H_2 Control for Sampled-Data Systems with Variable Sampling Rate

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Abstract

In this paper, H_2 control is considered for sampled-data systems with variable sampling rate, where the H_2 control performance is introduced according to a standard sampled-data setting. A discrete-time H_2 control problem is employed for solving the original problem. Its solvability condition is then established as a parameter-dependent linear matrix inequality. A probabilistic approach is taken for coping with the parameter-dependency.

1 Introduction

Since variable sampling rate enables us to utilize computational resource for communication and control with much flexibility, systems with variable sampling rate have been investigated [1, 2], which will be used in embedded systems or DC motor control, and so on. In this context, the authors of the present paper have also dealt with sampled-data systems with variable sampling rate, where sampling rate is managed by supervisor. For example, when disturbance is detected, high sampling rate is preferred to cope with the disturbance. On the contrary, when no disturbance is detected and/or output of the plant is near the equilibrium, low sampling rate is enough. Supervisor manages the sampling rate in this way, which leads to flexibility in utilizing resource. Under this setting, H_∞ control has been considered in [3].

In this paper, H_2 control problem for sampled-data systems with variable sampling rate is considered, where the H_2 control performance is introduced according to a standard sampled-data setting [4]. We first employ a discrete-time H_2 control problem [5] for solving the original problem. Then, following a standard LMI technique [6], we establish the solvability condition of the problem as a parameter-dependent LMI. We take a probabilistic approach [7, 8] for solving this LMI with a probabilistic guarantee.

This paper is organized as follows. In Section 2, sampled-data system with variable sampling rate is introduced. In Section 3, a design method for H_2 control is stated. In Subsection 3.1, a parameter-dependent linear matrix inequality (LMI) is derived as the solv-

ability condition following a standard LMI technique and the main result of the paper is stated. In Subsection 3.2, a randomized algorithm is employed for solving the parameter-dependent LMI derived in the previous subsection. In Section 4, we see numerical examples for showing performance of the controller. Then, in Section 5, we summarize the result of the paper.

2 Sampled-Data System with Variable Sampling rate

Let us consider a standard generalized plant in a sampled-data system in Fig. 1

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}, \quad (1)$$

where (A, B_2) is stabilizable, (C_2, A) is detectable, $C_1 e^{At} B_1 \equiv 0$, $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, $z(t) \in \mathbb{R}^{p_1}$, and $y(t) \in \mathbb{R}^{p_2}$. An impulse sampler $\mathcal{S}(h_k)$ and a zero-order-hold device $\mathcal{H}(h_k)$ work synchronously, where $h_k \in [\underline{h}, \bar{h}]$ denotes the k -th sampling interval and $0 < \underline{h} \leq \bar{h} < \infty$. $y_d[k]$ and $u_d[k]$ denote discrete-time signals, where $y_d[k] = y(t_k)$ and $u(t) = u(t_k) = u_d[k]$ for all $t \in [t_k, t_{k+1})$. We consider a sampling-rate dependent discrete-time controller $C(h)$

$$\begin{bmatrix} \xi[k+1] \\ u_d[k] \end{bmatrix} = \begin{bmatrix} A_c(h_k) & B_c \\ C_c & 0 \end{bmatrix} \begin{bmatrix} \xi[k] \\ y_d[k] \end{bmatrix}, \quad (2)$$

where $\xi[k] \in \mathbb{R}^n$. We assume that the next sampling time t_{k+1} is determined at time t_k , i.e., h_k is known at

Fig. 1: Sampled-data system with variable sampling rate

time t_k . This situation is consistent with the case that a supervisor manages the sampling rate. Important point is, supervisor cannot generate a sequence of sampling rates at once.

In this paper, we investigate a problem to find a $C(h)$ such that

$$\left[\frac{1}{h_0} \int_0^{h_0} \left\{ \int_\sigma^\infty \text{trace}(g(t, \sigma)g^T(t, \sigma)) dt \right\} d\sigma \right]^{1/2} < \gamma \quad (3)$$

for a given $\gamma > 0$ and for all possible sequence of $h_k \in [\underline{h}, \bar{h}]$. Here $g(t, \sigma)$ is the weighting function of the closed-loop system $\mathcal{G}$ composed of (1), $\mathcal{S}(h_k)$, $\mathcal{H}(h_k)$, and (2). That is, the input-output equation of $\mathcal{G}$ is given by

$$z(t) = \int_0^t g(t, \sigma)w(\sigma)d\sigma.$$

3 Controller design

3.1 Parameter-dependent LMI

Let us introduce a discrete-time plant

$$\begin{bmatrix} x_d[k+1] \\ \hat{z}_d[k] \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d(h_k) & \tilde{B}_{1d}(h_k) & B_{2d}(h_k) \\ \hat{C}_{1d}(h_k) & 0 & \hat{D}_{12d}(h_k) \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \hat{w}_d[k] \\ u_d[k] \end{bmatrix} \quad (4)$$

where $x_d[k] \in \mathbb{R}^n$, $\tilde{w}_d[k] \in \mathbb{R}^n$, $\hat{z}_d[k] \in \mathbb{R}^{n+m_2}$, and

$$A_d(h) = e^{Ah}, \quad \tilde{B}_{1d}(h) = \left(\frac{1}{h} \int_0^h e^{A\sigma} B_1 B_1^T e^{A^T \sigma} d\sigma \right)^{1/2},$$

$$B_{2d}(h) = \int_0^h e^{A\sigma} B_2 d\sigma, \quad C_{2d} = C_2,$$

$$[\hat{C}_{1d}(h) \quad \hat{D}_{12d}(h)] = \left(\int_0^h V^T(\theta)V(\theta)d\theta \right)^{1/2},$$

$$V(\theta) = \begin{bmatrix} C_1 e^{A\theta} & \int_0^\theta C_1 e^{A\sigma} B_2 d\sigma + D_{12} \end{bmatrix}.$$

Notice that this form is taken from a standard sampled-data control [5], though we here consider the variable sampling case.

We seek a discrete-time controller $C(h)$ such that the discrete-time closed-loop system composed of (2) and (4) achieves

$$\sum_{k=1}^{\infty} \text{trace} \{g_d(k)g_d^T(k)\} < \gamma^2 \quad (5)$$

for a given $\gamma > 0$ and for all possible sequence of $h_k \in [\underline{h}, \bar{h}]$, where $g_d(k)$ denotes the impulse response matrix for the closed-loop. This is actually possible if we employ an LMI technique [6] for discrete-time H_2 control. We emphasize that the technique has been

developed for time-invariant systems, while we use the LMI formula for the *sampling rate dependent* plant with *sampling rate independent* LMI variables. Then, we can obtain a parameter-dependent LMI

$$F(\eta, h) < 0, \quad \forall h \in [\underline{h}, \bar{h}] \quad (6)$$

as the solvability condition, where

$$F_1(\eta, h) = \begin{bmatrix} F_{1,11}(\eta) & F_{1,21}^T(\eta, h) & F_{1,31}^T(\eta, h) \\ F_{1,21}(\eta, h) & F_{1,22}(\eta) & 0 \\ F_{1,31}(\eta, h) & 0 & -I_n \end{bmatrix},$$

$$F_2(\eta, h) = \begin{bmatrix} -W & F_{2,21}^T(\eta, h) \\ F_{2,21}(\eta, h) & F_{2,22}(\eta) \end{bmatrix},$$

$$F_3(\eta) = \text{trace}(W) - \gamma^2,$$

$$F_{1,11}(\eta) = F_{1,22}(\eta) = F_{2,22}(\eta) = \begin{bmatrix} -X & -I_n \\ -I_n & -Y \end{bmatrix},$$

$$F_{1,21}(\eta, h) = \begin{bmatrix} A_d(h)X + B_{2d}(h)G & A_d(h) \\ J & Y A_d(h) + H C_{2d} \end{bmatrix},$$

$$F_{1,31}(\eta, h) = [\hat{C}_{1d}(h)X + \hat{D}_{12d}(h)G \quad \hat{C}_{1d}(h)],$$

$$F_{2,21}(\eta, h) = \begin{bmatrix} \tilde{B}_{1d}(h) \\ Y \tilde{B}_{1d}(h) \end{bmatrix},$$

$$F(\eta, h) = F_1(\eta, h) \oplus F_2(\eta, h) \oplus F_3(\eta).$$

Here, $A \oplus B$ denotes direct sum of matrices A and B , and η is a vector composed of the independent elements of the matrix variables $W = W^T$, $X = X^T$, $Y = Y^T$, G , H , and J . That is, if there exists η which satisfies the parameter-dependent LMI (6), there exists a sampling rate dependent discrete-time controller $C(h)$ which stabilizes the plant (4) and achieves (5) where one of them is given by

$$\begin{aligned} A_c(h) &= \{(A_d(h) - B_c C_{2d})X - Y^{-1}J\}(X - Y^{-1}) \\ &\quad + B_{2d}(h)C_c, \quad B_c = -Y^{-1}H, \quad C_c = G(X - Y^{-1})^{-1}. \end{aligned} \quad (7)$$

We can now state our main result.

Theorem 3.1 Suppose we have a controller (2), whose coefficient matrices are given by (7), where $X = X^T$, $Y = Y^T$, G , H , and J are matrix variables which satisfy a parameter-dependent LMI (6) for all $h \in [\underline{h}, \bar{h}]$. Then, the controller achieves (3) for a given $\gamma > 0$ and for all possible sequence of $h_k \in [\underline{h}, \bar{h}]$.

Proof of Theorem 3.1 is given in appendix A.

3.2 Randomized algorithm

We employ following Algorithm 1, which is a sequential randomized algorithm [8], because LMI (6) is parameter-dependent. In fact, parameter dependency makes it quite difficult to solve the LMI in a deterministic manner. We remark that the LPV design via randomized algorithms for a continuous-time plant has been proposed in [7].

Algorithm 1

- *Input:* LMI (6), p^* , δ

- *Output:* η_{pr}

1. Initialization

- (a) Initialize outer counter $i = 1$.
- (b) Set $\eta^{(1)}$ as a candidate of solution.
- (c) Determine number of samples $N(i)$.

2. Probabilistic oracle

- (a) Initialize inner counter $j = 1$.
- (b) Sample $h^{(j)}$.
- (c) If $F(\eta^{(i)}, h^{(j)}) \not\leq 0$, goto update.
- (d) Else if $j = N(i)$, output $\eta_{pr} = \eta^{(i)}$ and exit.
- (e) Else, set inner counter $j = j + 1$ and goto probabilistic oracle.

3. Update

- (a) Update $\eta^{(i)}$ based on a certain method (e.g., ellipsoid method) to get $\eta^{(i+1)}$.
- (b) Set outer counter $i = i + 1$ and goto probabilistic oracle.

In this type of sequential randomized algorithms, we usually set parameters $p^* \in (0, 1)$ and $\delta \in (0, 1)$. We then employ $N(i)$ random samples of $h \in [\underline{h}, \bar{h}]$, e.g., the one satisfies

$$N(i) \geq \left(\frac{1}{2} + 2 \ln i + \ln \frac{1}{\delta}\right) / \ln \frac{1}{p^*} \quad (8)$$

where i denotes ordinal of outer iteration. Then, if $f(\eta^{(i)}, h^{(j)}) < 0$, for all $N(i)$ samples, we can conclude that, with probability at least $1 - \delta$,

$$p = \text{Prob}\{f(\eta^{(i)}, h) < 0\} > p^* \quad (9)$$

where $f(\eta, h) = \lambda_{\max}(F(\eta, h))$ [9]. On the other hand, if $f(\eta^{(i)}, h^{(j)}) \geq 0$ for a random sample $h^{(j)}$, we update the solution candidate $\eta^{(i)}$ based on a subgradient $\partial_{\eta}\{f(\eta, h)\}$ at $\eta^{(i)}$ and $h^{(j)}$ which is given by, e.g.,

$$\eta^{(i+1)} = \eta^{(i)} - r^{(i)} \partial_{\eta}\{f(\eta^{(i)}, h^{(j)})\}$$

where $r^{(i)}$ is a step size suitably selected. Here, a subgradient is given as an explicit form

$$\begin{aligned} & \partial_{\eta}\{f(\eta^{(i)}, h^{(j)})\} \\ &= [\zeta_M^T F_1(h^{(j)}) \zeta_M \quad \cdots \quad \zeta_M^T F_{n_{\eta}}(h^{(j)}) \zeta_M]^T \end{aligned}$$

when we rewrite $F(\eta, h)$ of (6) in the form

$$F(\eta, h) = F_0(h) + \sum_{i=1}^{n_{\eta}} \eta_i F_i(h)$$

where ζ_M is a unit eigenvector associated with the largest eigenvalue of $F(\eta^{(i)}, h^{(j)})$. That is, when we select p^* as almost one and δ as almost zero, we can say that this $\eta^{(i)}$ satisfies (6) for almost all $h \in [\underline{h}, \bar{h}]$. In this way, we can solve the parameter dependent LMI (6) rigorously in a probabilistic sense.

4 Numerical Example

Let us consider a plant (1) whose coefficient matrices are given by

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We used MATLAB R2011b with packages SeDuMi [10], YALMIP [11], and RACT [12] for computation. When we set parameters $\gamma = 5$, $\underline{h} = 0.1$, $\bar{h} = 0.6$, $p^* = 0.99$, and $\delta = 0.001$ in (6) and (8), we got η as

$$W = \begin{bmatrix} 14.19 & 0.3427 & 0.3427 \\ 0.3427 & 0.9199 & 0.04084 \\ 0.3427 & 0.04084 & 0.9199 \end{bmatrix},$$

$$X = \begin{bmatrix} 13.45 & -0.4017 & -0.9072 \\ -0.4017 & 0.7314 & -0.1037 \\ -0.9072 & -0.1037 & 0.7564 \end{bmatrix},$$

$$Y = \begin{bmatrix} 13.12 & -6.297 & -5.415 \\ -6.297 & 11.87 & -5.302 \\ -5.415 & -5.302 & 28.31 \end{bmatrix},$$

$$G = [0.4574 \quad -0.4943 \quad -0.3857], \quad H = \begin{bmatrix} -8.098 \\ -0.3194 \\ 1.859 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.8692 & -0.1631 & -0.1158 \\ -0.07882 & 0.9826 & 0.1374 \\ -0.009507 & -0.1364 & 0.9845 \end{bmatrix}.$$

In order to confirm probabilistic accuracy (9) of this η , we employed the Chernoff bound [8], where we set N as 152,019 which satisfies

$$\tilde{N} \geq \frac{1}{2\tilde{\epsilon}^2} \log \frac{2}{\tilde{\delta}}$$

for $\tilde{\epsilon} = 0.005$ and $\tilde{\delta} = 0.001$. We generated $\tilde{N}$ random samples of h . Then, we saw that $f(\eta, h) < 0$ held for all $\tilde{N}$ samples, which means that the empirical probability of p in (9), denoted by $\hat{p}$, is 1. Since the Chernoff bound guarantees that $|p - \hat{p}| < \tilde{\epsilon}$ with probability at least $1 - \tilde{\delta}$, we can say that $p > 0.995$ holds with probability at least 0.999. That is, $p > p^* = 0.99$ actually holds with high probability.

Figs. 2, 3 and 4 show the responses of the control system with above η , where $w(t) = \exp(-0.1t) \sin(t)$ and $x(0) = 0$. A fixed sampling rate $h = 0.2$ was employed in Fig. 2, while a variable sampling rate

$$h = \begin{cases} 0.2 & t \in [0, 16) \\ 0.3 & t \in [16, 28) \\ 0.5 & t \in [28, \infty) \end{cases}$$

was employed in Fig. 3. We see that the disturbance responses are almost the same for both cases. Also, random sampling rate $h = 0.1 + 0.5q$ was employed in Fig. 4, where q is a random variable according to the uniform distribution $U(0, 1)$. We see that the plant is stabilized under the existence of disturbance.

Fig. 2: Disturbance response via fixed sampling rate

Fig. 3: Disturbance response via variable sampling rate

Fig. 4: Disturbance response via random sampling rate

5 Conclusion

In this paper, we considered H_2 control problem for SDSVSR. Time-varying discrete-time controller, which is dependent on sampling interval is proposed. Controller design was composed of three steps - employing discrete-time problem in the first step, deriving a parameter-dependent LMI as the solvability condition in the second step, and then employing a randomized algorithm for solving the LMI in the third step. Also we formulated main result of the paper as Theorem 3.1 and showed performance of the proposed controller via numerical example.

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Appendix A

Proof of Theorem 3.1 Proof is composed of two steps. In the first step, we show that a discrete-time controller (2), which satisfies the assumption of the theorem, achieves (5) for discrete-time plant (4). In the second step, we show that the same controller achieves H_2 performance (3) for the original continuous-time

plant (1) when it achieves (5) for the discrete-time plant (4).

We first examine the discrete-time closed-loop, whose state equation is given by

$$\begin{bmatrix} \Xi[k+1] \\ z_d[k] \end{bmatrix} = \begin{bmatrix} A_{cl}(h_k) & B_{cl}(h_k) \\ C_{cl}(h_k) & 0 \end{bmatrix} \begin{bmatrix} \Xi[k] \\ w_d[k] \end{bmatrix}$$

where

$$\begin{aligned} \Xi[k] &= \begin{bmatrix} x_d[k] \\ \xi[k] \end{bmatrix}, \quad A_{cl}(h) = \begin{bmatrix} A_d(h) & B_{2d}(h)C_c \\ B_cC_2 & A_c(h) \end{bmatrix}, \\ B_{cl}(h) &= \begin{bmatrix} \hat{B}_{1d}(h) \\ 0 \end{bmatrix}, \quad C_{cl}(h) = [\hat{C}_{1d}(h) \quad \hat{D}_{12d}(h)C_c]. \end{aligned}$$

Then, the impulse response matrix for the closed-loop is given by

$$g_d(k) := \begin{cases} 0 & (k=0) \\ C_{cl}(h_k)\Phi(k,1)B_{cl}(h_0) & (k=1,2,3,\dots) \end{cases}$$

where $k=0$ denotes the time-step which unit-impulse is inputted and

$$\Phi(k,1) = A_{cl}(h_{k-1}) \cdots A_{cl}(h_1), \Phi(k,k) = I.$$

By assumption of the theorem,

$$Y_{cl} - A_{cl}(h_k)^T Y_{cl} A_{cl}(h_k) - C_{cl}(h_k)^T C_{cl}(h_k) > 0 \quad (10)$$

$$\gamma^2 - \text{trace}(B_{cl}(h_k)^T Y_{cl} B_{cl}(h_k)) > 0 \quad (11)$$

holds for all $h_k \in [\underline{h}, \bar{h}]$ and for $Y_{cl} = Y_{cl}^T > 0$ which is given by

$$Y_{cl} = \begin{bmatrix} X & X - Y^{-1} \\ X - Y^{-1} & X - Y^{-1} \end{bmatrix}^{-1}$$

where X and Y are matrix variables which satisfy a parameter-dependent LMI (6).

Considering (10) for $k=1,2,\dots,l$,

$$\begin{aligned} Y_{cl} &> A_{cl}(h_1)^T Y_{cl} A_{cl}(h_1) + C_{cl}(h_1)^T C_{cl}(h_1) \\ &> A_{cl}(h_1)^T (A_{cl}(h_2)^T Y_{cl} A_{cl}(h_2) + C_{cl}(h_2)^T C_{cl}(h_2)) \\ &\quad \cdot A_{cl}(h_1) + C_{cl}(h_1)^T C_{cl}(h_1) \\ &= A_{cl}(h_1)^T A_{cl}(h_2)^T Y_{cl} A_{cl}(h_2) A_{cl}(h_1) \\ &\quad + \sum_{j=1}^2 \Phi(j,1)^T C_{cl}(h_j)^T C_{cl}(h_j) \Phi(j,1) \\ &> \cdots \\ &> A_{cl}(h_1)^T \cdots A_{cl}(h_l)^T Y_{cl} A_{cl}(h_l) \cdots A_{cl}(h_1) \\ &\quad + \sum_{j=1}^l \Phi(j,1)^T C_{cl}(h_j)^T C_{cl}(h_j) \Phi(j,1) \\ &\geq \sum_{j=1}^l \Phi(j,1)^T C_{cl}(h_j)^T C_{cl}(h_j) \Phi(j,1) \end{aligned} \quad (12)$$

is derived. Multiplying $B_{cl}(h_0)^T$ from left-side of (12) and $B_{cl}(h_0)$ from right-side of (12), we have

$$B_{cl}(h_0)^T Y_{cl} B_{cl}(h_0)$$

$$\begin{aligned} &> B_{cl}(h_0)^T \sum_{j=1}^l \Phi(j,1)^T C_j^T C_j \Phi(j,1) B_{cl}(h_0) \\ &= \sum_{j=1}^l g_d(j)^T g_d(j) \end{aligned} \quad (13)$$

for all l . Considering trace and the case ($l \rightarrow \infty$) for the right-side and the left-side of (13), (5) holds by (11).

Now we consider H_2 performance (3) for the original SDSVSR.

$$\begin{aligned} &\left[\frac{1}{h_0} \int_0^{h_0} \left\{ \int_{\sigma}^{\infty} \text{trace}(g(t,\sigma)g^T(t,\sigma)) dt \right\} d\sigma \right] \\ &= \left[\frac{1}{h_0} \int_0^{h_0} \left\{ \sum_{i=1}^{m_1} \int_{\sigma}^{\infty} \|g_i(t,\sigma)\|^2 dt \right\} d\sigma \right] \\ &= \left[\frac{1}{h_0} \int_0^{h_0} \left\{ \sum_{i=1}^{m_1} \int_{t_1}^{\infty} \|g_i(t,\sigma)\|^2 dt \right\} d\sigma \right] \\ &= \left[\frac{1}{h_0} \int_0^{h_0} \left\{ \sum_{i=1}^{m_1} \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} \|g_i(t,\sigma)\|^2 dt \right\} d\sigma \right] \end{aligned} \quad (14)$$

where $g_i(t,\sigma)$ denotes the i -th column of $g(t,\sigma)$. The second equality holds because $g_i(t,\sigma)$ is equivalent to 0 for $t \in [0, h_0]$, which is given by assumption $C_1 e^{A t} B_1 \equiv 0$. Considering each integral in range $[t_k, t_{k+1})$,

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \|g_i(t,\sigma)\|^2 dt \\ &= \int_{t_k}^{t_{k+1}} \left\{ \cdot \right\}^T \left\{ C_1 e^{A(t-t_k)} x(t_k) \right. \\ &\quad \left. + C_1 \int_{t_k}^t e^{A(t-\tau)} B_2 u(\tau) d\tau + D_{12} u(t) \right\} dt \\ &= \int_{t_k}^{t_{k+1}} \left\{ \cdot \right\}^T \left\{ C_1 e^{A(t-t_k)} x(t_k) \right. \\ &\quad \left. + C_1 \int_{t_k}^t e^{A(t-\tau)} B_2 u(t_k) d\tau + D_{12} u(t_k) \right\} dt \\ &(\because u(t) = u(t_k) \text{ for all } t \in [t_k, t_{k+1})) \\ &= \int_0^{h_k} \left\{ \cdot \right\}^T \left\{ C_1 e^{A\theta} x(t_k) \right. \\ &\quad \left. + \left(C_1 \int_0^{\theta} e^{A(\theta-\sigma)} B_2 d\sigma + D_{12} \right) u(t_k) \right\} d\theta \\ &= \begin{bmatrix} x(t_k) \\ u(t_k) \end{bmatrix}^T \int_0^{h_k} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}^T \\ &\quad \cdot \left[C_1 e^{A\theta} \left(C_1 \int_0^{\theta} e^{A(\theta-\sigma)} B_2 d\sigma + D_{12} \right) \right] d\theta \begin{bmatrix} x(t_k) \\ u(t_k) \end{bmatrix} \\ &= \begin{bmatrix} x(t_k) \\ C_c \xi(t_k) \end{bmatrix}^T [\cdot]^T \cdot [\hat{C}_{1d}(h_k) \quad \hat{D}_{12d}(h_k)] \begin{bmatrix} x(t_k) \\ C_c \xi(t_k) \end{bmatrix} \\ &= \begin{bmatrix} x(t_k) \\ \xi(t_k) \end{bmatrix}^T [\cdot]^T \cdot [\hat{C}_{1d}(h_k) \quad \hat{D}_{12d}(h_k) C_c] \begin{bmatrix} x(t_k) \\ \xi(t_k) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} x(t_k) \\ \xi(t_k) \end{bmatrix}^T C_{cl}^T(h_k) C_{cl}(h_k) \begin{bmatrix} x(t_k) \\ \xi(t_k) \end{bmatrix} \\
&= \begin{bmatrix} x(t_1) \\ \xi(t_1) \end{bmatrix}^T \Phi^T(k, 1) C_{cl}^T(h_k) C_{cl}(h_k) \Phi(k, 1) \begin{bmatrix} x(t_1) \\ \xi(t_1) \end{bmatrix} \\
&= \begin{bmatrix} e^{A(h_0-\sigma)} B_1 e_i \\ 0 \end{bmatrix}^T \Phi^T(k, 1) C_{cl}^T(h_k) \\
&\quad \cdot C_{cl}(h_k) \Phi(k, 1) \begin{bmatrix} e^{A(h_0-\sigma)} B_1 e_i \\ 0 \end{bmatrix} \\
&= \text{trace} \left\{ C_{cl}(h_k) \Phi(k, 1) \begin{bmatrix} e^{A(h_0-\sigma)} B_1 e_i \\ 0 \end{bmatrix} \right. \\
&\quad \cdot \left. \begin{bmatrix} e^{A(h_0-\sigma)} B_1 e_i \\ 0 \end{bmatrix}^T \Phi^T(k, 1) C_{cl}^T(h_k) \right\} \quad (15)
\end{aligned}$$

where the third equality holds when we replace $\tau - t_k$ by σ and $t - t_k$ by θ respectively, e_i ($i = 1, \dots, m_1$) denotes the i -th basis of $\mathbb{R}^{m_1}$ and $\{\cdot\}^T$ or $[\cdot]^T$ denotes the transpose of the corresponding matrix.

Substituting (15) to (14), we have

$$\begin{aligned}
&\left[\frac{1}{h_0} \int_0^{h_0} \left\{ \int_\sigma^\infty \text{trace} (g(t, \sigma) g^T(t, \sigma)) dt \right\} d\sigma \right] \\
&= \left[\frac{1}{h_0} \int_0^{h_0} \left\{ \sum_{i=1}^{m_1} \sum_{k=1}^\infty \text{trace} (C_{cl}(h_k) \Phi(k, 1) \right. \right. \\
&\quad \cdot \left. \begin{bmatrix} e^{A(h_0-\sigma)} B_1 e_i e_i^T B_1^T e^{A^T(h_0-\sigma)} & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \cdot \left. \left. \Phi^T(k, 1) C_{cl}^T(h_k) \right) \right\} d\sigma \right] \\
&= \left[\frac{1}{h_0} \int_0^{h_0} \left\{ \sum_{k=1}^\infty \text{trace} (C_{cl}(h_k) \Phi(k, 1) \right. \right. \\
&\quad \cdot \left. \begin{bmatrix} e^{A(h_0-\sigma)} B_1 B_1^T e^{A^T(h_0-\sigma)} & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \cdot \left. \left. \Phi^T(k, 1) C_{cl}^T(h_k) \right) \right\} d\sigma \right] \\
&= \sum_{k=1}^\infty \text{trace} (C_{cl}(h_k) \Phi(k, 1) \\
&\quad \cdot \left[\frac{1}{h_0} \int_0^{h_0} \begin{bmatrix} e^{A(h_0-\sigma)} B_1 B_1^T e^{A^T(h_0-\sigma)} d\sigma & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \cdot \left. \left. \Phi^T(k, 1) C_{cl}^T(h_k) \right) \right] \\
&= \sum_{k=1}^\infty \text{trace} \left(C_{cl}(h_k) \Phi(k, 1) \right. \\
&\quad \cdot \left. \begin{bmatrix} \tilde{B}_{1d}(h_0) \tilde{B}_{1d}^T(h_0) & 0 \\ 0 & 0 \end{bmatrix} \Phi^T(k, 1) C_{cl}^T(h_k) \right) \\
&= \sum_{k=1}^\infty \text{trace} (C_{cl}(h_k) \Phi(k, 1) B_{cl}(h_0) \\
&\quad \cdot B_{cl}^T(h_0) \Phi^T(k, 1) C_{cl}^T(h_k))
\end{aligned}$$

$$= \sum_{k=1}^\infty \text{trace} \{ g_d(k) g_d^T(k) \} \quad (16)$$

where the second equality holds because $\sum_{i=1}^{m_1} e_i e_i^T$ equals identity matrix of size m_1 .

Substituting (16) to (5), we have

$$\begin{aligned}
&\left[\frac{1}{h_0} \int_0^{h_0} \left\{ \int_\sigma^\infty \text{trace} (g(t, \sigma) g^T(t, \sigma)) dt \right\} d\sigma \right] \\
&= \sum_{k=1}^\infty \text{trace} \{ g_d(k) g_d^T(k) \} \\
&< \gamma^2.
\end{aligned}$$

Thus (3) holds and the proof is complete. $\square$