cdot P_u(\omega_i), \tag{14}
\]

where \( u = \sum_{i=1}^{M} 1_{\omega_i}(\omega) \cdot U_i \) is an element of \( \mathbb{R} \), and it should be noted that the probability \( P_u(\omega_i) \) depends on the value of the original point \( u_0 \).

Then, we have the following proposition (see [3, 4]).

Proposition 1. Let \( \tilde{U}(u_0, \omega) \) be an n-dimensional elementary random set, i.e., \( \tilde{U} \in \mathbb{F}_c^{(p)}(\mathbb{R}^n) \). Then, there exists a fuzzy set such that

\[
E_{\omega}\{\tilde{U}\} = \sum_{i=1}^{M} \tilde{U}_i(u_0) \cdot P_u(\omega_i) \in \mathbb{F}_c^{(p)}(\mathbb{R}^n). \tag{15}
\]

Remark 1. Let \( \mathbb{A} \) be the selection set defined by

\[
\mathbb{A} = \left\{ u | u(\omega) \in \mathbb{R} \right\} \text{ and } u(\omega) \in \left[\tilde{U}(u_0, \omega)\right]_{\alpha}. \tag{16}
\]

Then, the element of the set representation of \( E_{\omega}\{\tilde{U}\} \) given by (12) is rewritten as follows:

\[
\{E_{\omega}\{\tilde{U}\}_{\alpha} = E_{\omega}\{\tilde{U}\}_{\alpha} = \left\{E_{\omega}\{\tilde{U}\}_\alpha \right\} \tag{17}
\]

\[
E_{\omega}\{\tilde{U}\} = \sum_{i=1}^{M} \tilde{U}_i(u_0) \cdot P_u(\omega_i). \tag{14}
\]

The class of fuzzy random sets is generalized to that for the vague perceptions of random phenomena. Let \( (\Omega_1, \mathcal{A}_1, P_1) \) be an elementary probability space describing the randomness of capricious persons’ minds defined as \( (\Omega, \mathcal{A}, P) \) in Def.1, and let \( (\Omega_2, \mathcal{A}_2, P_2) \) be a probability space, on which an original random point \( u_0 \in \mathbb{R}^n \) as the model of a random phenomenon is defined. Then, the extended fuzzy random set as a capricious vague perception of the original random point \( u_0 \) is defined on \( (\Omega, \mathcal{A}, P) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_1 \times P_2) \) and given as follows:

Definition 3. An extended fuzzy random set \( \tilde{U}(\omega) \) on \( (\Omega, \mathcal{A}, P) \) obtained as the capricious vague perception of an original random point \( u_0(\omega^{(2)}) \) on \( (\Omega_2, \mathcal{A}_2, P_2) \) is defined by

\[
\tilde{U}(\omega) = (\mathbb{R}^n, [\tilde{U}(\omega)], s_{\tilde{U}}) \in \mathbb{F}_c^{(p)}(\mathbb{R}^n) \tag{18}
\]

with

\[
\{\tilde{U}(\omega)\} = \left\{[\tilde{U}(\omega)]_{\alpha} \right\}, \tag{19}
\]

where \( s_{\tilde{U}} \) is the predicate associated with the proposition such as

\[
s_{\tilde{U}}(u) = \left\{u \text{ in } \tilde{U} \text{ coincides with the original random point } u_0 \right\}. \tag{20}
\]

Then, we can rewrite \( \tilde{U}(\omega) \) in (18) by

\[
\tilde{U}(\omega) = \sum_{i=1}^{M} 1_{\omega^{(1)}}(\omega) \cdot \tilde{U}_i, \tag{21}
\]

where \( \{\tilde{U}_i ; i = 1, 2, \cdots, M\} \) is a collection of fuzzy random sets similar to (6)(but independent of the value of the original point) given by

\[
\tilde{U}_i = (\mathbb{R}^n, [\tilde{U}_i], s_{\tilde{U}_i}) \in \mathbb{F}_c^{(p)}(\mathbb{R}^n), \tag{22}
\]

and

\[
1_{\omega^{(1)}}(\omega) = \begin{cases} 
1 & \text{if } \omega \in \{\omega^{(1)}\} \times \Omega_2 \\
0 & \text{otherwise.}
\end{cases} \tag{22}
\]

The measurability of \( \tilde{U} \) is given through

\[
\tilde{U}^{-1}(B) \in \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \text{ for any } B \in \mathcal{B}, \tag{23}
\]

where \( \mathcal{B} \) is a \( \sigma \)-algebra generated by the subsets of \( \tilde{U} = \{\tilde{U}_1, \tilde{U}_2, \cdots, \tilde{U}_M\} \), and the admissible class of possible original random points \( \mathcal{A}_c \) is assumed to be given by

\[
\mathcal{A}_c = \left\{u | \text{integrable random variables on } (\Omega, \mathcal{A}, P) \right\} = \left\{u | u(\omega) = \sum_{i=1}^{M} 1_{\omega^{(1)}}(\omega) \cdot \xi(\omega^{(2)}); \right. \]

\[
\xi \text{ is the integrable on } (\Omega_2, \mathcal{A}_2, P_2) \right\}. \tag{24}
\]

Applying the similar procedure as that for \( \tilde{U}(u_0, \cdot) \), we can show that the expectation of a fuzzy random set \( \tilde{U} \) may be given as follows:
Definition 4. Let $\bar{U} = (\mathbb{R}^n, [\bar{U}], s_{\bar{U}})$ be an extended fuzzy random set given by (18). Then, the expectation of $\bar{U}$ is given by

$$E[\bar{U}] = \left( \mathbb{R}^n, [E[\bar{U}]], s_{E[\bar{U}]} \right)$$

with

$$[E[\bar{U}]] = \left\{ E[[\bar{U}]_{\alpha}] \ : \ \alpha \in I \right\},$$

where $s_{E[\bar{U}]}$ is the predicate associated with the proposition given by

$$s_{E[\bar{U}]}(x) = \left\{ x \text{ coincides with the expectation of } u_\alpha \right\},$$

and $[E[\bar{U}]]$ is the set representation of $E[\bar{U}]$ given through

$$E[[\bar{U}]_{\alpha}] = \int \sum_{i=1}^M [[\bar{U}]_{\alpha}]_i \cdot P(\omega_i^{(1)}, d\omega_i^{(2)})$$

$$= \int \sum_{i=1}^M [[\bar{U}]_{\alpha}]_i \cdot P(\omega_i^{(1)}),$$

where $P(\omega_i^{(1)})$ is the marginal probability, i.e., $P(\omega_i^{(1)}) = P(\omega_1^{(1)}, \omega_2^{(1)}).$

Let here $\mathcal{S}$ be the sub $\sigma$-algebra of $\mathcal{A}$ consisting all cylinder sets of the form $A = \Omega_1 \times A^{(2)}$ with $A^{(2)} \in \mathcal{A}_2.$ Then, the conditional expectation of $\bar{U}$ concerned with $\mathcal{S}$ should be given as follows:

$$E[\bar{U}|\mathcal{S}] = \left( \mathbb{R}^n, [E[\bar{U}]|\mathcal{S}], s_{E[\bar{U}]|\mathcal{S}} \right)$$

with

$$[E[\bar{U}]|\mathcal{S}] = \left\{ E[[\bar{U}]|\mathcal{S}]_{\alpha} \ : \ \alpha \in I \right\}.$$

Proposition 2. Let $\bar{U} = (\mathbb{R}^n, [\bar{U}], s_{\bar{U}})$ be an extended fuzzy random set given by (18). Then, it follows

$$E[\bar{U}] = E[E[\bar{U}]|\mathcal{S}]$$

where $E[\bar{U}]|\mathcal{S}$ defined by (29) is given by

$$E[\bar{U}]|\mathcal{S}] = \sum_{i=1}^M \bar{U}_i \cdot P(\omega_i^{(1)}|\mathcal{S}).$$

Remark 2. Let $\mathfrak{U}_{\epsilon, \alpha}$ be the selection set defined by

$$\mathfrak{U}_{\epsilon, \alpha} = \left\{ u \ : \ |u(\omega)| \in \mathcal{U}_i, \ and \ u(\omega) \in [\bar{U}(\omega)]_\alpha \right\}. $$

Then, the element of the set representation of $E[\bar{U}]$ given by (28) is rewritten as follows:

$$[E[\bar{U}]|\mathcal{S}] = \left\{ E(u) \ : \ u \in \mathfrak{U}_{\epsilon, \alpha} \right\},$$

where $E(u)$ is the expectation of $u$ given by $E(u) = \int u(\omega) dP(\omega).$

3 Convergence for Fuzzy Random Sets

The distribution of an elementary fuzzy random set $\bar{U}(u_\cdot) \in \bar{U}(u_\cdot)$ is a probability measure on $\bar{U}(u_\cdot)$ defined by

$$P_{\bar{U}(u_\cdot)}(B) = P(U^{-1}(u_\cdot); B)$$

for any $B \in \mathcal{B}(u_\cdot).$ Let $\mathcal{A}_{\bar{U}(u_\cdot)}$ be the $\sigma$-algebra generated by $U^{-1}(u_\cdot), B),$ i.e.,

$$\mathcal{A}_{\bar{U}(u_\cdot)} = \sigma \left\{ U^{-1}(u_\cdot); B \in \mathcal{B}(u_\cdot) \right\}.$$ (36)

Then, fuzzy random sets $\{\bar{U}(u_\cdot); i = 1, 2, \cdots \}$ are said to be independent if $\{\mathcal{A}_{\bar{U}(u_\cdot)}; i = 1, 2, \cdots \}$ are independent, and identically distributed (i.i.d.) if all $\{P_{\bar{U}(u_\cdot)}; i = 1, 2, \cdots \}$ are identical, and independent identically distributed (denoted by i.i.d. simply), if they are independent and identically distributed.

Then, it can be shown that every elementary fuzzy set $\bar{U} \in \mathbb{F}_p(\mathbb{R}^n)$ is embedded into the separable Banach space $L_p(\mathbb{F} \times \mathbb{R}^n, \| \|)$ by the mappings as follows[1, 2]:

$$j_{p_{\mathcal{F}(\mathbb{R}^n)}} : \mathbb{F}_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{F} \times \mathbb{R}^n, \| \|),$$

$$j_{p_{\mathcal{F}(\mathbb{R}^n)}}(\bar{U}) = \tilde{s}_p(\bar{U}, \alpha, x)$$

with the norm $\| \|_p$

$$\left\| j_{p_{\mathcal{F}(\mathbb{R}^n)}}(\bar{U}) - j_{p_{\mathcal{F}(\mathbb{R}^n)}}(\bar{V}) \right\|_p = \rho_p(\bar{U}, \bar{V}).$$ (39)

Then, we can make use of the standard strong law of large numbers (SLLN) in a separable Banach space[5].

Proposition 3. Let $(B, \| \|)$ be a separable Banach space and let $\{X_i; i = 1, 2, \cdots \}$ be a sequence of i.i.d. Borel random variables distributed as $X$ with values in $B.$ Then,

$$\left\| \frac{1}{N} \sum_{i=1}^N X_i - E(X) \right\|_p \rightarrow 0 \ \text{a.s. as } N \rightarrow \infty$$ (40)

if and only if $E(||X||) < +\infty.

Substituting $\tilde{s}_p(\bar{U}(u_\cdot), \alpha, x), \tilde{s}_p(\bar{U}(u_\cdot), \alpha, x)$ for $X_i$ and $X$ respectively, we have

$$\frac{1}{N} \sum_{i=1}^N \tilde{s}_p(\bar{U}(u_\cdot), \alpha, x) = \tilde{s}_p \left( \frac{1}{N} \sum_{i=1}^N \bar{U}(u_\cdot), \alpha, x \right)$$ (41)

and

$$E(\tilde{s}_p(\bar{U}(u_\cdot), \alpha, x)) = \tilde{s}_p(E_{\mathcal{A}}([\bar{U}(u_\cdot)], \alpha, x)).$$ (42)

Furthermore, we have

$$E(||\tilde{s}_p(\bar{U}(u_\cdot), \alpha, x)||) = \sum_{i=1}^M \rho_p(\bar{U}_i, (0)) \cdot P_{\mathcal{U}_i}(\omega_i) < +\infty$$ (43)

because of $\bar{U}_i(u_\cdot) \in \mathbb{F}_p(\mathbb{R}^n)$ for all $i = 1, 2, \cdots, M.$ Then, it follows

$$\rho_p \left( \frac{1}{N} \sum_{i=1}^N \bar{U}_i, E_{\mathcal{U}_i}([\bar{U}(u_\cdot)], \alpha, x) \right) \rightarrow 0 \ \text{a.s. as } N \rightarrow \infty$$ (44)

Hence, the following Proposition is obtained[1, 2].
Proposition 4. Let \( \{\tilde{U}(u_i); i = 1, 2, \cdots \} \) be a sequence of i.i.d. elementary fuzzy random sets distributed as \( \tilde{U}(u_i) \) with its expectation \( E[\tilde{U}(u_i)] \). Then, the SLLN for \( \{\tilde{U}(u_i); i \} \) is given by (44).

The distribution of an extended fuzzy random set \( \tilde{U}(\omega) \in \tilde{U} \) is a probability measure on \( \tilde{U} \) defined by

\[
P_{\tilde{U}}(B) = P(\tilde{U}^{-1}(B))
\]

for any \( B \in \mathcal{B} \). Let \( \mathcal{A}_{\tilde{U}} \) be the \( \sigma \)-algebra generated by \( \tilde{U}^{-1}(B) \), i.e.,

\[
\mathcal{A}_{\tilde{U}} = \sigma\{\tilde{U}^{-1}(B) \in \mathcal{A}_1 \otimes \mathcal{A}_2; B \in \mathcal{B}\}.
\]

Then, as same as that for the elementary fuzzy random sets, the extended fuzzy random sets \( \{\tilde{U}; i = 1, 2, \cdots \} \) are said to be independent if \( \{\mathcal{A}_{\tilde{U}}; i = 1, 2, \cdots \} \) are independent, and identically distributed(i.i.d.) if all \( \{P_{\tilde{U}}; i = 1, 2, \cdots \} \) are identical, and independent identically distributed, if they are independent and identically distributed. Furthermore the similar discussion as that for the i.i.d. elementary fuzzy random sets is possible, and the following proposition is obtained:

Proposition 5. Let \( \{\tilde{U}; i = 1, 2, \cdots \} \) be a sequence of i.i.d. extended fuzzy random sets distributed as \( \tilde{U}(\cdot) \) with its expectation \( E[\tilde{U}(\cdot)] \). Then, the SLLN for \( \{\tilde{U}; i \} \) is given by

\[
\rho_p\left(\frac{1}{N} \sum_{i=1}^{N} \tilde{U}^i, E[\tilde{U}(\cdot)]\right) \to 0 \ a.s. \ as \ N \to \infty \quad (47)
\]

### 4 Numerical Examples for Vague Perception of Random Phenomena

Since the simulation results for vague perceptions of crisp phenomena have already been shown in [1], those only for the vague perceptions of random phenomena are shown here, which are new results never presented anywhere.

#### 4.1 Approximation of Set Representation by Step-wise Levels

The set representation of an extended fuzzy random set as the vague perception of a random phenomenon, which is given by (18), is approximated by the step-wise membership levels, i.e.,

\[
[\tilde{U}(\omega)]_{a_k} = \left\{ x \in \mathbb{R}^n \wedge ((\tilde{U}(\omega))(x) \geq a_k) \right\}
\]

for \( k = 1, 2, \cdots, L \), and

\[
[\tilde{U}(\omega)]_{a_0} = [\tilde{U}(\omega)] = \text{cl.} \left\{ x \in \mathbb{R}^n \wedge ((\tilde{U}(\omega))(x) > 0) \right\}
\]

satisfying

\[
[\tilde{U}(\omega)]_1 = [\tilde{U}(\omega)]_{a_1} \subseteq [\tilde{U}(\omega)]_{a_2} \subseteq \cdots \subseteq [\tilde{U}(\omega)]_{a_L} \subseteq [\tilde{U}(\omega)]_{a_0} = [\tilde{U}(\omega)].
\]

Hence, the set representations of \( \{\tilde{U}_j; j = 1, 2, \cdots, M\} \) in (21) are also approximated by the step-wise membership levels, i.e.,

\[
[\tilde{U}_j] = \left\{ [\tilde{U}_j]_{a_k} \right\} = \left\{ x \in \mathbb{R}^n \wedge ((\tilde{U}_j)(x) \geq a_k) \right\}
\]

for \( k = 1, 2, \cdots, L \), and

\[
[\tilde{U}_j]_{a_0} = [\tilde{U}_j] = \text{cl.} \left\{ x \in \mathbb{R}^n \wedge ((\tilde{U}_j)(x) > 0) \right\}
\]

satisfying

\[
[\tilde{U}_j]_1 = [\tilde{U}_j]_{a_1} \subseteq [\tilde{U}_j]_{a_2} \subseteq \cdots \subseteq [\tilde{U}_j]_{a_L} \subseteq [\tilde{U}_j]_{a_0} = [\tilde{U}_j].
\]

Let \( \{\tilde{U}^i(\cdot); i = 1, 2, \cdots\} \) be a sequence of i.i.d. elementary fuzzy random sets distributed as \( \tilde{U}(\cdot) \) with its expectation \( E[\tilde{U}(\cdot)] \). Then, using the support function and the metric \( \rho_p \) for fuzzy sets, we have

\[
\rho_p\left(\frac{1}{N} \sum_{i=1}^{N} \tilde{U}^i, E[\tilde{U}(\cdot)]\right) = \left( \sum_{k=1}^{K} \Delta \alpha_k \int_{x_{a_k}}^{x_{a_{k-1}}} \frac{1}{N} \sum_{i=1}^{N} \text{sp} \left( x, [\tilde{U}^i]_{a_k} \right) - \text{sp} \left( x, E[\tilde{U}(\cdot)]_{a_k} \right) \right)^p d\mu_{x_{a_k}}(x) \right)^{\frac{1}{p}},
\]

where \( \Delta \alpha_k = a_k - a_{k-1} \) (\( k = 1, 2, \cdots, L \)).

#### 4.2 Simulation Studies

As stated in Sec. 1, \((\Omega_1, \mathcal{A}_1, P_1)\) is an elementary probability space describing the randomness of capricious persons’ minds, and \((\Omega_2, \mathcal{A}_2, P_2)\) is a probability space, on which the original random point \( u_0 \in \mathbb{R}^n \) as the model of a random phenomenon is defined. Then, the extended fuzzy random set as a vague perception of the original random point \( u_0 \) is defined on \((\Omega, \mathcal{A}, P) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_1 \times P_2)\) and given by (18). Hence, relabeling \( i = 1, 2, \cdots, M \) by \( i = -[M/2], -1, 0, 1, \cdots, [M/2] \) in (21) \((M \text{ is assumed to be an odd number without loss of generality})\), we can rewrite (18) by

\[
\tilde{U}(\omega) = \sum_{i=-[M/2]}^{[M/2]} I_{\omega_0}^{\mathcal{O}_1}(\omega_2) \cdot \tilde{U}_i.
\]

In simulation studies, the original random point \( u_0(\omega_2) \in \mathbb{R} \) is assumed to be a Gaussian random variable(i.e., \( n = 1 \))
with mean $m_u$, the variance $\sigma_u$ and the cut-off for the sufficiently large values of $|u_0|$, and its realization is recognized to be in some interval $I_j(f = 0, \pm 1, \pm 2, \cdots)$ such that

$$I_j = \left[ \frac{2j - 1}{4} \sigma_u, \frac{2j + 1}{4} \sigma_u \right]. \quad (56)$$

The subset $A_j$ be given by

$$A_j^{(2)} = \{ \omega^{(2)} \in \Omega_j | u_0(\omega^{(2)}) \in I_j \} \quad (57)$$

and

$$A_j = \Omega_1 \times A_j^{(2)} \quad (58)$$

Then, the sub $\sigma$-algebra $S \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ given in Sec. 2 is generated by the cylinder sets of the form $\{A_j | j = 0, \pm 1, \pm 2, \cdots, \pm [M/2]\}$. The conditional probability $P(B|S)$ is given by

$$P(B|S) = \sum_{j=-[M/2]}^{[M/2]} \mathbf{1}_{A_j}(\omega) \cdot \frac{P(B \cap A_j)}{P(A_j)} \quad (59)$$

for $\forall B \in \mathcal{A}$. Hence, if $\omega \in A_j$, it follows

$$P(B|S) = P(B|A_j) = \frac{P(B \cap A_j)}{P(A_j)}. \quad (60)$$

Let $B_i$ be given by

$$\Omega = \bigcup_i B_i \quad \text{and} \quad B_i = \{\omega^{(1)}\} \times \Omega_2. \quad (61)$$

Then, for $\omega \in A_j$, we have

$$E[ \tilde{U}_j | S](\omega) = E \left[ \sum_i \tilde{U}_i \cdot 1_{\omega^{(1)}}(\omega) | S \right] \quad (62)$$

The set representation of $\tilde{U}_i$ is assumed to be given by

$$\{\tilde{U}_i\}_{n_i} = \left[ a_i - \frac{\ell_i}{\pi} \cos^{-1}(2\sigma_k - 1), a_i + \frac{\ell_i}{\pi} \cos^{-1}(2\sigma_k - 1) \right].$$

(63)

The reconstructed membership function of $\tilde{U}_i$ from (63) is given by

$$\tilde{U}_i(x) = \begin{cases} \frac{1}{2} \left( \cos \left( \frac{\pi}{\ell_i} \left( x - a_i \right) + 1 \right) \right) & \text{for } a_i - \ell_i \leq x \leq a_i \\ \frac{1}{2} \left( \cos \left( \frac{\pi}{\ell_i} \left( x - a_i \right) + 1 \right) \right) & \text{for } a_i \leq x \leq a_i + \ell_i \\ 0 & \text{otherwise,} \end{cases}$$

(64)

and its shape is illustrated in Fig. 1.

In order to describe the capricious mind of each individual, which may fluctuates slightly but randomly, the conditional probabilities $P(B_i|A_j)$ $i = 0, \pm 1, \pm 2, \cdots$ in (62) are assumed to be given by

$$P(B_{j-2}|A_j) = 0.3, \quad P(B_{j-1}|A_j) = 0.2, \quad P(B_j|A_j) = 0.2, \quad P(B_{j+1}|A_j) = 0.2, \quad P(B_{j+2}|A_j) = 0.1 \quad (65)$$

at each $j$, and otherwise, $P(B_i|A_j) = 0$. The number of data $N$ is set as $N = 100$, the number of levels $L$ is set as $L = 25$, and $\Delta \alpha_k$ in (54) is given by $\Delta \alpha_k = 1/L = 0.04$. Furthermore, the mean $m_u$ and the variance $\sigma_u$ of the original random point $u_0$ are set as $m_u = 0$ and $\sigma_u = 10$, and the set of parameters $(a_i, \ell_i, r_i)$ in (63) are assumed to be given by

$$a_i = \frac{i}{4} \sigma_u, \quad \ell_i = \frac{3}{4} \sigma_u, \quad r_i = \frac{3}{4} \sigma_u. \quad (66)$$

Then, using (62) and (65), the conditional expectation of $\tilde{U}$ at $\omega \in A_j$ is given by

$$E[ \tilde{U}_i | S](\omega) = 0.3 \tilde{U}_{j-2} + 0.2 \tilde{U}_{j-1} + 0.2 \tilde{U}_j + 0.2 \tilde{U}_{j+1} + 0.1 \tilde{U}_{j+2} \quad (67)$$

with its set representation

$$[E[ \tilde{U}_i | S](\omega)]_{n_i} = \{E[ \tilde{U}_i | S](\omega) \}_{n_i} | k = 1, 2, \cdots, L \} \quad (68)$$

and

$$[E[ \tilde{U}_i | S](\omega)]_{n_i} = E[ \tilde{U}_i | A_j]_{n_i} | \omega = 0.3 \tilde{U}_{j-2} + 0.2 \tilde{U}_{j-1} + 0.2 \tilde{U}_j + 0.2 \tilde{U}_{j+1} + 0.1 \tilde{U}_{j+2} \quad (69)$$

Then, the expectation of $\tilde{U}$ is given by

$$E[ \tilde{U} ] = E[E[ \tilde{U} | S]] = \sum_{j=-[M/2]}^{[M/2]} E[ \tilde{U} | S](\omega) P(A_j). \quad (70)$$

with

$$P(A_j) = \frac{1}{\sqrt{2\pi \sigma_u}} \int_{\ell_i}^{\ell_i + \ell_i} \exp \left\{ -\frac{(x - m_u)^2}{2\sigma^2} \right\} dx. \quad (71)$$

The shapes of the membership functions of $\tilde{U}_i(i = 0, \pm 1, \pm 2, \cdots)$ and $E[ \tilde{U} ]$ are given in Fig. 2.

The membership functions of the estimated expectations of $E[ \tilde{U} ]$ are depicted in Fig. 3, where the black lines denote the estimated ones at every 5 data whereas the red lines denote the true one depicted only for the convenience of the comparisons between the estimated and true ones.
Since the support function of the closed interval $[a, b]$ is given by
\[ \text{sp}(x, [a, b]) = \begin{cases} \frac{a}{b} & \text{if } x = -1 \\ b & \text{if } x = 1, \end{cases} \]
the support functions of $\bar{U}_i$ are given by
\[ \text{sp}(x, [\bar{U}_i, a_k], x) = \begin{cases} \text{sp}(x, [\bar{U}_i, a_k]) & \text{for } k = 1, 2, \ldots, L \\ 0 & \text{for } k = 0, \end{cases} \quad (72) \]
and
\[ \text{sp}(x, [\bar{U}_i, a_k]) = \begin{cases} \left( a_i - \frac{a_k}{\pi} \cos^{-1}(2a_k) - 1 \right) & \text{for } x = -1 \\ \left( a_i + \frac{r_i}{\pi} \cos^{-1}(2a_k) + 1 \right) & \text{for } x = 1, \end{cases} \quad (73) \]
Furthermore, the estimation error (47) with $p = 1$ can be rewritten by
\[
\rho_1 \left\{ \frac{1}{N} \sum_{i=1}^{N} \bar{U}_i, \mathcal{E}[\bar{U}] \right\} 
= \frac{1}{L \times 2} \sum_{k=1}^{L} \left[ \left| \frac{1}{N} \sum_{i=1}^{N} \text{sp}(x, [\bar{U}_i, a_k]) - \text{sp}(x, [\mathcal{E}[\bar{U}], a_k]) \right| \right] 
+ \frac{1}{N} \sum_{i=1}^{N} \left| \text{sp}(x, [\bar{U}_i, a_k]) - \text{sp}(x, [\mathcal{E}[\bar{U}], a_k]) \right| \quad (74) \]

Then, using (67), (70), (71), (72), (73) and (74), the estimation error $\rho_1$ between the estimated expectation of the fuzzy random set $\bar{U}$ and true one is calculated. The simulation result up to $N = 100$ is depicted in Fig. 4.

**Fig. 4:** Estimation errors $\rho_p$ (20 sample paths)

**References**


