Higher order K-scheme and application to derivative pricing

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Abstract

The author presents a new discretization method of stochastic differential equations (SDEs) without proof. The method is a kind of K-scheme (Kusuoka approximation, Kusuoka-Lyons-Ninomiya-Victoir method). K-scheme is a new framework of the higher order discretization methods for approximating SDEs weakly, proposed by Kusuoka. Ninomiya-Victoir and Ninomiya-Ninomiya methods are known as practical realizations of K-scheme, and some extrapolations of these methods are proposed. The new method is higher order than these methods without the use of extrapolations, and the author applies it to a financial problem of derivative pricing.

1 Introduction

1.1 Background

Our problem is the calculation of $E[f(X(T, x))]$ where $X(t, x)$ is the value at time t of a N -dimensional diffusion process and f is a function from $\mathbb{R}^N$ into $\mathbb{R}$. This calculation is important for pricing financial derivatives. There are two approaches to this problem: the partial differential equation approach and the simulation approach. We focus on the simulation approach in this paper.

When we use the simulation approach, the calculation is implemented by following two steps. The first step is the discretization of the SDE, i.e., the construction of a set of random variables $\{X^{(n)}(t_i, x)\}_{i=0,1,\dots,n}$ ($0 = t_0 < t_1 < \dots < t_n = T$) that approximates $\{X(t, x)\}_{0 \leq t \leq T}$ in a precise sense. Here n represents the number of partitions of $[0, T]$. Euler-Maruyama and Milstein methods are well known discretization methods of SDEs. The second step is the calculation of $E[f(X^{(n)}(T, x))]$, which is the calculation of a finite dimensional integration. We denote by $D(n)$ the number of dimensions of the domain. One usually calculates the integral by Monte Carlo method or quasi-Monte Carlo method. In each two step, a certain kind of error is involved: the discretization error and the integration error. Here the discretization error represents the difference between $E[f(X(T, x))]$ and $E[f(X^{(n)}(T, x))]$, and the integration error the error of Monte Carlo method or quasi-

Monte Carlo method. When the discretization error is $O(p)$, i.e., there exists a constant $C > 0$ such that $|E[f(X(T, x))] - E[f(X^{(n)}(T, x))]| < C \frac{1}{n^p}$ holds, we say that the discretization method is of order p . Euler-Maruyama method is of order 1. By using higher order discretization methods, we can reduce n and $D(n)$. Therefore we can calculate $E[f(X^{(n)}(T, x))]$ with the quasi-Monte Carlo method efficiently. This point is very important, see remark 6.2 of [14].

K-scheme is a framework for higher order discretization methods based on the stochastic Taylor expansion, Malliavin calculus, free Lie algebra and rough path theory, etc [8, 10, 12]. Ninomiya-Victoir [15] and Ninomiya-Ninomiya [14] methods are known as a kind of K-scheme. They are practical discretization methods of order 2, and some applications of these methods are studied in [2, 1].

There are some techniques to attain higher order discretization methods. Romberg extrapolation is general technique for increasing the order of some methods, it is proved to be applicable to Euler-Maruyama method in [19] and Ninomiya-Victoir and Ninomiya-Ninomiya methods in [11]. Other type of extrapolation methods for Ninomiya-Victoir method which is a kind of classical Richardson extrapolation method is studied in [5, 16, 3]. However, in these extrapolation methods, negative weighted paths appear, and these methods are not numerically stable in some cases. Furthermore, the cubature method on Wiener space of higher order is proposed in [6].

1.2 Our result

In this paper, we give a discretization method of order 3 for the SDE driven by 2-dimensional Brownian motion without proof. The new method enables us to attain higher order discretization with only positive weighted paths.

In order to verify the numerical behavior of the new discretization method, we apply it to a financial problem of pricing an Asian call option under the Heston model. The new method is compared with other discretization methods and these each method with Romberg extrapolation. As numerical results show, we achieve the theoretical order by the new method.

2 Approximation theorem for K-scheme and existing methods

2.1 The problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $t \in [0, \infty)$, let $B^0(t) = t$ and $(B^1(t), \dots, B^d(t))$ be a d -dimensional standard Brownian motion. $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ denotes the space of $\mathbb{R}^N$ -valued smooth functions defined in $\mathbb{R}^N$ whose derivatives of any order are bounded.

Now, let $X : \Omega \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a solution of the SDE written in Stratonovich form as

$$X(t, x) = x + \sum_{i=0}^d \int_0^t V_i(X(s, x)) \circ dB^i(s), \quad (1)$$

where $x \in \mathbb{R}^N$, $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Here we can identify $V \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ as a smooth vector field on $\mathbb{R}^N$ via

$$Vf(x) = \sum_{j=1}^N V^{(j)}(x) \frac{\partial f}{\partial x_j}(x), \text{ for } f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}).$$

Our aim is the numerical evaluation of $E[f(X(T, x))]$, where $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$.

2.2 Some notations of free Lie algebra and vector field, iterated Wiener integral

Let $A = \{0, 1, \dots, d\}$ be a set of the index of vector fields and the Brownian motion, and we denote by A^* the set of all words consisting of elements of A (i.e. $A^* = \{a_1 a_2 \dots a_k \mid k \in \mathbb{N} \cup 0, a_i \in A\}$). Here the empty word ϕ is the identity of A^* . For $\alpha \in A^*$, we denote an iterated Wiener integral as follows.

$$B^\phi(t) = 1,$$

$$B^{\alpha}(t) = \int_0^t \int_0^{s_k} \dots \int_0^{s_2} \circ dB^{a_1}(s_1) \dots \circ dB^{a_{k-1}}(s_{k-1}) \circ dB^{a_k}(s_k) \\ \text{for } \alpha = a_1 a_2 \dots a_k \in A^* \setminus \{\phi\}.$$

Let $\mathbb{R}\langle A \rangle$ be the free $\mathbb{R}$ -algebra with basis A^* and $\mathbb{R}\langle\langle A \rangle\rangle$ be the set of all $\mathbb{R}$ -coefficient formal series with basis A^* (i.e. $\mathbb{R}\langle A \rangle = \{\sum_{k=1}^L r_k \alpha_k \mid L \in \mathbb{N}, r_k \in \mathbb{R}, \alpha_k \in A^*\}$ and $\mathbb{R}\langle\langle A \rangle\rangle = \{\sum_{\alpha \in A^*} r_\alpha \alpha \mid r_\alpha \in \mathbb{R}\}$).

For $P = \sum_{\alpha \in A^*} r_\alpha \alpha, Q = \sum_{\beta \in A^*} r_\beta \beta$, the product PQ is defined by

$$PQ = \sum_{\theta = \alpha\beta \in A^*} r_\alpha r_\beta \theta,$$

and the inner product $\langle P, Q \rangle$ by

$$\langle P, Q \rangle = \sum_{\alpha = \beta} r_\alpha r_\beta,$$

and let $\|P\|_2 = (\langle P, P \rangle)^{\frac{1}{2}}$. The Lie bracket is defined as $[P, Q] = PQ - QP$. We define $\mathcal{L}_{\mathbb{R}}(A)$ as the set of Lie

polynomials in $\mathbb{R}\langle A \rangle$, i.e., the smallest $\mathbb{R}$ -submodule of $\mathbb{R}\langle A \rangle$ including A and closed under the Lie bracket.

For $\alpha = a_1 a_2 \dots a_k \in A^*$, we define the length of α as $|\alpha| = k$, and the order of α as $\|\alpha\| = |\alpha| + \#\{1 \leq i \leq k \mid a_i = 0\}$. For $m \in \mathbb{N}$, we denote $A_m^* = \{\alpha \in A^* \mid \|\alpha\| = m\}$, $A_{\leq m}^* = \{\alpha \in A^* \mid \|\alpha\| \leq m\}$ and $\mathbb{R}\langle A \rangle_{\leq m} = \{\sum_{k=1}^L r_k \alpha_k \mid L \in \mathbb{N}, r_k \in \mathbb{R}, \alpha_k \in A_{\leq m}^*\}$. Then we define the projection $j_m : \mathbb{R}\langle\langle A \rangle\rangle \rightarrow \mathbb{R}\langle A \rangle_{\leq m}$ as follows:

$$j_m \left(\sum_{\alpha \in A^*} r_\alpha \alpha \right) = \sum_{\alpha \in A_{\leq m}^*} r_\alpha \alpha.$$

For $P \in \mathbb{R}\langle\langle A \rangle\rangle$ whose coefficient of ϕ is 0, we define

$$\exp(P) = 1 + \sum_{k=1}^{\infty} P^k / k!. \quad (2)$$

In addition, for $Q \in \mathbb{R}\langle\langle A \rangle\rangle$ whose coefficient of ϕ is 1, $\log(Q)$ is defined as $\log(Q) = \sum_{k=1}^{\infty} (-1)^{k-1} (Q - 1)^k / k$. Then we have the relations

$$\log(\exp(P)) = P \quad \text{and} \quad \exp(\log(Q)) = Q.$$

Let Φ be the homomorphism from $\mathbb{R}\langle A \rangle$ to the $\mathbb{R}$ -algebra consisting of smooth differential operators over $\mathbb{R}^N$ such that

$$\Phi(\phi) = \text{Id},$$

$$\Phi(\alpha) = V_{a_1} V_{a_2} \dots V_{a_k} \quad \text{for } \alpha = a_1 a_2 \dots a_k \in A^* \setminus \{\phi\}.$$

Here Id represents the identity operator. For smooth vector fields W_1, W_2 , we define the Lie bracket of vector fields as $[W_1, W_2] = W_1 W_2 - W_2 W_1$. Then it is well known that $[W_1, W_2]$ is also a smooth vector field on $\mathbb{R}^N$.

For $\alpha \in A^*$, we define a smooth differential operator V_α and a smooth vector field $V_{[\alpha]}$ as follows.

$$V_\phi = \text{Id},$$

$$V_\alpha = V_{a_1} V_{a_2} \dots V_{a_k} \quad \text{for } \alpha = a_1 a_2 \dots a_k \in A^* \setminus \{\phi\}.$$

$$V_{[\phi]} = \text{Id},$$

$$V_{[\alpha]} = [[\dots [V_{a_1}, V_{a_2}], \dots], V_{a_k}]$$

$$\text{for } \alpha = a_1 a_2 \dots a_k \in A^* \setminus \{\phi\}.$$

For a smooth vector field V and $x \in \mathbb{R}^N$, we consider the ordinary differential equation

$$\frac{dz(u)}{du} = V(z(u)), \quad z(0) = x,$$

and we denote the solution $z(s)$ by $\exp(sV)(x)$. Then we see $\exp(sV)(\cdot)$ is a map from $\mathbb{R}^N$ to $\mathbb{R}^N$, and we define a linear operator $\text{Exp}(V)$ on $C_b^\infty(\mathbb{R}^N; \mathbb{R})$ by

$$\text{Exp}(V)(f)(x) = f(\exp(V)(x)). \quad (3)$$

Then we have the relation by applying Taylor expansion as follows; for $P \in \mathcal{L}_{\mathbb{R}}(A)$, $\Phi(P)$ is a smooth vector field and

$$\text{Exp}(\Phi(tP)) = \Phi(j_m(\exp(tP))) + O(t^{m+1}).$$

We regard $\text{Exp}(V)$ as a smooth differential operator over $\mathbb{R}^N$ via Taylor expansion.

2.3 Approximation theorem for K-scheme

In this subsection we introduce approximation theorem for K-scheme, based on [8, 10].

We begin with the necessary condition of vector fields of the SDE (1) for applying K-scheme. This condition is necessary to obtain a precise estimation of derivatives of the solution of the SDE (1); for more details, see [9].

Definition 2.1 (UFG condition). We say that vector fields $V_0, V_1, \dots, V_d$ satisfy UFG condition, if there exists an integer l such that for all $\alpha \in A^* \setminus \{\emptyset, (0)\}$, there are $\{\beta \mid \|\beta\| \leq l\} \subset A^* \setminus \{\emptyset, (0)\}$ and $\{\varphi_{\alpha, \beta}\} \subset C_b^\infty(\mathbb{R}^N; \mathbb{R})$ which satisfy

$$V_{[\alpha]} = \sum_{\beta} \varphi_{\alpha, \beta} V_{[\beta]}$$

Then we define the condition of approximation operators as follows.

Definition 2.2 (m -similar operator). We say that Markov operator $Q_{(s)}$ is m -similar operator if there exists a $\mathcal{L}_{\mathbb{R}}(A)$ -valued random variable $\hat{L}_{(s)}$ such that

$$E[j_m(\exp(\hat{L}_{(s)}))] = E\left[j_m\left(\sum_{\alpha \in A^*} B^{\alpha}(s)\alpha\right)\right],$$

$$E\left[\left\|j_m(\hat{L}_{(s)})\right\|_2^n\right] < \infty \quad \text{for } n \in \mathbb{N}$$

and

$$Q_{(s)} = \text{Exp}\left(\Phi(\hat{L}_{(s)})\right)$$

hold.

Remark 2.1. This definition is slightly different from original Kusuoka's work [8, 10]. In fact, this definition is affected by the spirit of [12] in terms of not taking projection of $\hat{L}_{(s)}$.

The intuitive understanding of this condition is that the operator approximates the stochastic Taylor series of the solution of the SDE (1) up to order m . Then we describe the approximation theorem.

Theorem 2.1 (Approximation theorem for K-scheme [8, 10]). Suppose that vector fields of the SDE (1) satisfy UFG condition, and $Q_{(s)}$ is m -similar operator. Then there exists $C > 0$ such that

$$\|E[f(X(s, x))] - E[Q_{(s)}f(x)]\|_{\infty} \leq Cs^{\frac{m+1}{2}} \|\text{grad}(f)\|_{\infty}.$$

2.4 Existing discretization methods

Ninomiya-Victoir [15], Ninomiya-Ninomiya [14] methods are known as a kind of K-scheme, and their operators are as follows.

Proposition 2.2. (Ninomiya-Victoir [15], Ninomiya-Ninomiya [14] methods) Let $\eta_1, \eta_2, \dots, \eta_d$ and $\eta'_1, \eta'_2, \dots, \eta'_d$ are independent normal Gaussian random variables.

$$Q_{(s)}^{(NV)} = \frac{1}{2} \left(\text{Exp}(sV_0) \text{Exp}(\sqrt{s}\eta_1 V_1) \cdots \text{Exp}(\sqrt{s}\eta_d V_d) \right. \\ \left. + \text{Exp}(\sqrt{s}\eta_d V_d) \cdots \text{Exp}(\sqrt{s}\eta_1 V_1) \text{Exp}(sV_0) \right),$$

$$Q_{(s)}^{(NN)} = \text{Exp}\left(c_1 s V_0 + \sqrt{R_{11}} s \eta_1 V_1 + \cdots \sqrt{R_{11}} s \eta_d V_d\right) \\ \text{Exp}\left(c_2 s V_0 + \left(\frac{R_{12}}{\sqrt{R_{11}}} \eta_1 + \sqrt{R_{22} - \frac{R_{12}^2}{R_{11}}} \eta'_1\right) \sqrt{s} V_1 + \right. \\ \left. \cdots + \left(\frac{R_{12}}{\sqrt{R_{11}}} \eta_d + \sqrt{R_{22} - \frac{R_{12}^2}{R_{11}}} \eta'_d\right) \sqrt{s} V_d\right).$$

Here the parameters of Ninomiya-Ninomiya method are as follows:

$$c_1 = \frac{\mp \sqrt{2(2u-1)}}{2}, c_2 = 1 \pm \frac{\sqrt{2(2u-1)}}{2}, R_{11} = u, \\ R_{22} = 1 + u \pm \sqrt{2(2u-1)}, R_{12} = -u \mp \frac{\sqrt{2(2u-1)}}{2}$$

for some $u \geq 1/2$.

These operators are 5-similar.

Remark 2.2. Although $Q_{(s)}^{(NV)}, Q_{(s)}^{(NN)}$ are not classified into m -similar operator of the above definition, but these operators are just like m -similar operators and the evaluation in Theorem 2.1 can be obtained. Furthermore, in [11], these operators are analyzed more precisely and proved to have some nice properties. These properties ensure that Romberg extrapolation is applicable to Ninomiya-Victoir and Ninomiya-Ninomiya methods.

3 Main result

In this section, we present a 7-similar operator with $d = 2$. Here we omit details of the construction and proof, but the operator is constructed based on [8].

Theorem 3.1 (Approximation operator of order 3 for the SDE driven by 2-dimensional Brownian motion). Let $\eta_1, \eta_2, \eta_{12}, \eta_{01}, \eta_{112}, \eta_{122}$ are a set of independent normal Gaussian random variables.

$$Q_{(s)}^{(7,2)} = \text{Exp}\left((\eta_1 V_1 + \eta_2 V_2) \sqrt{s} + \right. \\ \left. \left(V_0 + \frac{1}{2\sqrt{3}}(\eta_{02}\eta_1 + \eta_{01}\eta_2 + \eta_{12})[V_1, V_2]\right)s \right. \\ \left. + \left(\frac{-\eta_{01}}{2\sqrt{3}}[V_0, V_1] + \frac{\eta_{02}}{2\sqrt{3}}[V_0, V_2] \right. \right. \\ \left. + \frac{\eta_2 + 2\eta_{112}}{12}[V_1, [V_1, V_2]] + \frac{\eta_1 + 2\eta_{122}}{12}[[V_1, V_2], V_2]\right)s^{\frac{3}{2}} \\ \left. + \left(\frac{1}{12}[[V_0, V_1], V_1] + \frac{1}{12}[[V_0, V_2], V_2] \right. \right. \\ \left. + \frac{(\eta_{02}\eta_1 + \eta_{01}\eta_2 + \eta_{12})}{36\sqrt{3}}([V_1, [V_1, [V_1, V_2]]] \right. \\ \left. + [[V_1, V_2], V_2], V_2])\right)s^2$$

$$\begin{aligned}
& + \left(\frac{\eta_2[V_1, [V_1, [V_1, [V_1, V_2]]]]}{360} + \frac{\eta_1[V_1, [V_1, [[V_1, V_2], V_2]]]}{360} \right. \\
& + \frac{\eta_1[[V_1, [V_1, V_2]], [V_1, V_2]]}{120} + \frac{\eta_2[V_1, [[[V_1, V_2], V_2], V_2]]]}{360} \\
& + \frac{\eta_2[[V_1, V_2], [[V_1, V_2], V_2]]]}{90} + \frac{\eta_1[[[[V_1, V_2], V_2], V_2], V_2]]]}{360} \Big) s^{\frac{5}{2}} \\
& + \left(\frac{[[[[V_0, V_1], V_1], V_1], V_1]]]}{360} + \frac{[[[[V_0, V_2], V_2], V_2], V_2]]]}{360} \right. \\
& + \frac{[V_0, [V_1, [[V_1, V_2], V_2]]]}{180} + \frac{[[V_0, [V_1, V_2]], [V_1, V_2]]]}{60} \\
& + \frac{[[V_0, [[V_1, V_2], V_2]], V_1]}{120} + \frac{[[V_0, V_2], [V_1, [V_1, V_2]]]}{120} \\
& \left. + \frac{[[[V_0, V_2], [V_1, V_2]], V_1]}{90} + \frac{[[[[V_0, V_2], V_2], V_1], V_1]]]}{180} \right) s^3.
\end{aligned}$$

Then $Q_{(s)}^{(7,2)}$ is 7-similar with $d = 2$.

We denote the smooth vector field $L_{(s)}^{(7,2)}$ such that $Q_{(s)}^{(7,2)} = \text{Exp} \left(L_{(s)}^{(7,2)} \right)$ holds.

Remark 3.1 (The construction of approximation operator). There are two key points of the construction. The first is the approximation of joint probability distribution of iterated Wiener integrals appearing in stochastic Taylor series. For this point, we construct a moment similar family introduced by Kusuoka [8], which is a family of random variables which mimic the family of iterated Wiener integrals in a sense.

The second is that we should avoid higher order differential operator appearing in stochastic Taylor series. To overcome this obstacle, we use the result of Takanobu [18]. He expresses the stochastic Taylor series of the solution of the SDE by an exponential of a Lie series as follows.

$$\begin{aligned}
& \sum_{\alpha \in A^*} B^{\alpha} \alpha = \\
& \exp \left(\sum_{m=1}^{\infty} \sum_{||\alpha||=m} \left(\sum_{\sigma \in S_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} B^{\circ \sigma^{-1}(\alpha)} [\alpha] \right) \right).
\end{aligned}$$

Here S_m is the symmetric group of degree m , $e(\sigma) = \#\{1 \leq j \leq m \mid \sigma(j) > \sigma(j+1)\}$. In addition, we change the basis of Lie series to Lyndon basis [17].

4 Numerical results - Pricing Asian option under the Heston model

4.1 Problem

We consider an Asian call option written on an asset whose price process follows the Heston stochastic volatility model [7]. Let $Y^{(1)}(t, x)$ be the price process and $Y^{(2)}(t, x)$ be the volatility process as follows.

$$\begin{aligned}
dY^{(1)}(t, x) &= \mu Y^{(1)}(t, x) dt + \\
& Y^{(1)}(t, x) \sqrt{Y^{(2)}(t, x)} dB^{(1)}(t),
\end{aligned}$$

$$\begin{aligned}
dY^{(2)}(t, x) &= \alpha(\theta - Y^{(2)}(t, x)) dt \\
& + \beta \sqrt{Y^{(2)}(t, x)} \left(\rho dB^{(1)}(t) + \sqrt{1 - \rho^2} dB^{(2)}(t) \right)
\end{aligned}$$

where $x = (x_1, x_2) \in (\mathbb{R}_{>0})^2$, $-1 \leq \rho \leq 1$, and α, β, θ are some positive constants such that

$$2\alpha\theta - \beta^2 > 0$$

to ensure the existence and uniqueness of a solution to the SDE [4].

The payoff of the Asian call option on this asset with maturity T and strike K is $\max(Y^{(3)}(T, x)/T - K, 0)$ where

$$Y^{(3)}(t, x) = \int_0^t Y^{(1)}(s, x) ds.$$

Let $Y(t, x) = {}^t(Y^{(1)}(t, x), \log(Y^{(2)}(t, x)), Y^{(3)}(t, x))$. Here we change the coordinate of $Y^{(2)}$ for numerical stability of the simulation. Then $Y(t, x)$ is the solution of the Stratonovich form stochastic differential equations

$$Y(t, x) = \sum_{i=0}^2 \int_0^t W_i(Y(s, x)) \circ dB^i(s)$$

where

$$\begin{aligned}
W_0 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} \left(\mu y_1 - \frac{\beta \rho}{4} \right) y_1 - \frac{1}{2} y_1 e^{y_2} \\ \left(\alpha \theta - \frac{\beta^2}{4} \right) e^{-y_2} - \alpha \\ y_1 \end{pmatrix}, \\
W_1 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} y_1 e^{\frac{y_2}{2}} \\ \beta \rho e^{-\frac{y_2}{2}} \\ 0 \end{pmatrix}, \\
W_2 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ \beta \sqrt{1 - \rho^2} e^{-\frac{y_2}{2}} \\ 0 \end{pmatrix}.
\end{aligned}$$

4.2 Implementation of the new method

We implement our method via (3), i.e.,

$$Q_{(s)}^{(7,2)} f(x) = \text{Exp} \left(L_{(s)}^{(7,2)} \right) f(x) = f \left(\exp \left(L_{(s)}^{(7,2)} \right) (x) \right).$$

In order to calculate $\exp \left(L_{(s)}^{(7,2)} \right) (x)$, we need to calculate Lie brackets of W_0, W_1, W_2 and derive the ordinary differential equation to be solved. Details of these calculations are given in appendix.

Then we solve the ordinary differential equation numerically. We remark that it is necessary to solve the ordinary differential equation by a certain order to realize higher order discretization of the SDE, for more details, see Theorem 1.3 of [14]. Here we need to solve the ordinary differential equation by order 7. There are some methods for this calculation, for example 7th-order explicit Runge-Kutta method, Yoshida algorithm [20]. Here we present numerical results obtained by applying 7th-order Runge-Kutta method. For the implementation of 7th-order Runge-Kutta method, see the appendix of [14].

Fig. 1: Discretization Error and Num. of Partitions

Fig. 2: Discretization Error and Num. of Partitions (with Romberg extrapolation)

Remark 4.1 (The partitions of time). According to the result of [8, 10], in order to obtain the order 3 discretization method by $Q_{(s)}^{(7,2)}$, we should take the partitions $0 = t_0 < t_1 \cdots < t_n = T$ as follows :

$$t_k = \frac{k^\gamma T}{n^\gamma}, \quad k = 0, 1, \dots, n, \quad \gamma > 6.$$

However, in the numerical experience, we take the even partition from the viewpoint of practical applications. In fact, it is proved that we can take the even partition to obtain the order 2 discretization method when we use Ninomiya-Victoir or Ninomiya-Ninomiya method in [11].

4.3 Numerical result

We set $T = 1, K = 1.05, \mu = 0.05, \alpha = 2.0, \beta = 0.1, \theta = 0.09, \rho = 0.3$, and $(x_1, x_2) = (1.0, 0.09)$ and take the true value as follows:

$$E[\max(Y_3(T, x)/T - K, 0)] = 0.06068740243939$$

This value is obtained by the new method and the quasi-Monte Carlo with the number of partitions $n = 100$ and the number of sample points $M = 10^9$.

The relations between discretization error and the number of partitions of each method are plotted in Fig.1 and Fig. 2.

Here we use the quasi-Monte Carlo method with the number of sample points $M = 10^8$ for existing methods and $M = 10^9$ for the new method.

As the numerical results show, we can attain the theoretical order by the new method.

5 Discussion

We can see some additional facts from the numerical results. The first is that we can reduce the discretization error by the new method with 1 partition in the same level by Euler-Maruyama method with 200 partitions in our case, but any mathematical justification of this fact is not established. Second, it seems that the new method with Romberg extrapolation is of order 4, i.e., Romberg extrapolation is applicable to the new method in our case, but any mathematical justification is not established as well.

A The implementation of the new method to pricing an Asian option under the Heston model

The Lie brackets of vector fields W_0, W_1, W_2 are for example,

$$\begin{aligned} [W_1, W_2] &= -\frac{\beta\sqrt{1-\rho^2}}{2}, \\ [W_1, [W_1, W_2]] &= [W_0, W_1] = 0. \end{aligned}$$

In these calculation, the relation formulated in [13] is very useful.

The ordinary differential equations to be solved are as follows:

$$\begin{cases} \frac{dz_1(t)}{dt} = z_1(t) \sum_{i=2,1,0,-1,-2,-4} A_i(s) \exp\left(\frac{i}{2}z_2(t)\right), \\ \frac{dz_2(t)}{dt} = \sum_{i=0,-1,-2,-3,-4,-6} B_i(s) \exp\left(\frac{i}{2}z_2(t)\right), \\ \frac{dz_3(t)}{dt} = z_1(t) \sum_{i=0,1,2,4} C_i(s) \exp\left(\frac{i}{2}z_2(t)\right). \end{cases}$$

Here the coefficients of the the ordinary differential equations $\{A_i(s)\}_{i=2,1,0,-1,-2,-4}$, $\{B_i(s)\}_{i=0,-1,-2,-3,-4,-6}$, $\{C_i(s)\}_{i=0,1,2,4}$ are random variables as follows. Let $\eta_1, \eta_2, \eta_{12}, \eta_{01}, \eta_{02}$ be a set of independent Gaussian random variables.

$$\begin{cases} A_2(s) = -\frac{1}{2}s \\ A_1(s) = \eta_1\sqrt{s} - \frac{(-\eta_{01}\alpha + \eta_{01}\beta\rho - \eta_{02}\beta\sqrt{1-\rho^2})s^{\frac{3}{2}}}{4\sqrt{3}} \\ A_0(s) = \left(\mu - \frac{\beta\rho}{4} - \frac{\beta\sqrt{1-\rho^2}(\eta_{02}\eta_1 + \eta_{01}\eta_2 + \eta_{12})}{4\sqrt{3}}\right)s + \frac{2\alpha\beta\rho - \beta^2}{48}s^2 \\ A_{-1}(s) = -\frac{(4\alpha\theta - \beta^2)\eta_{01}}{16\sqrt{3}}s^{\frac{3}{2}} \\ A_{-2}(s) = \frac{4\alpha\theta - \beta^2}{96}\beta\rho s^2 \\ A_{-4}(s) = \frac{4\alpha\theta - \beta^2}{960}\beta^3\rho s^3 \end{cases}$$

$$\begin{cases} B_0(s) = -\alpha s \\ B_{-1}(s) = (\eta_1\beta\rho + \eta_2\beta\sqrt{1-\rho^2})\sqrt{s} - \frac{\alpha(\eta_{01}\beta\rho - \eta_{02}\beta\sqrt{1-\rho^2})s^{\frac{3}{2}}}{4\sqrt{3}} \\ B_{-2}(s) = \frac{(4\alpha\theta - \beta^2)}{4}s \\ B_{-3}(s) = \frac{(4\alpha\theta - \beta^2)(-\eta_{01}\beta\rho + \eta_{02}\beta\sqrt{1-\rho^2})}{16\sqrt{3}}s^{\frac{3}{2}} \\ B_{-4}(s) = \frac{4\alpha\theta - \beta^2}{96}\beta^2 s^2 \\ B_{-6}(s) = \frac{4\alpha\theta - \beta^2}{960}\beta^4 s^3 \end{cases}$$

$$\begin{cases} C_0(s) = s + \frac{\beta\rho}{24}s^2 + \frac{\beta^2}{240}s^3 - \frac{\beta^2\rho^2}{480}s^3 \\ C_1(s) = \frac{\eta_{01}}{2\sqrt{3}}s^{\frac{3}{2}} \\ C_2(s) = \frac{1}{12}s^2 + \frac{\beta\rho}{120}s^3 \\ C_4(s) = \frac{1}{360}s^3 \end{cases}$$

References

- [1] Aurélien Alfonsi. High order discretization schemes for the CIR process: application to affine term structure and Heston models. *Mathematics of Computation*, 79(269):209–237, 2010.
- [2] Christian Bayer, Peter Friz, and Ronnie Loeffen. Semi-closed form cubature and applications to financial diffusion models. *Quantitative Finance*, 13(5):769–782, 2013.
- [3] Philipp Doersek, Josef Teichmann, and Dejan Velušček. Cubature methods for stochastic (partial) differential equations in weighted spaces. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(4):634–663, 2013.
- [4] William Feller. Two singular diffusion problems. *Annals of mathematics*, pages 173–182, 1951.
- [5] Takehiko Fujiwara. Sixth order methods of kusuoka approximation. *PreprintSeries, Graduate School of Mathematical Sciences, Univ. Tokyo*, 2006.
- [6] Lajos Gergely Gyurkó and Terry Lyons. Efficient and practical implementations of cubature on Wiener space. In *Stochastic analysis 2010*, pages 73–111. Springer, 2011.
- [7] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. In *Review of Financial Studies*, 1993.
- [8] Shigeo Kusuoka. Approximation of expectation of diffusion process and mathematical finance. In *Taniguchi Conference on Mathematics Nara*, volume 98, pages 147–165, 2001.
- [9] Shigeo Kusuoka. Malliavin calculus revisited. *J. Math. Sci. Univ. Tokyo*, 10(2):261–277, 2003.
- [10] Shigeo Kusuoka. Approximation of expectation of diffusion processes based on lie algebra and malliavin calculus. *Advances in Mathematical Economics*, 6:69–83, 2004.
- [11] Shigeo Kusuoka. Gaussian k-scheme: justification for klnv method. *Advances in Mathematical Economics Volume 17*, 17:71, 2013.
- [12] Terry Lyons and Nicolas Victoir. Cubature on Wiener space. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 460, pages 169–198. The Royal Society, 2004.
- [13] Yusuke Morimoto and Makiko Sasada. Algebraic structure of vector fields in financial diffusion models and its applications. *arXiv preprint arXiv:1510.02013*, 2015.
- [14] Mariko Ninomiya and Syoiti Ninomiya. A new higher-order weak approximation scheme for stochastic differential equations and the Runge–Kutta method. *Finance and Stochastics*, 13(3):415–443, 2009.
- [15] Syoiti Ninomiya and Nicolas Victoir. Weak approximation of stochastic differential equations and application to derivative pricing. *Applied Mathematical Finance*, 15(2):107–121, 2008.

- [16] Kojiro Oshima, Josef Teichmann, and Dejan Velušček. A new extrapolation method for weak approximation schemes with applications. *The Annals of Applied Probability*, 22(3):1008–1045, 2012.
- [17] C. Reutenauer. Free Lie Algebras. *LMS monographs*. Clarendon Press, 1993.
- [18] Satoshi Takanobu. Multiple stochastic integrals appearing in the stochastic Taylor expansions. *Journal of the Mathematical Society of Japan*, 47(1):67–92, 1995.
- [19] Denis Talay and Luciano Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic analysis and applications*, 8(4):483–509, 1990.
- [20] Haruo Yoshida. Construction of higher order symplectic integrators. *Physics Letters A*, 150(5):262–268, 1990.