Stochastic Optimal Tracking with Preview for Linear Continuous-Time Markovian Jump Systems by Output Feedback

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Abstract

The author has already presented stochastic optimal tracking control theory with preview for linear continuous-time Markovian jump systems on the finite time interval by state feedback ([21]). He has also presented optimal estimation theory for linear continuous-time Markovian jump systems on the finite time interval ([22]). In this paper we study stochastic optimal tracking control theory with preview for linear continuous-time Markovian jump systems on the finite time interval by output feedback. The necessary and sufficient conditions for the solvability of the stochastic optimal tracking problem with preview by state feedback are given by coupled Riccati differential equations with terminal conditions. In order to decide the gains of output feedback controllers, we need solutions of another type of coupled Riccati differential equations with initial conditions.

Key Words: Markovian jump systems; stochastic optimal tracking with preview; Coupled Riccati differential equations; Coupled preview compensators; Output feedback

1 Introduction

It is well known that, for design of tracking control systems, preview information of reference signals is very useful for improving the performance of the closed-loop systems, and much work has been done for preview control systems ([2, 8, 9, 12, 13, 14, 15, 16, 20, 21, 23]). Particularly, Shaked et al. have presented H∞ tracking control theory with preview for continuous- and discrete- time linear time-varying systems by a game theoretic approach ([2, 23]). It is also very important to consider the effects of stochastic noise or uncertainties for tracking control systems, and so, by Gershon et al, the theory of stochastic H∞ tracking with preview has been presented for linear continuous- and discrete-time systems respectively ([8, 9]). Recently the author has extended their theory to linear impulsive systems ([14, 15]). All these research results are concerned with the no mode transition cases. Tracking problems with preview for systems with mode transition are also very important issues to research.

Markovian jump systems ([1, 3, 4, 5, 6, 7, 11, 12, 16, 17, 20, 21, 24]) have abrupt random mode changes in their dynamics. The mode changes follow Markov processes. Such systems may be found in the area of power systems, manufacturing systems, communications, aerospace systems, financial engineering and so on. Such systems are classified into continuous-time ([1, 4, 6, 7, 11, 12, 21, 24]) and discrete-time ([3, 5, 16, 17, 20]) systems. The optimal and H∞ control theory has presented for each of these systems respectively ([3, 5, 11, 24]).

The author has presented the noncausal optimal tracking control theory for linear switched systems ([13]), which are a type of hybrid systems. However at this stage he didn’t consider any stochastic effects between the subsystems. Recently the author has presented the stochastic optimal and H∞ tracking theory with preview for linear continuous- and discrete-time Markovian jump systems by state feedback ([12, 16, 20, 21]). He has also presented the direct derivation method of preview and noncausal compensator dynamics for them ([18, 19]) by stochastic variational calculus. Moreover he has partially presented the stochastic optimal tracking theory with preview for discrete-time Markovian jump systems by output feedback ([17]) with an LMMSE filter. However, in the case that the measured output is only partially observed for continuous-time Markovian jump systems, the stochastic optimal tracking theory with preview have not been yet investigated.

In this paper we study the stochastic optimal tracking problems with preview by output feedback for a class of linear continuous-time Markovian jump systems on the finite time interval, in which some effects of the stochastic mode transitions are considered. The author has already presented the necessary and sufficient conditions for the solvability of the stochastic optimal tracking problems by state feedback and given the control strategies for them ([21]). The necessary and sufficient conditions for the solvability of the stochastic optimal tracking problem with preview by state feedback are given by coupled Riccati differential equations with terminal conditions. The solutions of them are used to decide the state feedback gains. In order to solve the output feedback problem, we introduce another type of coupled Riccati differential equations with initial conditions. The solutions of them are used to decide the
output feedback gains. In order to solve the stochastic optimal tracking control problems, we reduce the original tracking control problems to the optimal state estimation and consider the design problems of filter gains. In order to design the output feedback controllers we introduce a separation principle including noncausal information of reference signals, which shows that we can design the optimal tracking input and the filter gains independently. The forms of dynamic controllers by output feedback are uncoupled with each other because of reduction of the original tracking control problems to the causal filtering problems. However, corresponding to the coupling of the Riccati differential equations, compensators introducing future information are coupled with each other. Therefore the whole tracking systems are coupled with each other because of the effects of the stochastic mode transitions. It is a very important point in the stochastic optimal preview tracking theory because the coupling of the Riccati differential equations, compensators introducing future information are coupled with each other.

Notations: Throughout this paper the superscript \( ^{\prime} \) stands for the matrix transposition, \( \| \cdot \| \) denotes the Euclidean vector norm and \( \| v \|^2 \) also denotes the weighted norm \( v^T R v \). \( O \) denotes the matrix with all zero components.

2 Problem formulation

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and, on this space, consider the following linear continuous-time system with reference signal and Markovian mode transitions:

\[
dx(t) = A(m_t)x(t)dt + B_2(m_t)u(t)dt + B_3(m_t)v(t)dt,
\]

\[
x(0) = x_0, \quad m_0 = i_0
\]

\[
z_c(t) = C_1(m_t)x(t) + D_{12}(m_t)u(t) + D_{13}(m_t)v(t)
\]

\[
dy(t) = C_2(m_t)x(t) + v(t)dt
\]

where \( x \in \mathbb{R}^n \) is the state, \( \omega \in \mathbb{R}^p \) is the exogenous disturbances, \( v \in \mathbb{R}^k \) is the measurement noise, \( u \in \mathbb{R}^m \) is the control input, \( z_c \in \mathbb{R}^{k_c} \) is the controlled output, \( y \in \mathbb{R}^r \) is the measured output and \( r_c(\cdot) \in \mathbb{R}^{r_c} \) is a known or measurable reference signal. \( x_0 \) is an initial state with the Gaussian distribution and \( i_0 \) is a given initial mode. \( \omega \) and \( v \) are white Gaussian noise processes satisfying the assumption A1 described below.

\( \{m_t\} \) is a homogenous Markov process with right continuous trajectories and taking values on the finite set \( \phi = \{1, 2, \ldots, N\} \) with the following stationary transition probabilities:

\[
P\{m_{t+\Delta} = j | m_t = i\} = \left\{ \begin{array}{ll}
\pi_{ij} \Delta + o(\Delta) & i \neq j \\
1 + \pi_{ii} \Delta + o(\Delta) & i = j
\end{array} \right.
\]

where \( \pi_{ij} \geq 0 \) is the transition rate from the state \( i \) to \( j, \ i \neq j \), and \( \pi_{ii} = -\sum_{j=1, j \neq i}^N \pi_{ij} < 0 \). Let \( \Pi = \{[\pi_{ij}]\} \) be the transition matrix of \( m_t \) and define \( p_i(t) := P\{m_t = i\} > 0 \) for any \( i \in \phi \). Then \( P_t := \{p_1(t), \ldots, p_N(t)\}' \) satisfies the Kolmogorov forward differential equations \( dP_t/dt = \Pi P_t \) for a given initial probability \( P_0 \) ([4, 7]). Corresponding to each mode \( i \), we define \( A_i := A(m_t = i), B_{2,i} := B_2(m_t = i), B_{3,i} := B_3(m_t = i), G_i := G(m_t = i), C_{1,i} := C_1(m_t = i), C_{2,i} := C_2(m_t = i) \), \( D_{12,i} := D_{12}(m_t = i) \) and \( D_{13,i} := D_{13}(m_t = i) \), respectively. We assume that these matrices are constant for each \( i \). We also assume that they are of compatible dimensions.

For the system (1), we assume the following conditions:

**A1:**

\[
E\{\omega(t)\} = 0, \quad E\{v(t)\} = 0,
\]

\[
E\{\omega(t)x'(s)\}1_{\{m_{t-s}=1\}} = Q_0 \delta(t-s), \quad t, s \in [0, T],
\]

\[
E\{v(t)x'(s)\}1_{\{m_{t-s}=1\}} = R_c(t) \delta(t-s), \quad t, s \in [0, T],
\]

\[
E\{x(0)1_{\{m_{t-0}=i_0\}}\} = x_0,
\]

\[
E\{x(0)x'(0)1_{\{m_{t-0}=i_0\}}\} = Q_0,
\]

\[
E\{\omega(0)x'(0)1_{\{m_{t-0}=i_0\}}\} = 0,
\]

\[
E\{\omega(t)x'(t)1_{\{m_{t-0}=1\}}\} = 0,
\]

\[
E\{\omega(t)v'(t)1_{\{m_{t-0}=1\}}\} = 0,
\]

\[
E\{\omega(t)v'(t)1_{\{m_{t-0}=1\}}\} = 0
\]

where \( E \) is the expectation with respect to \( x(0), \omega, v \) and \( m_t \), which are mutually independent. The indicator function \( 1_{\{m_{t-0}=1\}} := 1 \) if \( m_t = i \), and \( 1_{\{m_{t-0}=0\}} := 0 \) if \( m_t \neq i \). For this system (1), we assume the following condition:

**A2:**

\[
D_{12}'(m_t)D_{12}(m_t) > O, \quad t \in [0, T]
\]

We assume that the mode \( m_t \) is known at each time \( t \). The optimal tracking problem we address in this paper for the system (1) is to design control laws \( u(\cdot) \in L_2[0, T] \) over the finite horizon \([0, T]\), using the information available on the known part of the reference signal \( r_c(t) \) and minimizing the sum of the energy of \( z_c(t) \), for the initial condition \( x_0 \) and given initial mode \( i_0 \). It is assumed that the reference signal \( r_c(\cdot) \) is stochastic, i.e., \( r_c(\cdot) \) has any distribution at the unknown part. Considering the effects of the stochastic mode transitions, the exogenous noises and the average of the performance indices over the statistics of the unknown parts of \( r_c \), we define the following performance index:

\[
J_{\text{GT}}(x_0, u, r_c) := E \left\{ \int_0^T E_{\hat{R}_s}[\|z_c(s)\|^2] ds + x'(T)Q_T x(T) \right\}
\]

where \( Q_T \geq O \). \( E_{\hat{R}_s} \) means the expectation over \( \hat{R}_{s+h} \) and \( \hat{R}_s \) denotes the future information on \( r_c \) at time \( s \), i.e., \( \hat{R}_s := \{r_c(l); s < l \leq T\} \) (cf. [23]).
Now we formulate the following stochastic optimization problem by state feedback for the system (1) and the performance index (2).

The Stochastic Optimal Fixed-Preview Tracking Problem by State Feedback:
It is assumed that at the current time \( t \), \( r_c(s) \) is known for \( s \leq \min(T, t+h) \), where \( h \) is the preview length of \( r_c(\cdot) \). It is also assumed that the mode \( m_t \) is available for the design of controllers at each time \( t \). Then find \( \{u^*\} \) minimizing the performance index (2):

\[
J_C(x_0, u^*, r_c) \leq J_C(x_0, u, r_c)
\]

where the control strategy \( u^*(t) \), \( 0 \leq t \leq T \), is based on the information \( R_{t+h} := \{r_c(l); 0 \leq l \leq t+h \} \) with \( 0 \leq h \leq T \) and the state information \( x(t) \) at the current time \( t \).

The Stochastic Optimal Fixed-Preview Tracking Problem by Output Feedback:
It is assumed that at the current time \( t \), \( r_c(s) \) is known for \( s \leq \min(T, t+h) \), where \( h \) is the preview length of \( r_c(\cdot) \). It is also assumed that the mode \( m_t \) is available for the design of controllers at each time \( t \). Then find \( \{u^*\} \) minimizing the performance index (2):

\[
J_C(x_0, u^*, r_c) \leq J_C(x_0, u, r_c)
\]

where the control strategy \( u^*(t) \), \( 0 \leq t \leq T \), is based on the information \( R_{t+h} := \{r_c(l); 0 \leq l \leq t+h \} \) with \( 0 \leq h \leq T \), the state information \( x(t) \) at the current time \( t \) and the measured output information \( y(t) \) for \( 0 \leq s \leq t \).

3 Design of Tracking Controllers by State feedback
Now we consider the coupled Riccati equations

\[
\begin{align*}
\dot{X}_i + A_i'X_i + X_iA_i + C_{1i}'C_{1i} &= 0, \quad i = 1, \ldots, N \quad (3)
\end{align*}
\]

where

\[
\begin{align*}
\dot{R}_i &= D_{12i}'D_{12i} + \tilde{S}_i(t), \\
\tilde{S}_i(t) &= B_{2i}'X_i(t) + D_{12i}'C_{1i}.
\end{align*}
\]

We also consider the following coupled scalar differential equations.

\[
\begin{align*}
\dot{\alpha}_i(t) + tr\{G_i(t)Q_{c,i}(t)G_{c,i}(t)X_i(t)\} \\
&\quad + \sum_{j=1}^N \pi_{ij}\alpha_j(t) = 0 \quad (4)
\end{align*}
\]

Remark 3.1 Note that these coupled Riccati differential equations (3) and scalar differential equations (4) are the same as those for the standard LQG optimization problem of linear continuous-time Markovian jump systems without considering any exogeneous reference signals nor any preview information ([11]).

We obtain the following necessary and sufficient conditions for the solvability of the stochastic LQ optimal tracking problem and an optimal control strategy for it.

Theorem 3.1 Consider the system (1) and the performance index (2). Suppose the conditions A1 and A2. Then the Stochastic Optimal Fixed-Preview Tracking Problem by State Feedback for (1) and (2) is solvable if and only if there exist positive semi-definite matrices \( X_i(t), i = 1, \ldots, N \) and scalar functions \( \alpha_i(t), i = 1, \ldots, N \), satisfying the conditions \( X_i(T) = Q_T \) and \( \alpha_i(T) = 0 \) such that the coupled Riccati equations (3) and the coupled scalar equations (4) hold over \([0, T]\). Moreover an optimal control strategy for the Stochastic Optimal Fixed-Preview Tracking Problem by State Feedback for (1) and (2) is given by

\[
u^*(t) = -\tilde{R}_i^{-1}\tilde{S}_i(t)x(t) - C_{u,i}r_c(t) - C_{\theta_u,i}\theta_{c,i}(t)
\]

for \( t \in [0, T] \) and \( i = 1, \ldots, N \) where \( C_{\theta_u,i} = \tilde{R}_i^{-1}B_{2i}', C_{u,i} = \tilde{R}_i^{-1}D_{12i}'D_{13i} \) and let \( \theta_i(t), i = 1, \ldots, N \), \( t \in [0, T] \) satisfy

\[
\begin{align*}
\dot{\theta}_i(t) &= -\tilde{A}_i(t)\theta_i(t) + \tilde{B}_i(t)r_c(t) - \sum_{j=1}^N \pi_{ij}\theta_j(t), \\
\theta_i(T) &= 0
\end{align*}
\]

where

\[
\begin{align*}
\tilde{A}_i &= A_i - B_{2i}'\tilde{R}_i^{-1}\tilde{S}_i, \\
\tilde{B}_i &= \tilde{S}_i'\theta_{u,i} - (X_iB_{3i} + C_{1i}'D_{13i}).
\end{align*}
\]

Then \( \theta_{c,i}(t) \) is the 'causal' part of \( \theta_i(\cdot) \) at time \( t \). This \( \theta_{c,i} \) is the expected value of \( \theta_i \) over \( \mathcal{R}_i \) and given by

\[
\begin{align*}
\dot{\theta}_{c,i}(s) &= -\tilde{A}_i(s)\theta_{c,i}(s) + \tilde{B}_i(s)r_c(s) - \sum_{j=1}^N \pi_{ij}\theta_{c,j}(t), \\
\theta_{c,i}(t_f) &= 0
\end{align*}
\]

where

\[
\begin{align*}
t_f &= t + h \quad \text{if } t + h < T, \\
&= T \quad \text{if } t + h \geq T.
\end{align*}
\]

Moreover, the optimal value of the performance index is

\[
J_C(x_0, u^*, r_c) = \text{tr}\{Q_{u,i}X_{0,i}\} + \alpha_{u,i}(0) + E\left[ E_{R_0}\{2\theta_{i}x_0\}\right] + \int_0^T E_{R_i}\{||\tilde{R}_i^{-1/2}C_{\theta_u}(m_s)\theta_1(s, m_s)||^2\}ds \quad (8)
\]

where \( \theta_1(t, m_s) = \theta(t, m_s) - \theta_0(t, m_s), \theta_1(t, m_s) = \theta_1(t, m_s - 1) = \theta_1(t, m_s - 2) = \cdots = \theta_1(t, m_s - i) = \theta_1(t, m_s - i - 1) = \theta_1(t, m_s - i - 2) = \cdots = \theta_1(t, m_s - i - (i-1)) = \theta_1(t, m_s - i - (i-2)) = \cdots = \theta_1(t, m_s - 1) = \theta_1(t, m_s) \) for
\( t \in [0, T] \) and \( i = 1, \ldots, N \),

\[
\bar{J}_c(r_c) = \mathbb{E} \left\{ \int_0^T \mathbb{E}_{\bar{R}_c} \left\{ \delta \bar{J}_c(r_c(s)) + 2\theta'(s, m_s)B_3(m_s)r_c(s) \\
-2\theta'(s, m_s)C_{\theta u}(m_s)\bar{R}(m_s)C_u r_c(s) \\
-||\bar{R}_1^2(m_s)C_{\theta u}(m_s)\theta(s, m_s)||^2 \right\} ds \right\},
\]

\[
\delta \bar{J}_c(r_c(s)) = ||D_{13} r_c(s)||^2 - ||\bar{R}_1^2(m_s)C_u r_c(s)||^2,
\]

\[
C_{\theta u}(m_t) = \bar{R}^{-1}(m_s)B'_2(m_t)
\]

and

\[
C_u(m_t) = \bar{R}^{-1}(m_s)D'_{12}(m_s)D_{13}(m_s).
\]

\( \text{(Proof)} \)

We have already presented the proof of the necessity and sufficiency for the solvability of this stochastic optimal preview control problem by state feedback in [21]. (Q.E.D.)

\[\text{Remark 3.2} \text{ The compensator dynamics (7) in the fixed preview case has the same form as the compensator dynamics (6), while the terminal conditions in these two cases are different.}\]

\[\text{4 Output Feedback Case}\]

In this section we consider the output feedback problem.

Define

\[
\bar{u}_c(t) := u(t) + C_{u,i}i_r(t) + C_{\theta u,i}\theta_{c,i}(t).
\]

Then the performance index (2) can be written as follows (Refer to the proof for the necessity for the solvability of the state feedback problem in [21]):

\[
\bar{J}_c(r_c) = \text{tr} \{Q_m X_m(0) + \alpha_m(0) + \mathbb{E} \{E_{\bar{R}_c}\{2\theta'_w x_0\} \} + \int_0^T \mathbb{E}_{\bar{R}_c} \{||\bar{u}_c(s) + \bar{R}^{-1}\dot{S}(m_s)x(s) + C_{\theta u}(m_s)\theta_1(s, m_s)||^2 \right\} ds + \bar{J}(r_c)\}
\]

Moreover the system dynamics of (1) can be written as follows:

\[
dx(t) = A_i x(t)dt + G_{c,i}\omega(t)dt \\
+ B_2,i\bar{u}_c(t)dt + \bar{r}_c(t)dt \\
\text{(9)}
\]

where

\[
\bar{r}_c(t) = -B_{2,i}\{C_{u,i}r_c(t) + C_{\theta u,i}\theta_{c,i}(t)\} + B_{3,i}r_c(t).
\]

Now we consider the following output feedback dynamic controller for the system (9):

\[
d\hat{e}_c(t) = A_c\hat{e}_c(t)dt + B_{2,i}\hat{u}_c(t)dt + \hat{r}_c(t)dt \\
+ M_i[y(t) - C_{2,i}\hat{e}_c(t)]dt
\]

where \( M(t, m_t) = M_i \), \( i = 1, \ldots, N \) are the output feedback gains to decide later. By Theorem 4.1 we can design \( u(t) \) and \( M_i \), \( i = 1, \ldots, N \) independently. On maximum likelihood (ML) approach for the derivation of optimal estimators, the problem to obtain the optimal estimate can be reduced to the problem to obtain \( \omega^*(t) \) maximizing the likelihood function. Let \( e(t) := x(t) - \hat{x}_c(t) \) and then we obtain the following error dynamics.

\[
de(t) = [A_i - M_i(t)C_{2,i}]e(t)dt \\
+ G_{c,i}\omega(t)dt - M_i(t)u(t)dt \\
\text{(10)}
\]

Now we consider the following type of coupled Riccati differential equations.

\[
\dot{Z}_i = A_iZ_i + Z_iA'_i + G_{c,i}Q_{c,i}G_{c,i}' - Z_iC_{2,i}R_{c,i}^{-1}C_{2,i}Z_i \\
+ \sum_{j=1}^N \pi_{ij}Z_j \\
\text{(11)}
\]

From (11), we can obtain the following coupled Riccati differential equations for \( Z_i^{-1} \), \( i = 1, \ldots, N \),

\[
\dot{Z}_i^{-1} = Z_i^{-1}A_i + A'_iZ_i^{-1} \\
- C_{2,i}R_{c,i}^{-1}C_{2,i} + Z_i^{-1}G_{c,i}Q_{c,i}G_{c,i}'Z_i^{-1} \\
+ \sum_{j=1}^N \pi_{ij}Z_j^{-1} \\
\text{(12)}
\]

The infinitesimal operator \( \mathcal{L}_u \) denotes

\[
\mathcal{L}_u V(x, i, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \mathbb{E} \{E_{\bar{R}_c}\{V(x(t + \Delta t), m_t\Delta t, t + \Delta t)\} \} - \mathbb{E} \{V(x(t, i, t))\} \right\} \\
= \mathbb{E}_{\bar{R}_c} \left\{ \frac{\partial V}{\partial t} + (A_i x + B_{2,i}u + B_{3,i}r_c + G_{c,i}\omega)\frac{\partial V}{\partial x} \\
+ \sum_{j=1}^N \pi_{ij}V(x, j, t) \right\}
\]

for a scalar function \( V(x, i, t) \) which is twice differentiable with respect to \( x \) and once differentiable with respect to \( t \), considering the effects of the stochastic mode transitions and the exogenous noises, and the average of the performance indices over the statistics of the unknown parts of \( r_c \). This operator \( \mathcal{L}_u \) is called the averaged derivative at point \( (x(t) = x, m_t = i, t) \) ([1]).

Then, by the error dynamics (10) and the coupled Riccati equations (12),

\[
\mathcal{L}_u \mathbb{E}_{\bar{R}_c} \{ e'(t)Z_i^{-1}(t)e(t) \}
\]
\[
\begin{align*}
= e'(t)Z^{-1}_i(t)e(t) + e'(t)\dot{Z}^{-1}_i(t)e(t) \\
+ e'(t)Z^{-1}_i(t)e(t) + \sum_{j=1}^{N} \pi_{ij} e'(t)Z^{-1}_j(t)e(t)
\end{align*}
\]

Therefore we obtain the following maximum likelihood function.

\[
E\left\{ e'(T)\dot{Z}^{-1}(T,m_T)e(T) \right\} - E_{R_0}\left\{ e'(0)\dot{Z}^{-1}(0,m_0)e(0) \right\}
\]

\[
= E\left\{ \int_0^T E_{R_1}\left[ \|\omega(s)\|^2_{R^{-1}_i(s,m_2)} + \|v(s)\|^2_{R^{-1}_i(s,m_2)} \right] ds \right\}
\]

Now let

\[
x_0^* = \mu_{i0}, \quad \omega^*(t) = Q_c,i(t)G'_c(t)Z_i^{-1}(t)e(t)
\]

and then the error dynamics (10) implies \( e(t) = 0, t \in [0,T] \). The 'causal' \( \tilde{u}_c(t), t \in [0,T] \), determined on-line based on the information \( R_{s+h} \) and \( Y_i \) at the current time \( t \) minimizing \( J_{c,T} \) is given by

\[
\tilde{u}_c^*(t) = -\tilde{R}_i^{-1}\tilde{S}_i \hat{x}_c(t)
\]

where \( \hat{x}_c(\cdot) \) is the causal part of \( \hat{x} \) determined on-line based on the information \( R_{s+h} \) and \( Y_i \) at the current time \( t \).

Then we obtain the following necessary and sufficient conditions for the solvability of the stochastic optimal fixed-preview tracking problem by output feedback and dynamic controllers for it.

**Theorem 4.1** Consider the system (1) and the performance index (2). Suppose A1 and A2. Then the Stochastic Optimal Fixed-Preview Tracking Problem by Output Feedback for (1) and (2) is solvable if and only if there exist positive semi-definite matrices \( X_i(t), Z_i(t) i = 1, \cdots, N \), and scalar functions \( \alpha_i(t), i = 1, \cdots, N \), satisfying the conditions \( X_i(T) = Q_T, Z_i(0) = Q_{i0}, \alpha_i(T) = 0 \) such that the coupled Riccati equations (3), (11) and the coupled scalar equations (4) hold over \([0,T]\).

Moreover, using the solutions \( X_i(t) \) and \( Z_i(t) \) of two types of the coupled Riccati differential equations (3) and (11) with the conditions \( X_i(T) = Q_T \) and \( Z_i(0) = Q_{i0} \), an output feedback controller for the stochastic optimal fixed-preview tracking is given as follows:

\[
d\hat{x}_c(t) = A_c\hat{x}_c(t)dt + B_{22}u^*_c(t)dt + B_{3,i}\hat{r}_c(t)dt \\
+ M_i(t)[dg(t) - C_{2,i}\hat{x}_c(t)dt], \quad \hat{x}_c(0) = \mu_{i0}
\]

\[
M_i(t) = Z_i(t)C_{3,i}R^{-1}_{c,i}(t)
\]

\[
u^*_c(t) = -\tilde{R}_i^{-1}\tilde{S}_i \hat{x}_c(t) - C_{u,i}\hat{r}_c(t) - C_{\theta_u,i}\theta_c(t), i = 1, \cdots, N, \quad t \in [0,T]
\]

where \( C_{\theta_u,i} = \tilde{R}_i^{-1}B_{2,i} \) and \( C_{u,i} = \tilde{R}_i^{-1}D_{12,i}D_{13,i}, \theta_c(t), i = 1, \cdots, N, \quad t \in [0,T] \) satisfies (6). \( \theta_c(t), i = 1, \cdots, N, \quad t \in [0,T] \) satisfies (7)

Then, since \( x_0 = e(0) + \hat{x}_c(0) \), the value of the performance index is

\[
J_{c,T}(x_0, \tilde{u}_c, r_c)
\]

\[
= tr\{Q_{i0}X_{i0}\} + \alpha_{i0}(0) + E\left\{ E_{R_0}\{2\theta'_{i0}(e(0) + \mu_{i0})\} \right\}
\]

\[
+ \int_0^T E_{R_1}\{2\theta'_{i0}(e(0) + \mu_{i0})\}
\]

(Proof) We have already described the sufficiency for the solvability of the stochastic optimal tracking problem. The optimal input \( u^*_c(t), t \in [0,T] \) can be adopted using only the causal parts \( \theta_c \) and \( \hat{x}_c \) determined on-line based on the information \( R_{s+h} \) and \( y(s), s \in [0,t] \) at the current time \( t \). With regard to the necessity, because of arbitrariness of the reference signal \( r_c(\cdot) \), by considering the case of \( r_c(\cdot) = 0 \), one can reduced the proof of the necessity for the solvability of the stochastic optimal tracking problem to that of the standard LQG optimization problem. ([11])

**Remark 4.1** The output feedback controllers (13) in Theorem 4.1 are uncoupled respectively. However the whole tracking systems are coupled because the dynamics (6) and (7) of \( \theta_c \) and \( \theta_{c,i} \), \( i = 1, \cdots, N \) introducing future information of the reference signals are coupled each other respectively.

**5 Numerical Examples**

In this section, we study numerical examples to demonstrate the effectiveness of the design theory presented in this paper.
We consider the following two mode systems and assume that
the system parameters are as follows: (cf. [2], [23])
\[
\begin{align*}
\dot{x}(t) &= A(m_1)x(t)dt + B_2u(t)dt \\
&\quad + B_3r_c(t)dt + G_c\omega(t)dt, \\
x(0) &= x_0, \ m_0 = i_0
\end{align*}
\]
where
\[
\begin{align*}
\cdot \text{ Mode 1:} & \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 1 \\ 0.8 & -0.2 \end{bmatrix}, \\
B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
C_1 = \begin{bmatrix} -0.5 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad D_{13} = \begin{bmatrix} -1.0 \\ 0 \end{bmatrix}
\end{align*}
\]
and
\[
C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
where it is assumed that \( r_c(\cdot) \) is not always \textit{a priori}
known over the whole time interval \([0, T]\) but has any
distribution at the unknown part. Let
\[
\Pi = \begin{bmatrix} -0.2 & 0.2 \\ 0.3 & -0.3 \end{bmatrix}
\]
be a transition matrix of \( \{m_t\} \) and the covariances
\( Q_c(t)=1, \ R_c(t)=1 \) and
\[
Q_{i0} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}
\]
We set \( \mu_{i0} = \text{col}(0,0) \) and \( i_0 = 1 \).

Then we introduce the following objective function considering
the input energy.
\[
J_{CT}(x_0, u, r_c) = \mathbf{E}\left\{\int_0^T \mathbf{E}_{R_c}\{\|C_1x(s) + D_{13}r_c(s)\|^2 + 0.1^2\|u(s)\|^2\}ds\right\}
\]
By the term \( B_3r_c(t) \), the tracking performance can be
expected to be improved as [2, 23] and so on. The paths of
\( m_t \) are generated randomly, and the performances
are compared under the same circumstance, that is, the
same set of the paths so that the performances can be
easily compared.

We consider the whole system (14) with mode transition rate \( \Pi \) over
the time interval \( t \in [0, 15] \). For this
system, we apply the results of the stochastic optimal
tracking theory for \( r_c(t) = \sin(\pi t/15) \) with various
lengths of preview, and show the simulation results.

We verify the effectiveness of the preview compensation
both by state feedback and output feedback, and compare the tracking performances for them. We define
\[
TE(h) := \frac{1}{T}\int_0^T \sqrt{\|C_1x(s) + D_{13}r_c(s)\|^2}ds
\]
for each \( h \) as an averaged value of tracking errors.

As shown in Table 1, it is clear that, as increasing the
preview lengths, the square values \( \|C_1x(t)+D_{13}r_c(t)\|^2 \)
of the tracking errors decrease and so the tracking performance is improved by both types of controllers. These simulation results show that we can obtain almost the same results for the state feedback cases
and the output feedback cases except for the effects of stochastic randomness.

6 Concluding Remarks

In this paper we have presented the stochastic optimal tracking control theory considering the preview
information by state feedback and output feedback for the linear continuous-time Markovian jump systems, which
are a class of stochastic switching systems. The
compensators introducing the preview information of the reference signals are coupled with each other.

The author had presented the solution of the stochastic
optimal preview tracking control theory by state feedback
for the linear continuous-time Markovian jump systems ([21]). However the stochastic optimal preview tracking theory by output feedback had not been yet fully investigated. Hence we have focused on it in this paper. We have introduced another type of coupled Riccati differential equations with initial
conditions in order to solve the output feedback problem, and reduced the problem to the optimal state estimation problem. Using the solutions of this type of coupled Riccati differential equations, we can design the output feedback controller gains independendly from the control input. Note that, in both cases of state feedback and output feedback, the causal parts of the controllers are uncoupled, but the noncausal parts of them, i.e., preview or noncausal compensators are coupled with each other.

Throughout this paper it is assumed that the modes of
the system are fully observable over the whole time interval. The preview tracking control theory in the case with inaccessible modes is a very important further research issue.

References


Table 1: Averaged values $TE(h)$ of tracking errors for Case (a) state feedback controller and Case (b) output feedback controller

<table>
<thead>
<tr>
<th>TE(h)</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) state feedback</td>
<td>0.9220</td>
<td>0.8471</td>
<td>0.6344</td>
<td>0.4189</td>
<td>0.2432</td>
<td>0.1386</td>
<td>0.0950</td>
</tr>
<tr>
<td>(b) output feedback</td>
<td>0.9158</td>
<td>0.8362</td>
<td>0.6550</td>
<td>0.4163</td>
<td>0.2558</td>
<td>0.1322</td>
<td>0.0992</td>
</tr>
</tbody>
</table>


Appendix 1: Separation Principle Including Noncausal Information of Reference Signals

Theorem 6.1 (Separation Principle Including Noncausal Information of Reference Signals)
Consider the system

\[
\dot{x}(t) = A(m_t)x(t)dt + B_2(m_t)u(t)dt \\
+ B_3(m_t)r_c(t)dt + G_c(m_t)\omega(t)dt,
\]

\[
x(0) = x_0, \quad m_0 = i_0
\]

(15)

\[
dy(t) = C_2(m_t)x(t)dt + v(t)dt
\]

Assume each mode \(m_t\) is known. Also assume the control input \(u(t), 0 \leq t \leq T\), is decided based on the information \(R_{t+h} := \{r_c(l); 0 \leq l \leq t + h\} \) with \(0 \leq h \leq T\), the state information \(x(t)\) at the current time \(t\) and the measured output information \(Y_t = \{y(s)|0 \leq s \leq t\}\). Then we can design the optimal control input \(u(t)\) and the optimal estimator of the state \(x(t)\) independently.

(Outline of proof)
We can divide the system state \(x(t)\) into the following two part, i.e., the part dependent on exogenous inputs \(u(t)\) and \(r_c(t)\), and the part independent from them.

\[
\begin{align*}
\dot{x}_d(t) &= A(m_t)x_d(t)dt + B_2(m_t)u(t)dt \\
&\quad + B_3(m_t)r_c(t)dt \\
\dot{x}_i(t) &= A(m_t)x_i(t)dt + G_c(m_t)\omega(t)dt \\
x_d(0) &= \mathbf{E}\{x(0)\}, \quad x_i(0) = x_0 - \mathbf{E}\{x(0)\}
\end{align*}
\]

(16)

\[
x(t) = x_d(t) + x_i(t)
\]

For a given realization of the mode transition, we indicate the mode sequence by \(i_0, i_1, \cdots, i_k\) and the corresponding jump instants \((m_{t_0} = i_0, m_{t_1} = i_1, \cdots, m_{t_k} = i_k)\) by \(t_0, t_1, \cdots, t_k\).

Then \(x_d(t)\) is represented by the following expression.

\[
x_d(t) = \exp A_{i_k}(t - t_k)x_d(t_k) + \int_{t_k}^{t} e^{A_{i_k}(t-s)}B_{2,i_k}u(s)ds + \int_{t_k}^{t} e^{A_{i_k}(t-s)}B_{3,i_k}r_c(s)ds
\]

with

\[
x_d(t_k) = e^{A_{i_{k-1}}(t_k - t_{k-1})}x_d(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{A_{i_{k-1}}(t-s)}B_{2,i_{k-1}}u(s)ds + \int_{t_{k-1}}^{t_k} e^{A_{i_{k-1}}(t-s)}B_{3,i_{k-1}}r_c(s)ds
\]

Iterating this decomposition we can write \(x_d(t)\) as

\[
x_d(t) = \varphi_{i_0} \left\{ \int_{t_0}^{t} e^{A_{i_0}(t-s)}B_{2,i_0}u(s)ds + \int_{t_0}^{t} e^{A_{i_0}(t-s)}B_{3,i_0}r_c(s)ds \right\}
\]

\[
+ \cdots + \int_{t_{k-1}}^{t} e^{A_{i_{k-1}}(t-s)}B_{2,i_{k-1}}u(s)ds + \int_{t_{k-1}}^{t} e^{A_{i_{k-1}}(t-s)}B_{3,i_{k-1}}r_c(s)ds \right\}
\]

\[
+ \int_{t_k}^{t} e^{A_{i_k}(t-s)}B_{2,i_k}u(s)ds
\]

\[
+ \int_{t_k}^{t} e^{A_{i_k}(t-s)}B_{3,i_k}r_c(s)ds
\]

where

\[
\varphi_{i_0} = \exp A_{i_0}(t - t_k)\exp A_{i_{k-1}}(t_k - t_{k-1}) \times \cdots \times \exp A_{i_0}(t_1 - t_0)
\]

and

\[
\varphi_{i_{k-1}} = \exp A_{i_k}(t - t_k).
\]

Since \(u(t)\) is decided based on the known part \(R_{t+h}\) of the reference signal, \(x(t), M_t = \{m_s|0 \leq s \leq t\} \) and \(Y_t\), \(x_d(t)\) is \((M_t, Y_t)\)-measurable. Therefore the estimated value \(\hat{x}(t)\) is expressed by the following equality.

\[
\hat{x}(t) = x_d(t) + \mathbf{E}\{x(t)|M_t, Y_t\} = x_d(t) + \hat{x}(t)
\]

Our objective can be reduced to obtaining the estimated value \(\hat{x}_s(t)\). Hence we define

\[
d_y(t) = dy - C_2(m_t)x_d(t)dt
\]

and then we have

\[
d_y(t) = C_2(m_t)x_d(t)dt + v(t)dt
\]

(17)

From (16) and (17), we can design the following optimal filter.

\[
d_{\hat{x}}(t) = A_{i}d_{\hat{x}}(t)dt + Z_{s,i}(t)c_{3,i}^{-1}(t)(d_y(t) - C_{s,i}\hat{x}(t)dt), \quad \hat{x}_s(t) = 0,
\]

\[
\hat{Z}_{s,i} = A_{i}Z_{s,i} + Z_{s,i}A_{s} + G_cQ_cC_{c,i}'
\]

\[
- Z_{s,i}C_{3,i}c_{3,i}^{-1}c_{2,i}Z_{s,i}
\]

\[
+ \sum_{j=1}^{N} \pi_{ij} Z_{s,j},
\]

\[
Z_{s,i}(0) = 0
\]
where
\[ Z_{s,i} := \mathbf{E}\{[x_s(t) - \hat{x}_s(t)][x_s(t) - \hat{x}_s(t)]'\} \]

Since
\[
x_s(t) - \hat{x}_s(t) = [x(t) - x_d(t)] - [\hat{x}(t) - x_d(t)]
= x(t) - \hat{x}(t),
\]
we obtain \( Z_{s,i}(t) = Z_i(t) \). Then we obtain
\[
d\hat{x}(t) = dx_d(t) + d\hat{x}_s(t)
= A_i[x_d(t) + \hat{x}_s(t)]dt + B_{2,i}u(t)dt + B_{3,i}r_c(t)dt
+ Z_{s,i}(t)C_{2,i}^{-1}(t)[dy(t) - C_{2,i}(x_d(t) + \hat{x}_s(t))dt].
\]

By \( \hat{x}(t) = x_d(t) + \hat{x}_s(t) \) and \( Z_{s,i}(t) = Z_i(t) \), we obtain
\[
d\hat{x}(t) = A_i\hat{x}(t)dt + B_{2,i}u(t)dt + B_{3,i}r_c(t)dt
+ Z_i(t)C_{2,i}^{-1}(t)[dy(t) - C_{2,i}\hat{x}(t)dt],
\]
\[
\hat{x}(0) = \mathbf{E}\{x(0)\}
\]
\[
\dot{Z}_i = A_iZ_i + Z_iA_i' + G_{c,i}Q_{c,i}G_{c,i}' - Z_iC_{2,i}^{-1}(t)C_{2,i}Z_i
+ \sum_{j=1}^{N} \pi_{ij}Z_j,
\]
\[
Z_i(0) = Q_{i0}.
\]

This result shows that we can design the control input \( u(t) \) and the optimal estimator independently.