Further Studies on the VPPP Algorithms by using Multiple Antennas

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1 INTRODUCTION

In this paper we present the carrier-phase-based Precise Point Positioning (PPP) algorithms with multiple antennas [1]-[3] based on Global navigation satellite system Regression modeling (referred as GR models) which have been developed in [4]-[10].

Precise point positioning (PPP) is an ultimately desirable technology in the GPS/GNSS positioning community [11]. Our proposed positioning algorithm achieve the positioning accuracy in decimeter level in the horizontal plane [6]-[8] without any external transmitted information such as from the wide area augmentation system (WAAS).

By presenting GR equations according to [9, 10], in this paper, we derive very precise point positioning (VPPP) algorithms with using multiple antennas which are disposed with solid geometrical distances and have the common receivers’ clock errors.

In the next section, we show the GR models and use them to derive a PPP algorithm. In the section 3, the PPP algorithm is extended to VPPP using two or more GNSS antennas with common clock errors and the known distances among receivers [1]-[3]. Then we show the estimation algorithms for using the constraints of the distances among antennas, based on the recursive Kalman filtering. These positioning algorithms may give very accurate positioning results than those in only one antenna due to the constraints of the antennas’ positions and common receivers’ clock errors [1]-[3].

2 GNSS REGRESSION MODELS

Similarly to [4]:[10], we only describe the case of L1 frequency GNSS Regression (GR) models for the observed positioning data consisting of L1 carrier phases and pseudoranges based on C/A code due to simpler description and the commercial usage of positioning. The natural extensions of GNSS regression models for multiple frequencies of Galileo, Compass/BeiDou, GLONASS and US-GPS modernization are also similarly formulated. Namely, we consider the following fundamental measurements of L1 band carrier phases \( \gamma_{L1,u}(t) \), and the pseudoranges \( \rho_{C,A,u}(t) \) based on C/A codes, respectively, as follows [15]-[19]:

\[
\rho_{C,A,u}(t) = r_p^u(t,t - \tau^u_0) + \delta T^u_0(t) + \delta T^u_{s}(t) + c[\delta t_u(t) - \delta t^u_0(t)] + \delta b_{C,A,u} - \delta b_{C,A}^p + \epsilon_{C,A,u}^p(t) \tag{1}
\]

\[
\Phi_{L1,u}(t) = \lambda_1 \gamma_{L1,u}(t) - r_p^u(t, t - \tau^u_0) - \delta T^u_0(t) + \delta T^u_{s}(t) + c[\delta t_u(t) - \delta t^u_0(t)] + \delta b_{L1,u} - \delta b_{L1}^p \tag{2}
\]

where \( c \approx 2.99792458 \times 10^8 [\text{MHz}] \) denotes the speed of light, and \( f_1 \) and \( \lambda_1 \) are the central frequency and the wave length of the L1 carrier wave.

\[
f_1 = 2 \times 77 \times 10.23 [\text{MHz}] = 1575.42 [\text{MHz}] \tag{3}
\]

In (1)-(2), the so-called receiver’s biases, \( \{\delta b_{C,A,u}, \delta b_{L1,u}\} \), and the satellite biases, \( \{\delta b_{C,A}^p, \delta b_{L1}^p\} \), are contained in the usual observed positioning data consisting of L1 carrier phase and the pseudorange based on C/A codes. Also \( r_p^u(t, t - \tau^u_0) \) is the geometric distance between the receiver at the time \( t \) and the satellite at the time \( t - \tau^u_0 \) (\( \tau^u_0 \) denotes the travel time from the satellite \( p \) to the receiver \( u \)). Namely,

\[
r_p^u(t) = r_p^u(t, t - \tau^u_0) = \sqrt{(x_u(t) - x^p(t - \tau^u_0))^2 + (y_u(t) - y^p(t - \tau^u_0))^2 + (z_u(t) - z^p(t - \tau^u_0))^2}^{1/2} = || u(t) - s^p(t - \tau^u_0) || \tag{4}
\]

where \( u = [x_u, y_u, z_u]^T \) and \( s^p = [x^p, y^p, z^p]^T \) are a user (unknown) and satellite positions, respectively. Also \( n_s \) shows the number of the observable satellites. Further in (1), (2), \( \delta T^u_0(t) \) and \( \delta T^u_{s}(t) \) reflect the delay or the advance.
associated with the transmission of the L1 signal through the ionosphere and the troposphere, respectively. \( \delta u(t) \) and \( \delta T(t) - T_u \) are the clock errors of the receiver \( u \) at the time \( t \) and the satellite \( p \) at the time \( t - T_u \). \( N_u \) denotes integer ambiguity between the satellite \( p \) and the receiver \( u \), and \( e^p(t) \) and \( e^T(t) \) denote measurement errors.

Eq. (4) contains the satellite orbital errors. The estimated satellite orbits are obtained from the navigation messages which are decoded from the transmitted L1 signal. Let us denote \( \hat{s}^p \) as the estimated position of the satellite \( s^p \) at the time \( t - T_u \).

We use the following relations of the derivatives

\[
\begin{align*}
\frac{\partial r^p_u}{\partial x_u} &= \frac{(x_u - x^u)}{r^u_0}, & \frac{\partial r^p_u}{\partial y_u} &= \frac{(y_u - y^p)}{r^u_0}, \\
\frac{\partial r^p_u}{\partial z_u} &= \frac{(z_u - z^p)}{r^u_0}, & (p = 1, 2, \ldots, n_s),
\end{align*}
\]

and also

\[
\begin{align*}
\frac{\partial r^p_u}{\partial x^p} &= -\frac{(x_u - x^p)}{r^u_0}, & \frac{\partial r^p_u}{\partial y^p} &= -\frac{(y_u - y^p)}{r^u_0}, \\
\frac{\partial r^p_u}{\partial z^p} &= -\frac{(z_u - z^p)}{r^u_0}, & (p = 1, 2, \ldots, n_s).
\end{align*}
\]

Then we have the relation:

\[
\frac{\partial r^p_u}{\partial u} = -\frac{\partial r^p_u}{\partial s^p}.
\]

Thus the 1st order Taylor series approximation of (4) around the previous estimated value \( u = \bar{u}^{(o)} \) and \( s^p = \hat{s}^p \) is given by

\[
r^p_u \approx r^p_u(\bar{u}^{(o)}) + g^p_u(\bar{u}^{(o)})(u - s^p - (\hat{u}^{(o)} - \hat{s}^p))
\]

\[
= ||\hat{u}^{(o)} - \hat{s}^p|| \frac{\partial \hat{u}^{(o)} - \hat{s}^p}{\partial (u - s^p)}
\]

\[(11)\]

for \( p = 1, 2, \ldots, n_s \), where

\[
g^p_u(\bar{u}^{(o)}) = \left[ \frac{\partial r^p_u}{\partial u} \right]_{u=\hat{u}^{(o)}, s^p=\hat{s}^p} = \left( \frac{\partial \hat{u}^{(o)} - \hat{s}^p}{\partial (u - s^p)} \right)
\]

\[(12)\]

From (1)-(2), therefore, we have the approximations:

\[
\begin{align*}
\rho^p_{CA,u} &\approx g^p_u(\bar{u}^{(o)})(u - s^p) + \delta I_u^p + \delta T_u^p + c(\delta t_u - \delta t^p) + \delta b_{CA,u} - \delta b_{CA,u}^p + e^p_{CA,u},
\end{align*}
\]

\[
\Phi^p_{L1,u} \approx g^p_u(\bar{u}^{(o)})(u - s^p) - \delta I_u^p + \delta T_u^p + c(\delta t_u - \delta t^p) + \delta b_{L1,u} - \delta b_{L1,u}^p + \lambda_1 N_{L1,u}^p + \lambda_1 e^p_{L1,u},
\]

\[(10)\]

Define the \( n_s \times 3 \) matrix:

\[
G^s_u(\bar{u}^{(o)}) = \begin{bmatrix}
g^s_u(\bar{u}^{(o)}) \\
g^s_u(\bar{u}^{(o)}) \\
\vdots \\
g^s_u(\bar{u}^{(o)})
\end{bmatrix},
\]

\[(12)\]

namely,

\[
G^s_u(\bar{u}^{(o)}) = \begin{bmatrix}
\frac{\partial r^1_{u}^{(o)}}{\partial x_u} & \frac{\partial r^1_{u}^{(o)}}{\partial y_u} & \frac{\partial r^1_{u}^{(o)}}{\partial z_u} \\
\frac{\partial r^2_{u}^{(o)}}{\partial x_u} & \frac{\partial r^2_{u}^{(o)}}{\partial y_u} & \frac{\partial r^2_{u}^{(o)}}{\partial z_u} \\
\vdots & \vdots & \vdots \\
\frac{\partial r^3_{u}^{(o)}}{\partial x_u} & \frac{\partial r^3_{u}^{(o)}}{\partial y_u} & \frac{\partial r^3_{u}^{(o)}}{\partial z_u}
\end{bmatrix}.
\]

\[(13)\]

Now let us discuss the satellite’s as well as receiver’s hardware biases \( \delta b^p_u \) and \( \delta b_{L1,u} \). It was pointed out in [12] that the magnitudes of the satellite’s hardware biases are usually in the range of (several nanosecond \( \times c \)) while the receiver’s hardware biases could exceed 10 nanoseconds \( \times c \). Therefore, we assume that the satellite’s hardware biases may be negligible. On the other hand, we do not disregard the receiver’s hardware biases such that we define the terms the \( 2 \times 1 \) vector

\[
\delta b_u = [\delta b_{CA,u}, \delta b_{L1,u}]^T.
\]

\[(14)\]

Furthermore, from (12), we define a block diagonal matrix with the size \( (n_s \times 3n_s) \):

\[
G^s_{D_u} \equiv \begin{bmatrix}
g^s_{u^{(o)}}(O) & O & \cdots & O \\
O & g^s_{u^{(o)}}(O) & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & \cdots & \cdots & O
\end{bmatrix}.
\]

\[(15)\]

Then from (10)–(15), we have the following vector matrix regression equation:

\[
y^{(o)}_u = H^{(o)}_u \theta_u + v_u,
\]

\[(16)\]

where

\[
\begin{bmatrix}
\theta_u \\
\delta b_{CA,u} \\
\delta b_{L1,u} \\
\delta I_u \\
\delta T_u \\
\lambda_1 N_{L1,u}
\end{bmatrix} =
\begin{bmatrix}
u \\
c\delta t_u \\
\delta b_{CA,u} \\
s \\
c\delta t_s \\
\delta I_u \\
\delta T_u \\
\lambda_1 N_{L1,u}
\end{bmatrix}
\]

\[(17)\]

Then we get some knowledge of the satellite position \( s \), the satellite clock error \( c\delta t_s \) as well as the delay or the advance due to the ionospheric and tropospheric effects, \( \delta I_u^p \) and \( \delta T_u^p \) are, for instance,

\[
\begin{align*}
\tilde{s} &= s + \epsilon_s, \\
\tilde{c}\delta t &= c\delta t^p + \epsilon_{c\delta t}, \\
\tilde{\delta I}_u &= \delta I_u + \epsilon_{\delta I_u}, \\
\tilde{\delta T}_u &= \delta T_u + \epsilon_{\delta T_u}.
\end{align*}
\]

\[(18)\]
Substituting the above relations into the GR equation (16), and neglect $s$, $c\delta t^s$, $\delta I_u$, $\delta T_u$, we have
\[
\begin{bmatrix}
  y_{CA,u}^{(v)} \\
y_{L1,u}^{(v)}
\end{bmatrix} = C_{u,v}^{(v)} \begin{bmatrix}
  u \\
c\delta t_u \\
\delta b_u \\
\lambda_1 N_{L1,u}
\end{bmatrix} + \nu (22)
\]
where
\[
C_{u,v}^{(v)} = \begin{bmatrix}
  G_{u,v}^{(v)} & 1 & 1 & 0 \\
  G_{u,v}^{(v)} & 1 & 1 & I
\end{bmatrix},
\]
and
\[
\begin{align*}
y_{CA,u}^{(v)} &= \rho_{CA,u} + G_{D,u}^{(v)} \hat{s} + c\delta t^s - \delta I_u - \delta T_u, \\
y_{L1,u}^{(v)} &= \phi_{L1,u} + G_{D,u}^{(v)} \hat{s} + c\delta t^s + \delta I_u - \delta T_u.
\end{align*}
\]

3 VERY PRECISE POINT POSITIONING

Now let us consider that multiple antennas $u_i$, $i = 1, \ldots, n_r$, are disposed with the given distance $d_{ij}$ between any two antennas $u_i$ and $u_j$. In this assumption, we have the following $n_r$ single frequency GR equations:
\[
y_{u_i}^{(v)} = C_{u,v}^{(v)} \theta_{u_i} + v_{u_i}, (25)
\]
for $i = 1, \ldots, n_r$, where
\[
\theta_{u_i} = \begin{bmatrix}
  u_i \\
c\delta t_{u_i} \\
\delta b_{u_i} \\
\lambda_1 N_{L1,u_i}
\end{bmatrix}
\]
with the constraint
\[
d_{ij} = ||u_i - u_j|| + \text{small noise} \\
\quad = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} + \text{small noise}. (27)
\]

Now we show state equations and measurement equations for applying Kalman filtering. Let us show the Kalman filtering algorithm for VPPP with the constraint conditions in (27). For this purpose, we show the state equations for the static case (for the kinematic case of automobiles, see [20]-[23].

3.1 STATE EQUATIONS

Let us show the state equation for each component in (26).

3.1.1 State equation for receiver’s clock error

Also the receiver’s clock errors are generally modeled as follows [13, 14, 15] (let us call the following model as the A-model of the receiver’s clock error),
\[
\begin{align*}
c\delta t_{u,t+1} &= c\delta t_{u,t} + \Delta_e c\delta t_{u,t} + w_{c\delta t_{u,t}} \\
c\delta t_{u,t+1} &= c\delta t_{u,t} + w_{c\delta t_{u,t}}.
\end{align*}
\]
where $\Delta_e$ denotes the sampling interval of the receiver’s clock error, and the noise $w_{c\delta t_{u,t}}$ are assumed as white Gaussian processes with zero means and covariances $q_{c\delta t}$ and $q_{c\delta t}$, respectively. Then we can write
\[
\begin{align*}
\begin{bmatrix}
  c\delta t_{u,t+1} \\
c\delta t_{u,t+1}
\end{bmatrix} &= \begin{bmatrix}
  1 & \Delta_e \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  c\delta t_{u,t} \\
c\delta t_{u,t}
\end{bmatrix} + \begin{bmatrix}
  w_{c\delta t_{u,t}} \\
w_{c\delta t_{u,t}}
\end{bmatrix} \\
&= F_{A,c\delta t} \theta_{c\delta t_{u,t}} + \begin{bmatrix}
  w_{c\delta t_{u,t}} \\
w_{c\delta t_{u,t}}
\end{bmatrix} (30)
\end{align*}
\]

Also we consider another statistical model of the receiver’s clock error from the standpoint of time-series analysis. Namely we assume the time derivative of the receiver’s clock error is a first order markov process as follows:
\[
\begin{align*}
c\delta t_{u,t+1} &= c\delta t_{u,t} + \Delta_e c\delta t_{u,t} \\
c\delta t_{u,t+1} &= \kappa c\delta t_{u,t} + w_{c\delta t_{u,t}}. (31)
\end{align*}
\]
where $\kappa$ denotes the regression coefficient. By the vector-matrix form, we can also describe the model as follows (let call this model as the B-model of the receiver’s clock error)
\[
\begin{align*}
\begin{bmatrix}
  c\delta t_{u,t+1} \\
c\delta t_{u,t+1}
\end{bmatrix} &= \begin{bmatrix}
  1 & \Delta_e \\
  0 & \kappa
\end{bmatrix} \begin{bmatrix}
  c\delta t_{u,t} \\
c\delta t_{u,t}
\end{bmatrix} + \begin{bmatrix}
  0 \\
  w_{c\delta t_{u,t}}
\end{bmatrix} \\
&= F_{B,c\delta t} \theta_{c\delta t_{u,t}} + \begin{bmatrix}
  0 \\
w_{c\delta t_{u,t}}
\end{bmatrix} (32)
\end{align*}
\]

3.2 Approximated Method of Estimation (AP-Method)

Now we present an approximated but simpler estimation algorithm of $\eta_t$. Namely define
\[
\eta_t = \begin{bmatrix}
  \eta_{t,1} \\
  \vdots \\
  \eta_{t,n_r}
\end{bmatrix}, (33)
\]
where
\[
\eta_{i,t} = \begin{bmatrix}
  u_i \\
c\delta t_{u_i} \\
\delta b_{u_i} \\
N_{L1,u_i}
\end{bmatrix}, \quad i = 1, \ldots, n_r. (34)
\]

Then the state equation and measurement equation are given by
\[
\begin{align*}
\eta_{i,t+1} &= A_{i,t} \eta_{i,t} + w_{i,t}, \\
y_{u_i,t}^{(v)} &= C_{i,t} \eta_{i,t} + v_{i,t},
\end{align*}
\]
where

\[
A_{i,t} \equiv \begin{bmatrix} I_{3 \times 3} & F_{x,t} \end{bmatrix}
\]

(37)

\[
C_{i,t} \equiv \begin{bmatrix} G_{\hat{\theta}^{(i)}} & 1 & 0 & 1 & \lambda_{i}I \end{bmatrix}
\]

(38)

\[
w_{i,t} \equiv \begin{bmatrix} \Omega_I \\
\ast \\
w_{v,i,t}^T \\
0_{2+n_i,i}
\end{bmatrix}.
\]

(39)

Thus, for each state \(\hat{\eta}_{i,t}\), we can obtain the filtering estimate \(\hat{\eta}_{i,t}\) and its error covariance matrix \(\Sigma_{i,t|t}\) by applying the Kalman filter. The approximated estimate \(\hat{\eta}_{i,t}\) and its error covariance are obtained by

\[
\hat{\eta}_{i,t} = \begin{bmatrix} \hat{\eta}_{1,t|t} \\
\vdots \\
\hat{\eta}_{n_{r},t|t}
\end{bmatrix},
\]

(40)

and

\[
R_{\eta,t|t} = \begin{bmatrix} \Sigma_{i,t|t} & O \\
O & \Sigma_{2,t|t}
\end{bmatrix}
\]

(41)

### 3.3 Updating by the Constraints

The constraint conditions are applied to update PPP estimates as follows. Namely, when we have obtained the filtering estimates \(\hat{\eta}_{i,t}\) and the error covariance matrix \(R_{\eta,t}\), we apply the constraint conditions:

\[
d_{ij} \approx \kappa_{ij}^{T}(u_i - u_j)
\]

\[
c_{i,j} \approx c_{i,j}
\]

where

\[
\kappa_{ij} = \frac{(\hat{u}^{(i)}_j - \hat{u}^{(j)}_i)}{||\hat{u}^{(i)} - \hat{u}^{(j)}||}
\]

(42)

Equivalently, we assume

\[
d_{ij} = \kappa_{ij}^{T}(u_i - u_j) + e_{d_{ij}}
\]

(43)

\[
0 = c_{i,j} - c_{i,j} + e_{c_{i,j}}
\]

(44)

where \(e_{d_{ij}}\) and \(e_{c_{i,j}}\) are mutually independent Gaussian white noises with

\[
e_{d_{ij}} \sim N(0, \sigma_{d_{ij}}^2), \quad e_{c_{i,j}} \sim N(0, \sigma_{c_{i,j}}^2).
\]

Define

\[
\gamma_{n_{r},t} \equiv [\gamma_{12,t}, \ldots, \gamma_{1n_{r},t}, \gamma_{23,t}, \ldots, \gamma_{n_{r},n_{r},t}]
\]

(45)

and consider the following relations of the conditional probability density function (CPDF)

\[
p(\eta_{t}|Y^t, \gamma_{n_{r}}) = \frac{p(\eta_{t}, \gamma_{n_{r},t}|Y^t)p(Y^t)}{p(Y^t, \gamma_{n_{r},t})} = \frac{p(\eta_{t}, \gamma_{n_{r},t}|Y^t)p(Y^t)}{p(Y^t, \gamma_{n_{r},t})} = K_0(Y^t, \gamma_{n_{r},t})p(\eta_{t}|Y^t)p(\gamma_{n_{r},t} | \eta_{t}).
\]

(46)

Then we have relations:

\[
p(\eta_{t}|Y^t) = \frac{1}{(2\pi)^{n/2}|R_{\eta,t|t}|} \exp \left\{-\frac{1}{2}[\eta_{t} - \hat{\eta}_{t|t}]^{T} R_{\eta,t|t}^{-1} [\eta_{t} - \hat{\eta}_{t|t}] \right\},
\]

(47)

\[
p(\gamma_{n_{r},t} | \eta_{t}) = \prod_{i<j} \frac{1}{\sqrt{2\pi r_{d_{ij}}} \sigma_{d_{ij}}} \exp \left\{-\frac{1}{2r_{d_{ij}}} \left[\frac{d_{ij} - \kappa_{ij}^{T}(u_i - u_j)}{2} \right]^{2} \right\}
\]

(48)

Therefore, \(p(\gamma_{n_{r},t} | Y^t, \eta_{t})\) in (46) is expressed as follows:

\[
p(\eta_{t}|Y^t, \gamma_{n_{r}}) = K_0(Y^t, \gamma_{n_{r},t}) \frac{1}{(2\pi)^{n/2}|R_{\eta,t|t}|} \exp \left\{-\frac{1}{2}[\eta_{t} - \hat{\eta}_{t|t}]^{T} R_{\eta,t|t}^{-1} [\eta_{t} - \hat{\eta}_{t|t}] \right\}
\]

\[
\times \prod_{i<j} \frac{1}{\sqrt{2\pi r_{d_{ij}}} \sigma_{d_{ij}}} \exp \left\{-\frac{1}{2r_{d_{ij}}} \left[\frac{d_{ij} - \kappa_{ij}^{T}(u_i - u_j)}{2} \right]^{2} \right\}
\]

(49)

Then the constraints appeared in the power forms of the exponential function are expressed by the quadratic form of \(\eta\).
as follows:

\[
\frac{[d_{ij} - \kappa^T_{ij}(u_i - u_j)]^2}{2r_{u_{ij}}}
\]

\[= \frac{1}{2r_{u_{ij}}} \{d^2_{ij} + \kappa^T_{ij}(u_i - u_j)(u_i - u_j)^T \kappa_{ij} - 2d_{ij} \kappa^T_{ij}(u_i - u_j) \}
\]

\[= \frac{1}{2r_{u_{ij}}} \{d^2_{ij} + (u_i - u_j)^T \kappa_{ij} \kappa^T_{ij} (u_i - u_j) - 2d_{ij} \kappa^T_{ij} (u_i - u_j) \}
\]

\[= \frac{1}{2r_{u_{ij}}} \{d^2_{ij} + u_i^T \kappa_{ij} \kappa^T_{ij} u_i - u_i^T \kappa_{ij} \kappa^T_{ij} u_i - u_j^T \kappa_{ij} \kappa^T_{ij} u_j \}
\]

\[+ u_j^T \kappa_{ij} \kappa^T_{ij} u_j + 2d_{ij} \kappa^T_{ij} (u_i - u_j) \}
\]

\[= \frac{1}{2} \frac{d^2_{ij}}{r_{u_{ij}}} + u_i^T \kappa_{ij} \kappa^T_{ij} u_i - u_i^T \kappa_{ij} \kappa^T_{ij} u_i - u_j^T \kappa_{ij} \kappa^T_{ij} u_j
\]

\[+ u_j^T \kappa_{ij} \kappa^T_{ij} u_j + d_{ij}^T \kappa^T_{ij} (u_i - u_j),
\]

(50)

Therefore, in the case of \(n_r = 3\), we have

\[K_{12} = \frac{1}{r_{u_{12}}} \kappa^T_{12} \kappa^T_{12} \quad (: 3 \times 3),
\]

\[d_{12}^T = \frac{2d_{12}^T \kappa^T_{12}}{r_{u_{12}}} \quad (: 1 \times 3),
\]

\[c_{r_{12}} = \frac{1}{r_{c_{o_{12}}}},
\]

(52)

\[K_{13} = \frac{1}{r_{u_{13}}} \kappa^T_{13} \kappa^T_{13} \quad (: 3 \times 3),
\]

\[d_{13}^T = \frac{2d_{13}^T \kappa^T_{13}}{r_{u_{13}}} \quad (: 1 \times 3),
\]

\[c_{r_{13}} = \frac{1}{r_{c_{o_{13}}}},
\]

(53)

\[K_{23} = \frac{1}{r_{u_{23}}} \kappa^T_{23} \kappa^T_{23} \quad (: 3 \times 3),
\]

\[d_{23}^T = \frac{2d_{23}^T \kappa^T_{23}}{r_{u_{23}}} \quad (: 1 \times 3),
\]

\[c_{r_{23}} = \frac{1}{r_{c_{o_{23}}}},
\]

(54)

where

\[K_{ij} = \frac{1}{r_{u_{ij}}} \kappa_{ij} \kappa^T_{ij} \quad (: 3 \times 3),
\]

\[d_{ij}^T = \frac{2d_{ij}^T \kappa^T_{ij}}{r_{u_{ij}}} \quad (: 1 \times 3),
\]

\[c_{r_{ij}} = \frac{1}{r_{c_{o_{ij}}}},
\]

(51)

Then

\[
\frac{1}{2} \sum_{i < j} \left[ \frac{[d_{ij} - \kappa^T_{ij}(u_i - u_j)]^2}{2r_{u_{ij}}} + (c_{o_{ij}} - c_{o_{ij}})^2 \right]
\]

\[= \frac{1}{2} \eta^T M_{123} \eta + c_{123}^T \eta + d_{123}^T \eta,
\]

(55)

where

\[
M_{123} \equiv \begin{bmatrix}
K_{12} + K_{13} & c_{r_{12}} + c_{r_{13}} & -K_{12} & -c_{r_{12}} & -K_{13} & -c_{r_{13}} \\
& O_{n_r+3}
\end{bmatrix}
\]

\[
M_{123} \equiv \begin{bmatrix}
-K_{12} & K_{12} + K_{23} & c_{r_{12}} + c_{r_{23}} & -K_{23} & -c_{r_{12}} & O_{n_r+3} \\
& O_{n_r+3}
\end{bmatrix}
\]

\[
M_{123} \equiv \begin{bmatrix}
-K_{13} & K_{13} + K_{23} & c_{r_{13}} + c_{r_{23}} & O_{n_r+3} \\
& O_{n_r+3}
\end{bmatrix}
\]

and

\[c_{123} = \left[ -d_{123}^T - d_{123}^T \quad 0^T_{n_r+4} \right] \begin{bmatrix} d_{123}^T - d_{123}^T \quad 0^T_{n_r+4} \quad d_{123}^T + d_{123}^T \quad 0^T_{n_r+4} \end{bmatrix}.
\]

(56)
\[ d_{r123} = \frac{d_{12}^2}{r_{u12}} + \frac{d_{13}^2}{r_{u13}} + \frac{d_{23}^2}{r_{u23}}, \]  

(57)

where \( O_{n_s+3} \) and \( 0_{n_s+3} \) denote the \((n_s + 3) \times (n_s + 3)\) zero matrix and the \((n_s + 3) \times 1\) zero vector, respectively. Therefore, finally we have the following quadratic form as the power term of (49):

\[
- \frac{1}{2} (\eta - \tilde{\eta}) R^{-1} (\eta - \tilde{\eta}) - \sum_{i<j} \left( \frac{1}{2} \left( \frac{1}{r_{u12}} + \frac{1}{r_{u13}} + \frac{1}{r_{u23}} \right) d_{ij} - \kappa_{ij}^T (u_i - u_j) \right)^2 
\]

\[
+ \sum_{i<j} \left( \frac{1}{2} \left( \frac{1}{r_{u12}} + \frac{1}{r_{u13}} + \frac{1}{r_{u23}} \right) 2 \varepsilon_{ij} d_{r123} \right) 
\]

\[
= - \frac{1}{2} \left\{ \eta^T R^{-1} \eta - \eta^T R^{-1} \tilde{\eta} - \tilde{\eta}^T R^{-1} \eta 
+ \tilde{\eta}^T R^{-1} \tilde{\eta} + \eta^T M_{123} \eta + \tilde{\eta}^T M_{123} \tilde{\eta} + d_{r123} \right\} 
\]

\[
= - \frac{1}{2} \left\{ \eta^T R^{-1} \eta - \eta^T R^{-1} \tilde{\eta} - \tilde{\eta}^T R^{-1} \eta - \frac{1}{2} c_{123} \right\} 
\]

\[
= - \frac{1}{2} \left\{ \eta^T R^{-1} \eta - \frac{1}{2} c_{123} \right\} 
\]

Then the updated estimates \( \tilde{\eta} \) and the updated error covariance \( \tilde{\eta} \) are given by

\[ \tilde{\eta}_{123} = \left( R^{-1} + M_{123} \right)^{-1} \left( R^{-1} \tilde{\eta} - \frac{1}{2} c_{123} \right), \]  

(58)

\[ \tilde{\eta} = \left( R^{-1} + M_{123} \right)^{-1}. \]  

(59)

4 APPLICATIONS TO ATTITUDE ESTIMATION

Using multiple antennas, we will show the attitude estimation method by obtaining the baseline vector between two of any antennas \( u_1, \ldots, u_n \) located in same plane. From (12) and (14), by signal difference between two anten-

\[ \rho_{CA,ui}^p - \rho_{CA,uj}^p \equiv g_{\tilde{u}_{123}}^p (u_i - s^p) - g_{\tilde{u}_{123}}^p (u_j - s^p) \]

\[ + \delta b_{CA,ui} - \delta b_{CA,uj} \]

\[ + e_{\tilde{u}_{123}}^p (u_i - s^p) - e_{\tilde{u}_{123}}^p (u_j - s^p). \]  

(60)

\[ \Phi_{L1,ui}^p - \Phi_{L1,uj}^p \equiv g_{\tilde{u}_{123}}^p (u_i - s^p) - g_{\tilde{u}_{123}}^p (u_j - s^p) \]

\[ + \delta b_{CA,ui} - \delta b_{CA,uj} \]

\[ + \lambda_1 (N_{L1,ui}^p - N_{L1,uj}^p) \]

\[ + \lambda_1 (e_{\tilde{u}_{123}}^p (u_i - s^p) - e_{\tilde{u}_{123}}^p (u_j - s^p)). \]  

(61)

If we can assume

\[ \delta I_{ui} \equiv \delta I_{ui}, \]

\[ \delta T_{ui} \equiv \delta T_{ui}, \]

and also we can assume

\[ g_{\tilde{u}_{123}}^p \equiv g_{\tilde{u}_{123}}^p + \frac{1}{2} g_{\tilde{u}_{123}}^p + g_{\tilde{u}_{123}}^p \equiv g_{\tilde{u}_{123}}^p, \]  

(62)

then we have

\[ \rho_{CA,ui_{i,j}}^p \equiv g_{\tilde{u}_{123}}^p (u_i - j + \delta b_{CA,ui_{i,j}} + e_{\tilde{u}_{123}}^p) \]  

(63)

\[ \Phi_{L1,ui_{i,j}}^p \equiv g_{\tilde{u}_{123}}^p (u_i - j + \delta b_{CA,ui_{i,j}} + e_{\tilde{u}_{123}}^p) \]  

(64)

where

\[ \rho_{CA,ui_{i,j}}^p \equiv \rho_{CA,ui}^p - \rho_{CA,uj}^p \]  

(65)

\[ \Phi_{L1,ui_{i,j}}^p \equiv \Phi_{L1,ui}^p - \Phi_{L1,uj}^p \]  

(66)

\[ u_i_{i,j} \equiv u_i - u_j \]  

(67)

\[ N_{L1,ui_{i,j}}^p \equiv N_{L1,ui}^p - N_{L1,uj}^p \]  

(68)

\[ e_{\tilde{u}_{123}}^p \equiv e_{\tilde{u}_{123}}^p \]  

(69)

\[ e_{\tilde{u}_{123}}^p \equiv e_{\tilde{u}_{123}}^p \]  

(70)

Then we can easily derive the Kalman filtering algorithm for estimating the baseline vectors \( u_{i,j} \) with the constraints. The baseline vector \( u_{i,j} \) is linearly approximated by using an unimated value \( \tilde{u}_{i,j} \), namely

\[ d_{ij} = \kappa_{ij} (u_{i,j} - s_{ij}), \]  

(71)

where

\[ \kappa_{ij} = \frac{\tilde{u}_{i,j}}{||u_{i,j}||}. \]  

(72)

After obtaining the estimate \( \tilde{u}_{i,j} \) of \( u_{i,j} \) by applying Kalman filter using measurements (65) and (66), we can update estimates by using the constraints (71). After getting the estimates \( \tilde{u}_{i,j} \) and for the corresponding reference baseline vectors \( u_{i,j} \), we define the normalized vectors:

\[ \tilde{u}_{N,i,j} \equiv \frac{\tilde{u}_{i,j}}{||u_{i,j}||}. \]  

(73)

\[ u_{N,i,j} \equiv \frac{u_{i,j}}{||u_{i,j}||}. \]  

(74)
Thus we can find the rotational matrix $M$ (i.e. the orthogonal matrix with determinant $+1$) which minimizes

$$\sum_{i<j} ||\hat{u}_{N,i,j} - M\hat{u}_{N,i,j}^{(R)}||$$ (75)

Then we can estimate the Euler’s angles by using the matrix $M$ [28]-[30].

5 EXPERIMENTAL RESULTS

We carried out the preliminary experiment of the following large triangle of the antennas’ positions in Kyoto (see Table 1 for their coordinates in the WGS84 system): ($r=1$) Fushimi, ($r=2$) Saikyo, and ($r=3$) Sakyo-2, of GNSS Earth Observation Network System (GEONET) of Japan.

<table>
<thead>
<tr>
<th></th>
<th>WGS-84</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X [m]</td>
</tr>
<tr>
<td>(r=1)</td>
<td></td>
</tr>
<tr>
<td>Fushimi</td>
<td>-3750194.59</td>
</tr>
<tr>
<td>(r=2)</td>
<td></td>
</tr>
<tr>
<td>Saikyo</td>
<td>-3741822.81</td>
</tr>
<tr>
<td>(r=3)</td>
<td></td>
</tr>
<tr>
<td>Sakyo-2</td>
<td>-3745690.57</td>
</tr>
</tbody>
</table>

Table 2: Kyoto RMS-ERRORS

<table>
<thead>
<tr>
<th></th>
<th>PPP</th>
<th>VPPP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fushimi</td>
<td>E 0.2387</td>
<td>E 0.1095</td>
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<tr>
<td></td>
<td>N 1.4671</td>
<td>N 1.0564</td>
</tr>
<tr>
<td></td>
<td>U 1.3993</td>
<td>U 0.1837</td>
</tr>
<tr>
<td>Saikyo</td>
<td>E 0.1064</td>
<td>E 0.1225</td>
</tr>
<tr>
<td></td>
<td>N 1.4493</td>
<td>N 0.9730</td>
</tr>
<tr>
<td></td>
<td>U 1.2885</td>
<td>U 0.4935</td>
</tr>
<tr>
<td>Sakyo-2</td>
<td>E 0.1815</td>
<td>E 0.1818</td>
</tr>
<tr>
<td></td>
<td>N 1.4211</td>
<td>N 1.0599</td>
</tr>
<tr>
<td></td>
<td>U 1.4356</td>
<td>U 0.6615</td>
</tr>
</tbody>
</table>
Table 3: Hokkaido RMS-ERRORS

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>N</th>
<th>U</th>
<th>RMS [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3895</td>
<td>1.0541</td>
<td>1.4683</td>
<td>1.0675</td>
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<td>Sapporo-1</td>
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<tr>
<td></td>
<td>0.4183</td>
<td>0.7499</td>
<td>0.4383</td>
<td>0.6223</td>
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<tr>
<td></td>
<td>0.5703</td>
<td>1.3651</td>
<td>0.7886</td>
<td>1.5359</td>
</tr>
<tr>
<td></td>
<td>0.1483</td>
<td>0.6223</td>
<td>0.4383</td>
<td>0.5198</td>
</tr>
<tr>
<td></td>
<td>0.1483</td>
<td>0.6223</td>
<td>0.4383</td>
<td>0.5198</td>
</tr>
</tbody>
</table>

Table 4: Okinawa RMS-ERRORS

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>N</th>
<th>U</th>
<th>RMS [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naha</td>
<td>0.1181</td>
<td>0.5689</td>
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<td>0.7628</td>
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<td></td>
<td>0.0748</td>
<td>0.9241</td>
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<td>0.527</td>
</tr>
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<td></td>
<td>0.1049</td>
<td>0.6782</td>
<td>1.1378</td>
<td>0.7672</td>
</tr>
<tr>
<td></td>
<td>0.0589</td>
<td>0.7572</td>
<td>0.2146</td>
<td>0.4556</td>
</tr>
</tbody>
</table>

For the experiment, total 41 epochs (L1) single frequency data were collected at 1/30 [Hz] rate on June 1, 2014, from 04:00’00 to 04:20’00 UTC (20 minutes). The number of available satellites was between 4 to 6 during this period. In this experiment, we could not apply the constraint condition of common receiver’s clock errors: \(c_{\delta t_i} \approx c_{\delta t_j}\), we only use the contraints of distances between each two antennas of 3 antennas. When we apply the Kalman filter (by the AP-Method in this experiment) to the measurement equation, the covariance matrice of the observation noise \(v\) is calculated from the general standard deviation of the each error source shown in [27].

Fig. 1 shows the positioning errors with the local level axes (East, North and Height (Up)), where the dotted blue lines show the errors for PPP results and the solid red lines show the errors for VPPP.

The errors were computed by difference between the estimated positions and the corresponding positions in the coordinates of Tables 2-4. We can observe from Figs. 1 - 3 that the positioning quality is considerably improved by using VPPP. The statistics of the positioning results are summarized in Tables 2-4. RMS errors of PPP and VPPP for three antennas are, 1.19[m] by PPP, and 0.66[m] by VPPP in Kyoto area, 1.18[m] by PPP, and 0.51[m] by VPPP in Hokkaido area, 0.78[m] by PPP, and 0.52[m] by VPPP in Okinawa area.

6 CONCLUSIONS

Applying the coupled GR equations for multiple antennas in the case of unknown positions, we have derived the PPP as well as VPPP algorithms. The experiments in the static situation were carried out. As a result, the VPPP algorithm achieve higher accuracy than the PPP algorithm. Approximately, the VPPP algorithm decreses 40% of positioning errors with comparing to the PPP algorithm. In the near future, we will examine and will show the results for the kinematic environment, and also will present the precise algorithms of the attitude determination by obtaining the rotational matrix and Euler’s angles.

REFERENCES


