On the Differential Equation of Fishing Net.*

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Introduction. Recently Prof. M. TAUTI¹⁾ has given a differential equation satisfied by a fishing net subjected to a current, and has obtained the relation between a model and a full scale of fishing net. However, the expression of tension of net is not given in his equations. In the present paper the author wishes to introduce a differential equation satisfied by a fishing net placed in a current, especially with the view to introduce in it the force acting upon the elemental portion of fishing net on account of the tension of netting cords.

Fig. 1.

Let us assume that a fishing net can be considered as a continuous membrane. This assumption is approximately satisfied, in the case, where differences in physical states between neighbouring meshes are so small as can be neglected as infinitely small quantities. An arbitrary point upon the net is represented by a radius vector $\mathbf{r}^2$ whose components are denoted by x , y , and z . We consider two sets of curves which connect the knots of net situated at both ends of each diagonal of mesh as shown in Fig. 1. They are named u -line and v -line respectively. Every point upon the net is, therefor, repre-

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1) TAUTI, M.: A relation between experiments on model and on full scale of fishing net. This Bull. 3 (4), 1934, pp. 171-177.

2) Vector quantities are represented by german letters.

sented by a set of two values (u, v) , and, for the sake of simplicity, their values are so chosen that a knot is represented by a set of integers or half integers as illustrated in the figure. And we also assume that u - and v -lines are functions of x, y and z continuous up to their second order derivatives. A net is usually so constructed as to make every leg of mesh has equal lengths, therefore we can generally assume that a mesh is lozenge-shaped. Thus, u - and v -lines, which connect the knots, form an orthogonal coordinate system.

Stress of net. A normal of the net is represented by ξ and is given by

$$\xi = \mathbf{r}_u \times \mathbf{r}_v / \sqrt{EG}, \quad (1)$$

where

$$E = \mathbf{r}_u^2, \quad G = \mathbf{r}_v^2, \quad (2), (3)$$

and

$$F = \mathbf{r}_u \mathbf{r}_v = 0, \quad (4)$$

on account of the orthogonality of u - and v -lines.

We consider a small rectangular elemental portion of net bounded by two u -lines, $u = u$ and $u = u + du$, and two v -lines, $v = v$ and $v = v + dv$. The lengths of side are given by $ds_u = \sqrt{E} du$ and $ds_v = \sqrt{G} dv$ respectively. The direction of side is represented by the tangent of net perpendicular to the side, and is given by $\mathbf{v} = \mathbf{r}_v / \sqrt{G}$ and $\mathbf{u} = \mathbf{r}_u / \sqrt{E}$ respectively.

We also consider a tensor T representing a stress of net. The tensor is so chosen that, by operating a vector representing a line-element, a tension of net acting through the line-element from the positive to the negative side of it is obtained. In this operation, the direction of the vector is represented, differently from an ordinary usage, by a tangent of net perpendicular to the line-element and whose magnitude is equal to its length. Hence, the force $\mathfrak{T}$ acting upon the elemental portion of net from the surrounding part is given by

Fig. 2.

$$\begin{aligned} \mathfrak{T} &= [T_v ds_u]_v^{v+dv} + [T_u ds_v]_u^{u+du} \\ &= [T \mathbf{r}_v \sqrt{E/G}]_v^{v+dv} du + [T \mathbf{r}_u \sqrt{G/E}]_u^{u+du} dv. \end{aligned} \quad (5)$$

This internal force must balance the external forces acting upon the elemental portion. Considering the external forces in the following paragraph, we

now consider the expression of the tensor representing the stress of net.

The tensor T is written in a form of sum of unit tensors as follows :

$$T = T_{11}u \cdot u + T_{12}u \cdot v + T_{21}v \cdot u + T_{22}v \cdot v, \quad (6)$$

where T_{ij} is a component of the tensor. The tensor must be represented by the tensions acting upon the netting cords, which are denoted by T_1 or T_2 as illustrated in Fig. 2. Now, we consider a straight line, $\mathfrak{S}$, whose length is taken to be unit, for the sake of simplicity ; number of legs intersect the straight line is given by $\sin \left\{ \left(\frac{\pi - \varphi}{2} \right) + \psi \right\} / l \sin \varphi$ for the legs having the tension T_1 and $\sin \left\{ \left(\frac{\pi + \varphi}{2} \right) + \psi \right\} / l \sin \varphi$ for T_2 , where φ and ψ are the angle between the two legs intersecting the v -line and that between the straight line and v -line, respectively, and l is the length of leg.

The direction of the legs are given by $u \sin (\varphi/2) + v \cos (\varphi/2)$ for that having the tension T_1 and $-u \sin (\varphi/2) + v \cos (\varphi/2)$ for T_2 , respectively, and the force acting upon the negative side of net from the positive side through the straight line can be given by the sum of tensions acting upon the legs intersecting the line $\mathfrak{S}$, then we have

$$\begin{aligned} \mathfrak{F} &= T_1 \frac{\cos \left(\frac{\varphi}{2} - \psi \right)}{l \sin \varphi} \left(u \sin \frac{\varphi}{2} + v \cos \frac{\varphi}{2} \right) \\ &\quad + T_2 \frac{\cos \left(\frac{\varphi}{2} + \psi \right)}{l \sin \varphi} \left(-u \sin \frac{\varphi}{2} + v \cos \frac{\varphi}{2} \right) \\ &= \frac{1}{2l} \left[\left\{ (T_1 - T_2) \cos \psi + (T_1 + T_2) \tan \frac{\varphi}{2} \sin \psi \right\} u \right. \\ &\quad \left. + \left\{ (T_1 + T_2) \cot \frac{\varphi}{2} \cos \psi + (T_1 - T_2) \sin \psi \right\} v \right]. \quad (7) \end{aligned}$$

While this force can also be obtained by operating the tensor T on the vector $\mathfrak{S}$, and we have

$$\mathfrak{F} = T\mathfrak{S} = (T_{11} \sin \psi + T_{12} \cos \psi)u + (T_{21} \sin \psi + T_{22} \cos \psi)v. \quad (8)$$

The formulae (7) and (8) must be identical for arbitrary value of ψ , and we have the following expression of T_{ij} ,

$$T_{11} = \frac{T_1 + T_2}{2l} \tan \frac{\varphi}{2}, \quad T_{12} = T_{21} = \frac{T_1 - T_2}{2l}, \quad T_{22} = \frac{T_1 + T_2}{2l} \cot \frac{\varphi}{2}. \quad (9)$$

The lengths of diagonals of mesh are given by $\sqrt{E}$ and $\sqrt{G}$, because the difference between the values of u and v at the opposit ends of the diagonals are unit. Therefore, we have the following relations.

$$\tan \varphi/2 = \sqrt{E/G}, \quad \cot \varphi/2 = \sqrt{G/E}. \quad (10)$$

Substituting this relation into the formulae (9), we get

$$T_{11}=(\tau_1+\tau_2)\sqrt{E/G}, \quad T_{12}=T_{21}=\tau_1-\tau_2, \quad T_{22}=(\tau_1+\tau_2)\sqrt{G/E}, \quad (11)$$

where $T_1/2l$ and $T_2/2l$ are replaced by τ_1 and τ_2 respectively. Then, the tensor representing the stress of net is expressed by

$$T=\frac{1}{\sqrt{EG}}\{(\tau_1+\tau_2)\mathfrak{x}_u\cdot\mathfrak{x}_u+(\tau_1-\tau_2)(\mathfrak{x}_u\cdot\mathfrak{x}_v+\mathfrak{x}_v\cdot\mathfrak{x}_u)+(\tau_1+\tau_2)\mathfrak{x}_v\cdot\mathfrak{x}_v\} \quad (12)$$

From the formulae (5) and (12), we have the force acting upon the small rectangular elemental portion of net from the neighbouring part, as follows

$$\begin{aligned} \mathfrak{X} &= [(\tau_1+\tau_2)\mathfrak{x}_u+(\tau_1-\tau_2)\mathfrak{x}_v]_u^{u+du}dv + [(\tau_1-\tau_2)\mathfrak{x}_u+(\tau_1+\tau_2)\mathfrak{x}_v]_v^{v+dv}du \\ &= \left[(\tau_1+\tau_2)(\mathfrak{x}_{uu}+\mathfrak{x}_{vv}) + 2(\tau_1-\tau_2)\mathfrak{x}_{uv} + \left\{ \frac{\partial}{\partial u}(\tau_1+\tau_2) + \frac{\partial}{\partial v}(\tau_1-\tau_2) \right\} \mathfrak{x}_u \right. \\ &\quad \left. + \left\{ \frac{\partial}{\partial v}(\tau_1+\tau_2) + \frac{\partial}{\partial u}(\tau_1-\tau_2) \right\} \mathfrak{x}_v \right] du dv \end{aligned} \quad (13)$$

Differential equation of fishing net. The external forces acting upon the elemental portion of net are consisted of following parts:

1. Resistance of the relative velocity of water against the net, which is denoted by $\mathfrak{R}$ per mesh,

2. Apparent weight of netting cord in water, which can be written as $2\Lambda\sigma g$,

3. D'ALEMBERT's force acting upon the net, which can be written as $2\Lambda\rho\ddot{x}$;

where ρ is the linear density of netting cord and $\sigma=\rho-1$, 2Λ is the length of netting cord consisting a mesh, two legs and a knot, and g is the acceleration of gravity.

The external force must be in equilibrium with the force due to the tension of net given by the formula (13), therefore, we get, as a fundamental differential equation of fishing net, following formula:

$$\begin{aligned} &(\tau_1+\tau_2)(\mathfrak{x}_{uu}+\mathfrak{x}_{vv}) + 2(\tau_1-\tau_2)\mathfrak{x}_{uv} + \left\{ \frac{\partial(\tau_1+\tau_2)}{\partial u} + \frac{\partial(\tau_1-\tau_2)}{\partial v} \right\} \mathfrak{x}_u \\ &+ \left\{ \frac{\partial(\tau_1+\tau_2)}{\partial v} + \frac{\partial(\tau_1-\tau_2)}{\partial u} \right\} \mathfrak{x}_v + \mathfrak{R} + 2\Lambda\sigma g - 2\Lambda\rho\ddot{x} = 0. \end{aligned} \quad (14)$$

Considering the following formula of the differential geometry,

$$\begin{aligned} \mathfrak{x}_{uu} &= (E_u/2E)\mathfrak{x}_u - (E_v/2G)\mathfrak{x}_v + L\xi, \\ \mathfrak{x}_{uv} &= (E_v/2E)\mathfrak{x}_u + (G_v/2G)\mathfrak{x}_v + M\xi, \\ \mathfrak{x}_{vv} &= -(G_u/2E)\mathfrak{x}_u + (G_v/2G)\mathfrak{x}_v + N\xi, \end{aligned}$$

we get a set of scalar equation referred to the direction of $\mathfrak{x}_u$, $\mathfrak{x}_v$ and ξ :

$$\left. \begin{aligned} & \frac{\tau_1 + \tau_2}{2}(E_u - G_u) + (\tau_1 - \tau_2)E_v + E \left\{ \frac{\partial(\tau_1 + \tau_2)}{\partial u} + \frac{\partial(\tau_1 - \tau_2)}{\partial v} \right\} \\ & \quad + R_{\xi u} + 2A\sigma g_{\xi u} - 2A\rho \ddot{\xi} \xi_u = 0, \\ & \frac{\tau_1 + \tau_2}{2}(-E_v + G_v) + (\tau_1 - \tau_2)G_u + G \left\{ \frac{\partial(\tau_1 + \tau_2)}{\partial v} + \frac{\partial(\tau_1 - \tau_2)}{\partial u} \right\} \\ & \quad + R_{\xi v} + 2A\sigma g_{\xi v} - 2A\rho \ddot{\xi} \xi_v = 0, \\ & (\tau_1 + \tau_2)(L + N)2M(\tau_1 - \tau_2) + R_{\xi} + 2A\sigma g_{\xi} - 2A\rho \ddot{\xi} \xi = 0. \end{aligned} \right\} \quad (15)$$

In order to solve the above equations, the expression of the resistance and the relation between the length of leg and the tension acting upon it are necessary. As already mentioned, $\sqrt{E}$ and $\sqrt{G}$ are the lengths of two diagonals of mesh, following relation is obtained,

$$E + G = 4l^2, \quad (16)$$

where l must be a function of the tension acting upon the leg. However, usually a strong uniform initial tension acts upon the net by means of bouys and weights laid on the net, and netting cord is stretched almost to their maximum length, we can neglect the change of length due to an additional tension owing to the resistance of current, then we have $E + G = \text{const}$, and

$$E_u = -G_u, \quad E_v = -G_v. \quad (17)$$

The tension acting upon the legs circumscribing a mesh are generally seemed to be almost identical, we can put

$$\tau_1 + \tau_2 = 2\tau, \quad \tau_1 - \tau_2 = 0. \quad (18)$$

Prof. M. TAUTI¹⁾ has given the expression of the resistance of the relative velocity of water and he also pointed out that the resistance acting upon the knot is very small comparing with that upon the legs. Following his result, the resistance coefficients are given by the formulae $a(D/l)\lambda \cot \varphi/2$, $a(D/l)\mu \tan \varphi/2$ and $a(D/l)\nu/\sin(\varphi/2) \cos(\varphi/2)$ for the direction of ξ_u , ξ_v and ξ respectively, where λ, μ, ν are the direction cosines of the current and D , a diameter of netting cord and a , a constant l and φ have the same meaning used in the preceeding paragraphs. Representing a unit vector in the direction of current by w , the resistance acting upon a mesh is written as follows:

$$K_u = 2k(w_{\xi u})G/\sqrt{E}, \quad K_v = 2k(w_{\xi v})E/\sqrt{G}, \quad K_{\xi} = 2k(w_{\xi})(E + G), \quad (19)$$

where $k = (a/4)w^2(D/l)$ is a new constant, where w is the velocity of current.

Substituting these formulae (17), (18), and (19) into (15) we get

$$\left. \begin{aligned} & \tau E_u + E \tau_u - k(w_{\xi u})G + A\sigma(g_{\xi u}) - A\rho(\ddot{\xi} \xi_u) = 0, \\ & \tau G_v + G \tau_v + k(w_{\xi v})E + A\sigma(g_{\xi v}) - A\rho(\ddot{\xi} \xi_v) = 0, \\ & (L + N)\tau + 4l^2 k(w_{\xi}) + A\sigma(g_{\xi}) - A\rho(\ddot{\xi} \xi) = 0. \end{aligned} \right\} \quad (20)$$

1) M. TAUTI: The force acting on the plane net in motion through the water. This Bull 3 (1), 1934, pp. 1-4.

Summary. Under the following assumptions:

1. That a net can be considered as a continuous membrane,
2. That a net is furnished with suitable weights and bouys and the netting cords are subjected to the large initial uniform tension and are stretched to their maximum length,
3. That tensions acting upon the legs circumscribing a mesh are almost identical, and
4. That a resistance of current acting upon the knot of net is negligibly small against that upon the legs,

we have a differential equation of a fishing net

$$\left. \begin{aligned} \tau E_u + E\tau_u + k(w_{\xi u})G + A\sigma(g_{\xi u}) - A\rho(\ddot{\xi}\xi_u) &= 0, \\ \tau G_v + G\tau_v + k(w_{\xi v})E + A\sigma(g_{\xi v}) - A\rho(\ddot{\xi}\xi_v) &= 0, \\ (L + N)\tau + 4l^2k(w_{\xi\xi}) + A\sigma(g_{\xi\xi}) - A\rho(\ddot{\xi}\xi) &= 0. \end{aligned} \right\} \quad (20)$$

And boundary conditions as well as additional conditions

$$F = \xi_u \xi_v = 0, \quad (3)$$

$$E + G = 4l^2, \quad (16)$$

are necessary to solve the equations.