Effects on Hedging Errors of First-to-Default Swap from Parameter Estimation Errors

Hirofumi. Hebiguchi
Graduate School of Science and Engineering, Hosei University
3-7-2, Kajino-cho, Koganei-shi, Tokyo, 184-8584, Japan
E-mail: hirofumi.hebiguchi.9k@stu.hosei.ac.jp

Kazuhiro Yasuda
Faculty of Science and Engineering, Hosei University
3-7-2, Kajino-cho, Koganei-shi, Tokyo, 184-8584, Japan
E-mail: k_yasuda@hosei.ac.jp

Abstract
In this paper, we consider effects on hedging errors of a first-to-default swap (FTDS) under a hazard model with deterministic intensities from parameter estimation errors. Here parameter means default intensities of FTDS reference companies. As our mathematical model of FTDS and a credit default swap (CDS) market, we mainly follow Bielecki et al. [1] and [2] and use copula functions to represent the correlation of default times. We give some analytical and numerical results of hedging errors from parameter estimation errors.

1 Introduction
In this paper we attempt to analytically and numerically estimate hedging errors of a first-to-default swap (FTDS) from parameter estimation errors. Constructions of hedging strategies of credit derivatives are investigated by a lot of researchers and practitioners under several settings of defaults and products. For example, see Bielecki et al. [2], Cherubini et al. [3], O’Kane [7] and Schönbucher [8]. We here rely on mathematical models of FTDS and a credit derivative swap (CDS) market, which are given in Bielecki et al. [1]. In their papers, hedging strategies of basket credit derivatives under a general framework of defaults, in which a hazard model with deterministic default intensities is included, are constructed using the CDS market in which they assume that the same reference company’s CDSs as credit names of the credit derivative are traded. However their replicability holds under a restriction that the joint distribution of defaults are known, that is, in the case of a hazard model with deterministic intensities, it means that the default intensities are known. The assumption is unrealistic, namely they are commonly unobservable and we need to estimate them and unfortunately have estimation errors. Therefore in this paper, we attempt to estimate effects on hedging errors of FTDS using hedging strategies given in Bielecki et al. [1] with intensities containing errors.

In this paper we treat with CDS and FTDS, which are categorized in credit derivatives. CDS is explained as “A credit default swap is an insurance contract in which the party providing protection stands ready to pay the loss on a given notational amount the first time that a credit event involves a reference entity, called the name, within a given time horizon” in Cherubini et al. [3]. And kth-to-default swap is explained as “A kth-to-default swap is a basket credit instrument backed by a portfolio of single-name CDSs” in Bielecki et al. [2], which only guarantee the first k-times defaults in n-credit names. In those credit derivatives, instead of regularly paying guarantees charges, called spread, buyers of the derivatives can receive compensation of a default loss, called protection, from sellers of the derivatives when a default occurs before the maturity. In other words buyers of the derivatives can transfer a credit risk of the reference companies to sellers of the derivatives. Therefore sellers of the derivatives need to prepare to pay a protection, namely they need hedge. In this paper, based on Bielecki et al. [1] and [2], we assume that each CDS of reference companies of the basket credit derivative is traded in a CDS market and the seller use them in order to hedge.

In the case of basket credit derivatives, one of important points of modelling is how to model dependence between default times. As pointed out in the beginning of section 5 of Bielecki et al. [1], “copula-based credit risk models are not fully suitable for a dynamical approach to credit risk”, however it also says that “these models are of a common use in practical valuation credit derivatives”. Based on this idea, we also use copulas to represent dependence between default times in this paper. We mainly consider Clayton and Gumbel copulas in Archimedean copula. For more details about copula, see Cherubini et al. [3], McNeil et al. [5] and Nelsen [6].

This paper is organized as follows: In section 2, we introduce mathematical models according to Bielecki et al. [1] and [2]. In section 3, we incorporate estimation errors in the...
model which we have mentioned in section 2 and some of hedging errors with respect to FTDS are explicitly given. In section 4, we provide various numerical results which cannot be analytically obtained. And we comment on results and cause of hedging errors using figures. In section 5, we give some conclusions and future problems.

2 Characterization of Credit Derivatives

Following Bielecki et al. [1] and [2], we introduce credit models, price processes of credit derivatives and hedging strategies. Let \( T \) be a maturity of CDSS and FTDS, that is, we assume that the maturity of FTDS and market CDS are the same for simplicity. Note that as pointed out in the beginning of section 4.3 of Bielecki et al. [1], this assumption can be relaxed, and that their starting points are different (commonly the starting points of market CDS are earlier than FTDS's one) in this paper. Let \((\Omega, \mathcal{F}, \mathbb{Q})\) be a probability space and \(\mathbb{Q}\) be the spot martingale probability measure with respect to saving account. For simplicity, we suppose that the interest rate of saving account is zero. For every \( i = 1, 2, \ldots, n \) \((n \geq 2)\), let \( \tau_i > 0 \) be the default time of \( i \)-th company with a constant default intensity \( \lambda_i(> 0) \), where we assume \( \mathbb{Q}(\tau_i = \tau_j) = 0 \) for any \( i \neq j \). Therefore we have the marginal survivor function which satisfies

\[
G_i(t) = \mathbb{Q}(\tau_i > t), \quad (0 \leq t \leq T). \tag{1}
\]

Let \( H_i^t = \mathbf{1}_{[\tau_i \leq t]} \) be the default indicator process, where \( \mathbf{1}_0 \) denotes the indicator function. Set \( \mathcal{H}_i = \sigma(H_i^t) \; 0 \leq t \leq T \) and \( \mathcal{G}_i = \mathcal{H}_i \vee \mathcal{H}_j \vee \cdots \vee \mathcal{H}_n \), where we assume \( \mathcal{G}_i \subset \mathcal{G} \). Note that \( \tau_i \) is an \( \{\mathcal{H}_i\} \)-stopping time and an \( \{\mathcal{H}_i\} \)-stopping time. We denote by

\[
\tau(i) = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n = \min(\tau_1, \tau_2, \ldots, \tau_n) \tag{2}
\]

the time of first default in the \( n \) credit names, and the default indicator process with respect to the first default time \( \tau(i) \) by \( H_i^{\tau(i)} = \mathbf{1}_{[\tau(i) \leq t]} \). Let \( f(t_1, t_2, \ldots, t_n) \) be the joint probability density function of \( \tau_1, \ldots, \tau_n \) and

\[
G(t_1, t_2, \ldots, t_n) = \mathbb{Q}(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_n > t_n)
\]

be the joint probability function that the names \( 1, 2, \ldots, n \) survive until times \( t_1, t_2, \ldots, t_n \). In particular, the joint survival function is given as

\[
G(t, \ldots, t) = \mathbb{Q}(\tau_i > t, \tau_2 > t, \ldots, \tau_n > t) = \mathbb{Q}(\tau(i) > t) = G_i(t), \tag{3}
\]

for every \( t \in \mathbb{R}_+ \).

**Definition 2.1.** (Definition 3.1 in Bielecki et al. [1]) A defaultable claim expiring at time \( T \) is a \( T \)-maturity contract given by a quadruple \((X, A, Z, \tau)\) where \( X \) is a constant which is the \( rm \) promised payoff at maturity \( T \), \( A \) is a function of finite variation with \( A(0) = 0 \) which represents the promised dividends, \( Z \) is some function which specifies the recovery payoff at default time, and \( \tau \) is a random time which denotes a default time.

From p.251 of Bielecki et al. [2], the *fair spread of CDS or FTDS at time \( t \) is such that the value of the contract at time \( t \) is exactly zero*. Note that an FTDS in this paper are defined as \((-\kappa_1(0) \circ \tau_i, Z, \tau(i))\) using the quadruple representation given above, where \( \kappa_1(0) \) is a positive constant which is the fair spread of FTDS at time 0 and \( Z \) is a function such that \( Z = Z_i \) when \( \tau(i) = \tau_i \), where \( Z_i \) is a positive constant which denotes FTDS protection received at time \( \tau(i) \) if the \( i \)-th name is the first defaulted name.

2.1 First-to-Default Intensities

**Definition 2.2.** (Definition 4.1 in Bielecki et al. [1]) The \( i \)-th first-to-default intensity is defined as

\[
\tilde{\lambda}_i(t) = \lim_{h \to 0} \frac{1}{h} \mathbb{Q}(t < \tau_i \leq t + h | \tau(i) > t).
\]

The \( i \)-th first-to-default intensity means strength that the \( i \)-th reference company goes the first bankrupt in \( n \)-credit names. Let us denote

\[
\partial_i G(t, \ldots, t) = \frac{\partial G(t_1, t_2, \ldots, t_n)}{\partial t_i}_{t_1 = u_2 = \cdots = u_n = t} \tag{4}
\]

From Lemma 4.2 in Bielecki et al. [1], the \( i \)-th first-to-default intensity \( \tilde{\lambda}_i \) satisfies

\[
\tilde{\lambda}_i(t) = -\frac{\partial_i G(t, \ldots, t)}{G_i(t)}. \tag{5}
\]

We can define the first-to-default intensity \( \tilde{\lambda}_i \), which denotes the intensity of \( \tau(i) \), using the similar argument above. For more details about the \( i \)-th first-to-default intensity \( \tilde{\lambda}_i \) and the first-to-default intensity \( \lambda_1 \), see Bielecki et al. [1] and [2].

2.2 Price dynamics of the \( i \)-th market CDS and an FTDS

Let \( \kappa_1 \) be a constant which denotes \( i \)-th \( T \)-maturity market CDS fair spread at time \( s \) \((s \leq 0)\), and \( \delta_i \) be a positive constant which denotes \( i \)-th market CDS default protection. Note that the \( i \)-th \( T \)-maturity market CDS is expressed by the quadruple \((0, -\kappa_1 f, \delta_i, \tau_i)\) given in Definition 2.1. Let \( S^i(\kappa_1) \) be the price at time \( t \) of the \( i \)-th \( T \)-maturity market CDS starting at time \( s \), \( S^i(\kappa_1) \) be its pre-default price at time \( t \), \( S^i(\delta_i) \) be its price on the event \( \{\tau(i) = \tau_i\} \) for \( j \neq i \), and \( S^i(\kappa_1) \) be the cumulative price at time \( t \) of \( i \)-th FTDS stopped at time \( \tau(i) \). From equation (42), Lemma 4.4 and Lemma 4.5 in Bielecki et al. [1], they respectively satisfy the followings: for \( t \in [0, T] \),

\[
S_i^1(\kappa_1) = \delta_i \mathbb{E}_\mathbb{Q} \left[ \mathbf{1}_{[\tau(i) \leq T]}(\tau_1 \wedge T) \mathcal{G}_i \right] - \kappa_1 \mathbb{E}_\mathbb{Q} \left[ \mathbf{1}_{[\tau(i) \leq T]}(\tau_1 \wedge T) \mathcal{G}_i \right],
\]

\[
S_i^1(\kappa_1) = \mathbf{1}_{[\tau(i) \leq T]}(\tau_1 \wedge T) \mathcal{G}_i,
\]

\[
S_i^1(\kappa_1) = \int_0^T \int_0^{\tau_1} \delta_i f(u_1, \ldots, u_{n-1}, t, u_{n+1}, \ldots, u_n) u_t^{\mu(t)} du_t
\]

\[
- \kappa_1 \int_0^T \int_0^{\tau_1} f(u_1, \ldots, u_n, t, u_{n+1}, \ldots, u_n) u_t^{\mu(t)} du_t.
\]
and
\[ S_t^I(\kappa_i) = S_t^C(\kappa_i) + \int_0^t \delta_i dH^\kappa_i_{\tau(1)}, \]

where we set \( du^{(j)} = du_1 \ldots du_{j-1} du_{j+1} \ldots du_n \) and \( du^{(0)} = du_1 \ldots du_{j-1} du_{j+1} \ldots du_n \). Note that as mentioned in Example 3.1 of Bielecki et al. [1], if \( \kappa_i = \delta_i \lambda_i \) holds, then \( S_t^I(\kappa_i) = 0 \) and \( \kappa_i \) is the fair spread of the \( i \)th market CDS at every \( t \in [0, T] \).

Let \( \tau(t) \) be the risk-neutral value of the FTDS in this paper, that is, the quadruple is given as \((0, -\kappa(0), 0, \tau(1))\), and \( \hat{\tau}(t) \) be the pre-default Price of the FTDS. From Lemma 4.6 in Bielecki et al. [1], \( \hat{\tau}(t) \) satisfies on \( \{ \tau(1) > t \}\),

\[ \hat{\tau}(t) = \sum_{i=1}^n \mathbb{E}_Q \left[ Z_i(\tau_i) \right]_{\{ t \leq \tau(1) \} \cap \{ t \leq \tau \}} + \mathbb{1}_{\{ t \leq \tau(1) \}} (1 - \hat{H}^{(1)}_t) dA(t) \bigg| \mathcal{G}_t \bigg] \]

\[ = \frac{1}{G^{(1)}(t)} \left( -\sum_{i=1}^n Z_i \int_t^T \sigma_i G(u, \ldots, u) du - \kappa(0) \int_t^T G^{(1)}(u) du \right). \]  

(6)

For each fixed \( v \in [0, T] \), the fair spread \( \kappa^{(1)}(v) \) of FTDS at time \( v \), which is the constant from \( v \) to \( T \), satisfies

\[ \kappa^{(1)}(v) = -\sum_{i=1}^n Z_i \int_v^T \sigma_i G(u, \ldots, u) du \int_v^T G^{(1)}(u) du. \]  

(7)

From the definition of the fair spread of FTDS at time \( v \), the pre-default price of FTDS starting at time \( v \) satisfies \( \hat{\tau}(v) = 0 \). Note that when \( v = 0 \) in equation (7), we obtain \( \kappa^{(0)}(0) \).

For more details about \( S_t^I(\kappa_i), S_t^C(\kappa_i), S_t^{I'}(\kappa_i) \) and \( \hat{\tau}(t) \), see Bielecki et al. [1] and [2].

2.3 Replication of FTDS

Hedging strategies of FTDS are given in section 4.5 of Bielecki et al. [1]. We recall them in this subsection.

Let \( \phi = (\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{n}) \) and \( C \) be a \( \{ \mathcal{G}_t \} \)-predictable process and a function of finite variation with \( C(0) = 0 \) respectively. \( \phi \) denotes a hedging strategies in a CDS market and saving account, and \( C \) denotes a dividend stream on \([0, \tau(1) \wedge T]\).

Based on the beginning of section 4.5 of Bielecki et al. [1], we introduce a self-financing strategy with dividend stream \( C \), if the wealth process \( V_t(\phi, C) \) at time \( t \) satisfies

\[ dV_t(\phi, C) = \sum_{i=1}^n \phi_i^{(1)} \left( \delta_i - S_t^{I'}(\kappa_i) \right) d\hat{M}_t^i \]

\[ + \sum_{j=1}^n \left( S_t^{I'}(\kappa_j) - S_t^{I'}(\kappa_j) \right) \right) - dA(\tau(1) \wedge t), \]

where for \( i = 1, 2, \ldots, n, \hat{M}_t^i \) is defined as

\[ \hat{M}_t^i = H^{\kappa_i, \tau(1)}_t - \int_0^\tau(1) \hat{\lambda}(u) du \]

Note that \( \hat{M}_t^i \) is \( \{ \mathcal{G}_t \} \)-martingale stopped at \( \tau(1) \), which is mentioned in Corollary 4.1 of Bielecki et al. [1]. From equation (8), we thus obtain

\[ dV_t(\phi, C) = \sum_{i=1}^n (Z_i - \hat{\tau}(t) - \int_0^\tau(1) \hat{\lambda}(u) du) - dA(\tau(1) \wedge t) \]

\[ = \sum_{i=1}^n (Z_i - \hat{\tau}(t)) (dH^{\kappa_i, \tau(1)}_t - \hat{\lambda}(t) dt) - dA(\tau(1) \wedge t). \]

3 Hedging Errors of First-to-Default Swap from Parameter Estimation Errors

In our study, we use the hedging strategy introduced in the previous section, however as mentioned above, the default intensities are usually needed to estimate from data, so
that estimated intensities commonly have some estimation errors.

As previous sections, our correct default intensities are \( \lambda_i \) \((i = 1, \ldots, n)\) which is unknown, and in this section we consider the FTDS with \((0, A', Z, \tau_{(1)})\) which is obtained under \( \lambda_i \) and is the correct quadruple. For \( i = 1, \ldots, n \), let \( \lambda'_i \) be an estimated default intensity of \( i \)th reference company, which is used in pricing and hedging the FTDS by the FTDS seller. Note that the 15th market CDS follows the real default intensity \( \lambda_i \), which is commonly not equal to \( \lambda'_i \), and that the fair spread \( \kappa_{(1)}(0) \) at time 0 of the FTDS is calculated incorrectly using \( \lambda'_i \), which also derives \( A(t) \neq A'(t) \), where \( A'(t) \) is the promised dividends under \( \lambda'_i \). Then the FTDS seller considers the (wrong) quadruple of the FTDS as \((0, A', Z, \tau_{(1)})\) due to estimation of the default intensities and calculate their (wrong) promised dividend process \( A'(t) \) and hedging strategies \( \phi' \) under the (wrong) default intensities \( \lambda'_i \), which \( \phi' \) is the hedging strategy which is obtained as previous sections with respect to \((0, A', Z, \tau_{(1)})\).

Using the similar arguments to previous sections, the wealth process \( V_i(\phi', A') \) of the FTDS seller with estimated default intensities \( \lambda'_i \) follows

\[
V_i(\phi', A') = \tilde{\pi}'(0) + \sum_{i=1}^{n} \int_{(0, \tau_{(1)} A)} \phi'(u) dS^*_i(\kappa) - A'(\tau_{(1)} \land t),
\]

where \( \tilde{\pi}'(0) \) denotes the pre-default price of the FTDS with the quadruple \((0, A', Z, \tau_{(1)})\), which is obtained basically using the same argument as the previous sections. And as previous sections, we obtain the dynamics of the wealth process:

\[
dV_i(\phi', A') = \sum_{i=1}^{n} \left( Z_i \tilde{\pi}'(t^-) \right) dH^*_{i, \tau_{(1)} t} - \bar{A}_i(t) dt - A'(\tau_{(1)} \land t).
\]

Now we define hedging errors as

\[
h(\tau_{(1)}) = V_{\tau_{(1)}}(\phi', A') - \sum_{i=1}^{n} Z_i 1_{[\tau_{(1)} = \tau_{(1)} < \tau_{(1)}]} \quad (11)
\]

when \( \tau_{(1)} \leq T \), that is, the first-to-default occurs before the maturity \( T \), and \( h(T) = V_{\tau_{(1)}}(\phi', A') \) when \( \tau_{(1)} > T \), that is, there is no default until \( T \). Note that the first term in the left hand side of (11) is the wealth process at the first default time \( \tau_{(1)} \) and the second term denotes the guarantee to the default, and that when \( \tau_{(1)} > T \) and \( X = 0 \), the seller does not need to pay anything to the buyer at the maturity \( T \), so that the wealth \( V_{\tau_{(1)}}(\phi', A') \) at the maturity \( T \) just denotes the hedging error.

### 3.1 Independent Default Times

In the case of independent default times, we easily obtain from equations (1) - (3),

\[
G(t_1, t_2, \ldots, t_n) = \mathbb{Q}(\tau_1 > t_1) \mathbb{Q}(\tau_2 > t_2) \ldots \mathbb{Q}(\tau_n > t_n) = \exp\left( - \sum_{i=1}^{n} \lambda_i t_i \right).
\]

**Proposition 3.1.** Assume that \( \tau_1, \ldots, \tau_n \) are independent of each other. Under the estimated default intensities \( \lambda'_i \), the pre-default value \( \tilde{\pi}'(t) \) at time \( t \in [0, \tau_{(1)}] \) and the fair spread \( \kappa'_{(1)}(0) \) at time 0 of the FTDS with \((0, -\kappa'_{(1)}, Z, \tau_{(1)})\) at maturity \( T \), where \( \kappa_{(1)} \) is a constant, are respectively given as

\[
\tilde{\pi}'(t) = \sum_{i=1}^{n} Z_i \lambda_i' - \kappa_{(1)}(1 - e^{-(T-t) \sum_{i=1}^{n} \lambda_i'}),
\]

\[
\kappa'_{(1)}(0) = \sum_{i=1}^{n} Z_i \lambda_i' \quad \text{Proof.}
\]

Using the similar arguments to section 5.1 of Bielecki et al. [1], we have, from equations (3), (4) and (6),

\[
\tilde{\pi}'(t) = \frac{1}{e^{\sum_{i=1}^{n} \lambda_i' t}} \left( \sum_{i=1}^{n} Z_i \int_{0}^{t} \lambda_i' e^{-\sum_{j=1}^{n} \lambda_j' u} du \right) - \kappa_{(1)}(1 - e^{-(T-t) \sum_{i=1}^{n} \lambda_i'}).
\]

In order to satisfy \( \tilde{\pi}'(0) = 0 \), which is the definition of the fair spread at time 0, we have

\[
\kappa_{(1)} = \sum_{i=1}^{n} Z_i \lambda_i' = \kappa_{(1)}(0).
\]

**Remark 3.1.** (1) Note that the fair spread \( \kappa'_{(1)}(0) \) at time 0 does not depend on time 0, so that it holds for any \( v \), namely \( \kappa'_{(1)}(0) \) is the fair spread at time \( v \) of the FTDS with \((0, -\kappa'_{(1)}(0), Z, \tau_{(1)})\).

(2) The above proposition is basically the same as around the bottom of p.125 and the top of p.126 in Bielecki et al. [1] under the correct default intensity \( \lambda_i \) and \( n = 2 \). In Proposition 5.1 of Bielecki et al. [1], they give the replicating strategy and pre-default price for an FTDS with the quadruple \((X, 0, Z, \tau_{(1)})\) under the correct default intensity \( \lambda_i \) and \( n = 2 \).

**Theorem 3.1.** Assume that \( \tau_1, \ldots, \tau_n \) are independent of each other. Under the estimated default intensities \( \lambda'_i \), hedging error \( h \) is given as

\[
h(\tau_{(1)}) = \sum_{i=1}^{n} Z_i (\lambda_i' - \lambda_i) \tau_{(1)} \quad \text{if } \tau_{(1)} \leq T
\]

\[
h(T) = \sum_{i=1}^{n} Z_i (\lambda_i' - \lambda_i) T 
\]

**Proof.** In this case, \( \tilde{\pi}'(t) = 0 \) for every \( t > 0 \) under the fair spread of FTDS since it does not depend on time and first-to-default intensities \( \lambda_i(t) = \lambda_i \) for \( i = 1, 2, \ldots, n \) from equation (5). Therefore from equation (10), the dynamics of hedging portfolio are

\[
dV_i = \sum_{i=1}^{n} Z_i (dH^*_{i, \tau_{(1)} t} - \lambda_i dt) + \sum_{i=1}^{n} Z_i \lambda_i' 1(\tau_{(1)} \geq t)dt.
\]
Before first-to-default, the dynamics of hedging portfolio are given as
\[ dV^*_t = -\sum_{i=1}^{n} Z_i \lambda_i dt + \sum_{i=1}^{n} Z_i \bar{\lambda}^*_i dt = \sum_{i=1}^{n} Z_i (\lambda'_i - \lambda_i). \]

Hence
\[ V'(t) = \int_0^t dV'(t) = \sum_{i=1}^{n} Z_i (\lambda'_i - \lambda_i)t. \]

Assume first-to-default is i-th name’s default. The change of hedging portfolio is
\[ V'_{\tau(1)} = V'(t) - \lambda_i \tau(1) \]
\[ = -\sum_{j \neq i} Z_j \lambda_j dt + Z_i (1 - \lambda_i dt) + \sum_{j \neq i} Z_j \lambda_j dt \]
\[ = Z_i + \sum_{j \neq i} Z_j (\lambda'_i - \lambda_i)dt. \]

Therefore hedging error is given as
\[ h(\tau(1)) = \sum_{j=1}^{n} Z_i (\lambda'_i - \lambda_i)\tau(1). \]

**Remark 3.2.** Obviously (i) when \( \lambda'_i = \lambda_i \), \( h(t) = 0 \) holds for all \( t \in [0, T] \). (ii) when \( \lambda'_i > \lambda_i \), \( h(t) > 0 \) holds for all \( t \in [0, T] \). (iii) when \( \lambda'_i < \lambda_i \), \( h(t) < 0 \) holds for all \( t \in [0, T] \). Therefore, we can interpret it as follows: FTDS seller receives profit if estimation error is estimated rather high, and in contrast, FTDS seller receives loss if estimation error is estimated rather low. Besides, the fair spread is sum of CDS spread in the independent default. This fact is mentioned in Schönbucher [8].

### 3.2 Copula Based Models

Describing the dependence of default times, we use Archimedean copula and Gaussian copula in this paper. In Bielecki et al. [1] and [2], they only considered the hedging strategy and pre-default price of First-to-Default-Claim in the bivariate situation. Furthermore, they did not discuss the effect on hedging error from correlation.

Here we consider the hedging error of FTDS caused by the estimated default intensities in the multivariate situation. In terms of the effect caused by correlation, we report in section 5 as numerical experience. As applications of copula functions to credit derivatives, we referred O’Kane [7], Li [4], Schönbucher [8], Sintani et al. [10].

#### 3.2.1 Gumbel Copula

As an example of Archimedean copula, we consider the Gumbel copula. Gumbel copula can be represent as follows

\[ C(u_1, u_2, \ldots, u_n) = \exp \left\{ -\left( \sum_{i=1}^{n} (-\log u_i)^\theta \right)^{\frac{1}{\theta}} \right\}, \]

where parameter \( \theta \geq 1 \) which represent the degree of correlation. The corresponding joint survival function \( G(u_1, u_2, \ldots, u_n) \) is given as

\[ G(u_1, u_2, \ldots, u_n) = C(G_1(u_1), G_2(u_2), \ldots, G_n(u_n)) \]
\[ = \exp \left\{ -\left( \sum_{i=1}^{n} \lambda'^* u_i^\theta \right)^{\frac{1}{\theta}} \right\}. \]
Remark 3.3. (1) Note that as the case of independent defaults, the fair spread $\kappa'_i(0)$ at time 0 does not depend on time 0, so that it holds for any $v$, namely $\kappa'_i(0)$ is the fair spread at time $v$ of the FTDS with $(0,-\kappa'_i(0)u,Z,\tau(1))$.

(2) Calculations in the proof of Proposition 3.2 are similar to Proposition 5.3 in Bielecki et al. [1]. In Proposition 5.3 of Bielecki et al. [1], they give the replicating strategy and pre-default price for an FTDS with the quadruple $(\lambda,0,Z,\tau(1))$ under the correct default intensity $\lambda_i$ and $n = 2$.

Theorem 3.2. Assume that the joint distribution of $(\tau_1,\tau_2,\ldots,\tau_n)$ is given as the Gumbel copula (12) with parameter $\theta$. Under the estimated default intensities $\hat{\lambda}_i$, hedging error $h$ is given as

$$h(\tau_i) = \sum_{i=1}^{n} Z_i (\hat{\lambda}_i - \lambda_i) \tau_i, \quad \text{if } \tau_i \leq T$$

$$h(T) = \sum_{i=1}^{n} Z_i (\hat{\lambda}_i - \lambda_i) T, \quad \text{if } \tau_i > T.$$  

Proof. In the case of Gumbel copula, the fair spread of FTDS does not depend on time as the case of independent default times. Then $\hat{\pi}(t) = 0$ for every $t > 0$ under the fair spread at time 0 of FTDS holds. Therefore we proceed as in proof of Theorem 3.1, hedging portfolio and hedging error are driven as for $\tau_i \leq T$ and $\tau_i > T$, respectively

$$h(\tau_i) = \sum_{i=1}^{n} Z_i (\hat{\lambda}_i - \lambda_i) \tau_i,$$

$$h(T) = \sum_{i=1}^{n} Z_i (\hat{\lambda}_i - \lambda_i) T.$$  

Remark 3.4. Thus if $\lambda'_i = \lambda_i$, $i = 1,2,\ldots,n$, hold, $h(t) = 0$ holds for all $t$. Having default or not does not affect to hedging errors.

3.2.2 Clayton Copula

As another example of Archimedean copula, we consider the Clayton copula. Clayton copula can be represented as follows

$$C(u_1,u_2,\ldots,u_n) = \left( \sum_{i=1}^{n} u_i^{-\alpha} - n + 1 \right)^{-\frac{1}{\alpha}},$$

where parameter $\alpha \geq 0$ represents degree of correlation. The corresponding joint survival function $G(u_1,u_2,\ldots,u_n)$ equals

$$G(u_1,u_2,\ldots,u_n) = C(G_1(u_1),G_2(u_2),\ldots,G_n(u_n)) = \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}.$$  

(14)

Proposition 3.3. Assume that the joint distribution of $(\tau_1,\tau_2,\ldots,\tau_n)$ is given as the Clayton copula (14) with parameter $\theta$. Under the estimated default intensities $\hat{\lambda}_i$, the pre-default value $\tilde{\pi}(t)$ at time $t \in [0,\tau(1)]$ and the fair spread $\kappa'_i(0)$ at time 0 of the FTDS with $(0,-\kappa'_i(0)u,Z,\tau(1))$ with start at time 0 and maturity $T$, where $\kappa'_i$ is a constant, are respectively given as

$$\tilde{\pi}(t) = \frac{\sum_{i=1}^{n} Z_i \int_{t}^{T} e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}} \frac{1}{u} du}{\left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}} - \frac{1}{\kappa'_i(0) \int_{0}^{\tau(1)} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}} du},$$

$$\kappa'_i(0) = \frac{\sum_{i=1}^{n} Z_i \int_{0}^{\tau(1)} e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}} \frac{1}{u} du}{\left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}}.$$  

And also the $i$th first-to-default intensity is obtained as

$$\hat{\lambda}_i(t) = \lambda_i e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}.$$  

Proof. As previous two propositions, from equation (4),

$$\hat{\theta}G(t,\ldots,t) = -\lambda_i e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}.$$  

(15)

To combine equations (3), (6) and (15), the pre-default price of FTDS satisfies

$$\hat{\pi}(t) = \frac{\sum_{i=1}^{n} Z_i \int_{t}^{T} e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}} \frac{1}{u} du}{\left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}} - \frac{1}{\kappa'_i(0) \int_{0}^{\tau(1)} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}} du},$$

The fair spread at time 0 is calculated from equations (7) and (15). We obtain from equations (4) and (5)

$$\tilde{\pi}(t) = \lambda_i e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}.$$  

Proof. As previous two propositions, from equation (4),

$$\hat{\theta}G(t,\ldots,t) = -\lambda_i e^{\alpha u_i} \left( \sum_{i=1}^{n} e^{\alpha u_i} - n + 1 \right)^{-\frac{1}{\alpha}}.$$  

(15)

Remark 3.5. (1) Note that the Clayton copula case is not like the independent and Gumbel copula cases, that is, the fair spread $\kappa'_i(0)$ at time 0 depends on time 0, so that it does not hold for all $v$ except 0. $\kappa'_i(0)$ is the fair spread at time $v > 0$ of the FTDS with $(0,-\kappa'_i(0)u,Z,\tau(1))$.

(2) Calculations in the proof of Proposition 3.3 are similar to Proposition 5.2 in Bielecki et al. [1]. In Proposition 5.2 of Bielecki et al. [1], they give the replicating strategy and pre-default price for an FTDS with the quadruple $(\lambda,0,Z,\tau(1))$ under the correct default intensity $\lambda_i$ and $n = 2$.

(3) In this case, we still have some integrals which are not analytically calculated in hedging portfolios and hedging errors, so that we need to calculate them numerically.
3.2.3 One-Factor Gaussian Copula

Finally we consider the one-factor Gaussian copula case, which is considered in a lot of papers for CDO and so on. The random variable $X_i$ is given as

$$X_i = \rho Y + \sqrt{1 - \rho^2} Y_i,$$

where the random variable $Y$ and $Y_i$ ($i = 1, 2, \ldots, n$) are independent of each other and are Gaussian random variables with mean 0, variance 1 and correlation $\rho$. Let $N(x)$ be the cumulative density function $X_i$. One-factor Gaussian copula can be represented as

$$C(u_1, u_2, \ldots, u_n) = \mathbb{Q}\left( X_i < N^{-1}(u_1), \ldots, X_n < N^{-1}(u_n) \right).$$

The $i$-th default time $\tau_i, i = 1, 2, \ldots, n$ is given as

$$\tau_i = \inf\{t \in \mathbb{R}_+ : 1 - F_i(t) < 1 - N(X_i)\},$$

where $F_i(t)$ is the marginal distribution function of $\tau_i$. Then the joint survival function $G(u_1, u_2, \ldots, u_n)$ equals to

$$G(u_1, u_2, \ldots, u_n) = \mathbb{Q}(\tau_1 \leq u_1, \tau_2 \leq u_2, \ldots, \tau_n \leq u_n)$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} N \left( \frac{N^{-1}(F_i(u_i)) - y \rho}{\sqrt{1 - \rho^2}} \right) dy.$$

In the one-factor Gaussian copula case, we calculate the pre-default price, the fair spread of FTDS and first to default intensity using Monte Carlo method since integrals cannot be calculated explicitly. For more details of Monte Carlo method with respect to the Gaussian copula, see Li [4] and Sheng [9].

4 Numerical Illustrations

In this section, we give some numerical results with respect to hedging errors. As already mentioned, we can calculate hedging error analytically in the case of independent default times and default times with Gumbel copula. However in the cases of default times with Clayton copula and One factor Gaussian copula we have not analytically calculated hedging errors not to mention the pre-default price and the fair spread of FTDS. Accordingly, we use a discrete approximation method in order to calculate integrals in the Clayton copula case, and use the Monte-Carlo method in the one-factor Gaussian copula case.

We give our parameter values which are used in the following simulations in Table 1. We assume that the estimated default intensities $\lambda'_i$ ($i = 1, 2, \ldots, n$) are expressed as $\lambda'_i = e \lambda_i$ where $e > 0$, that is, we assume that $\lambda'_i$ has 100$(e - 1)$% estimation error from the true value $\lambda_i$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity ($T$)</td>
<td>5</td>
</tr>
<tr>
<td>Number of basket ($n$)</td>
<td>6</td>
</tr>
<tr>
<td>Default protection ($Z_i$)</td>
<td>0.6 for all $i = 1, 2, \ldots, 6$</td>
</tr>
<tr>
<td>Default intensities ($\lambda_i$)</td>
<td>0.01 $\times$ $i$, for $i = 1, 2, \ldots, 6$</td>
</tr>
<tr>
<td>Correlation between any two assets (Kendall’s $\tau$)</td>
<td>0, 0.2, 0.5, 0.75, 0.98</td>
</tr>
</tbody>
</table>

FTDS seller lose $Z_i$ as a default payment. Hence FTDS seller get $-\hat{\pi}(t-)$ which is positive when default occurs. In other words, the difference of hedging error and hedging portfolio is $-\hat{\pi}(t-)$ Therefore FTDS seller gains a profit when default occurs.

4.1 Features of the Fair Spread and Hedging Portfolio under Clayton copula without Parameter estimation error

In this section, we consider features of the fair spread at time $t$ and hedging portfolio under Clayton copula model without parameter estimation error in order to grab the characters of the Clayton copula model since, as already mentioned in section 3.1 and 3.2, in cases of independent default times and default times with the Gumbel copula, we explicitly obtain their features, so that we can easily find the structures, however in the case of default times with the Clayton copula, not like that. And also we will give a numerical result which shows approximation errors under the Clayton copula model to know how much error come from our approximation of integrals.

First, we show a feature of the fair spread of FTDS at each time $t$ in Figure 1. From the figure, we find that the fair spread slowly decreases as time passes, and from time 4 to time 5, a relation between the fair spread and Kendall’s $\tau$ is not monotone.

Figure 4.1 shows trajectories of wealth of hedging portfolios given in (9) (Theorem 4.1 of Bielecki et al. [1]) with several Kendall’s $\tau$. A primary feature is as follows: at the beginning, the portfolio wealth starts from 0, monotonically decreases halfway, after the minimum point monotonically increases afterwards and finally, as mentioned before, get close to 0 at the maturity time $T$, namely $V_T(\phi, A) = 0$ holds. When Kendall’s $\tau$ is close to 1, the minimum point is close to time 0 and when Kendall’s $\tau$ is close to 0, the minimum point is close to the maturity. And when Kendall’s $\tau$ is around from 0.5 to 0.75 in our simulation, minimum value is the smallest in minimum values of other Kendall’s $\tau$. In our simulation, this shape is the typical shape of hedging portfolio wealth, however we have not reached this reason clearly.

From Figure 3, all hedging errors from approximations are relatively small, and errors are slightly large at the beginning and when Kendall’s $\tau$ close to 0, so that simulations in the following section are not much affected from numerical approximation. Hedging errors become smaller and smaller as time passes and when Kendall’s $\tau$ is close to 0.
4.2 Effect of parameter estimation errors

4.2.1 Without default

In this section, we give some numerical results of hedging errors by parameter estimation errors when no default until the maturity in the case of Clayton copula model. In the cases of independent defaults and the Gumbel copula model, those results have been analytically obtained.

In Table 2, we give numerical results on hedging errors at the maturity \( T \) without default in the case of Clayton copula model. As we can see from the table, the relationships between Kendall’s \( \tau \) and hedging errors or rate of parameter estimation errors \( \varepsilon \) and hedging error are nonlinear. If \( \varepsilon > 1 \), hedging errors are positive and if \( \varepsilon < 1 \), hedging errors are negative. If \( \varepsilon = 1 \), they should be 0, however as we see the row of \( \varepsilon = 1 \), they are not equal to 0 due to discrete approximation errors. On the other hand, if the parameter estimation error \( \varepsilon \) is the same (fixed), hedging errors in case small Kendall’s \( \tau \) are larger.

In Figure 4, we give the dynamics of hedging portfolio wealth in the case of the Clayton copula model when Kendall’s \( \tau = 0.5 \). Those graphs show how much each hedging portfolio wealth is effected by parameter estimation errors. The line \( \varepsilon = 1 \) in Figure 4 is the same as the line \( \tau = 0.5 \) in Figure 4.1. Even if \( \varepsilon > 1 \), namely at the maturity the hedging errors are positive, we have negative wealth of hedging portfolio during the term. However their features are basically similar.

4.2.2 With default

In this section, we numerically investigate effects on hedging errors from parameter estimation errors in cases of Gumbel copula model and Clayton copula model.

Figures 5 and 6 show that hedging errors at the maturity in cases of Gumbel copula model and Clayton copula model respectively. Both features are similar, that is, when \( \varepsilon \) is
close to 0, hedging errors become dramatically large and when \( \varepsilon \) is larger than 1, hedging errors become gradually large. Therefore we can interpret FTDS seller receive profits if estimation error is estimated rather high. In contrast, FTDS seller receives loss if estimation error is estimated rather low. Note that, as mentioned in Theorem 3.2, having default or not does not affect hedging errors in the case of Gumbel copula model, so that this relation at the maturity is the same as the case without any default until the maturity, however in the case of Clayton model, not like that.

Figure 7 shows that the difference between hedging errors in cases of default times with Gumbel copula and Clayton copula. More precisely, the difference of hedging errors is defined as

\[ h_{\text{Clayton}}(t) - h_{\text{Gumbel}}(t) \]

where \( h_{\text{Gumbel}}(t) \) and \( h_{\text{Clayton}}(t) \) mean hedging errors of Gumbel copula and Clayton copula at time \( t \) respectively. In most cases, we can confirm that hedging errors in the Clayton copula is higher than ones in the Gumbel copula since hedging errors are positive for all most \( \varepsilon \). However, when Kendall’s \( \tau \) is close to 1, then features are not like above. Hence, there are fewer risks of estimation error for the FTDS seller in Gumbel copula model than Clayton copula model. Besides, there are little difference in hedging error when correlation is close to 0 or 1. This is due to FTDS spread.

Figure 8 shows the relationship between hedging errors and default time in the case of Clayton copula model

Figure 8 shows the relationship between hedging errors and default time in the case of Clayton copula model with Kendall’s \( \tau = 0.5 \). We find out that hedging errors grow larger over time in the case of \( \varepsilon > 1 \) from Figure 8. In contrast, hedging errors grow smaller with time in the case of \( \varepsilon < 1 \).

In addition, we also show the numerical result of the spread about One-Factor Gaussian copula.

From Figure 9, there are little difference in the fair spread of FTDS at time 0 when correlation is close to 0 or 1. In Schönbucher [8], the fair spread of FTDS is close to sum of CDS spread and is close to the highest CDS spread in the basket generally. This result of FTDS spread is showed numerically in O’Kane [7]. In addition, the fair spread of FTDS at time 0 is calculated higher in Clayton copula than Gaussian copula and Gumbel copula. We consider this is caused by tail dependence. According to the Sintani [10]
and Tozaka [11], Romano [12], Clayton copula have lower tail dependence. Contrary to this, Gaussian copula and Gumbel copula do not have lower tail dependence. Therefore we guess that default probability is calculated higher in Clayton copula.

5 Conclusion

The aim of this paper is to study the effect on hedging error of FTDS from estimation error of default intensities. Then we can find out the effect of parameter estimation error in cases of Gumbel copula and Clayton copula. Hedging errors from parameter estimation error are approximately similar in both copulas. Furthermore, hedging errors are positive when estimation error is estimated rather higher than true default intensities. In contrast, hedging errors are negative when estimation error is estimated lower than true default intensities. Additionally, FTDS seller can make more profit when estimation error is rather higher and can make less loss when estimation error is lower than true default intensities in Clayton copula.

In the case of Gaussian copula, we can not find out the effect of hedging error from estimation error of default intensities because of first to default intensities. The numerical solution of first to default intensities includes many noises by general Monte Carlo method. Although we also tried to calculate by numerical differentiation, a value is not so good and it takes much time to calculate by Mathematica.

References


