Attitude and Rate Estimation of a Spacecraft by Using Magnetometers

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New attitude and attitude rate estimation algorithms are presented for a spacecraft using magnetometers. An attitude rate estimation algorithm solves an optimization problem to identify the unknown angular rate from the short period observations of the earth magnetic field vectors, and then, estimates the angular velocity vector of the spacecraft. An attitude estimation algorithm solves another optimization problem to identify the unknown angle from the long period observation of the earth magnetic field vector and the estimated attitude rate, and then, estimates the spacecraft attitude in terms of Euler parameters. The effectiveness of the proposed algorithms is evaluated by numerical simulations.

Key Words: Spacecraft, Attitude, Attitude Rate, Determination, Magnetometers

Nomenclature

$\mathbf{q}_{CD}$: Euler parameters representing the attitude of frame D with respect to frame C
 $\mathbf{m}$: earth magnetic field vector
 $\hat{\mathbf{m}}$: normalized earth magnetic field vector ($\hat{\mathbf{m}} = \mathbf{m}/|\mathbf{m}|$)
 $\boldsymbol{\omega}$: angular velocity of spacecraft
 $\mathbf{J}$: inertia tensor of spacecraft body around its center of mass
 $\mathbf{h}$: angular momentum of the spacecraft ($\mathbf{h} = \mathbf{J}\boldsymbol{\omega}$)
 $*$: quaternion multiplication
 $V(\mathbf{q})$: vector part of quaternion $\mathbf{q}$
 $S(\mathbf{q})$: scalar part of quaternion $\mathbf{q}$
 $\mathbf{q}^\dagger$: conjugate of quaternion $\mathbf{q}$

Euler parameters are a unit quaternion and three dimensional vector is a quaternion whose scalar part is 0. When the superscript B (or I) is attached to a vector, the vector is represented in the body frame (or in the inertial frame).

1. Introduction

An attitude of a spacecraft is typically estimated by using star trackers, sun sensors, earth sensors, and magnetometers. Although the star trackers are highly accurate, they are expensive and require the complicated system structure. The other sensors do not determine the attitude from a single observation. Hence, they are used to determine the attitude from the combination of observations. An attitude rate of the spacecraft is typically estimated by using rate gyros. Although the rate gyros are also highly accurate, they are sometimes expensive, space-consuming, and require a large amount of power.

Among the several approaches, we consider the estimation of both the attitude and attitude rate of the spacecraft by using magnetometers. A number of methods have been developed for both estimation problems. A typical scenario is a batch estimation called the quaternion estimation (QUEST) for the attitude estimation^{1, 4, 6)}. The other typical scenario is a recursive estimation by applying an extended Kalman filter (EKF) technique for the attitude rate

estimation^{2, 3)}. In many cases, the Euler parameters representing the attitude or the angular velocity vector are directly estimated. However, as pointed out by Reynolds⁴⁾, the Euler parameters undetermined from the magnetic field vectors can be parametrized with respect to a rotation along the magnetic field vector in the inertial frame.

In this paper, we extend the Reynolds's approach as follows: Firstly, we develop a batch attitude rate estimation algorithm from the short period of observations of the magnetic field vectors in the body frame. It is supposed that the angular momentum and the magnetic field vector are quasi stationary in the inertial frame; then, the angular velocity vector is estimated by solving an optimization problem, where the cost function is based on the conservation of the angular momentum. Although a similar problem was discussed by Psiaki²⁾, the cost function is based on the trapezoidal integration of the Euler equation of motion. Moreover, in order to overcome the numerical differentiation problem³⁾ in the batch attitude rate estimation algorithm, we propose an alternative numerical differentiation method by combining the second order low pass filter with the time derivative filter.

Next, we develop the batch attitude estimation algorithm from the long period observations of the magnetic field vectors in the body frame and from the estimated angular velocity vectors. It is supposed that the magnetic field vector is time varying in the inertial frame and its changes in the inertial frame is known; then, the Euler parameters are estimated by solving an optimization problem, where the cost function is constructed by integrating the attitude change of the spacecraft. Special attention should be paid to the case where the estimated angular velocity is also obtained from magnetometers. In this case, a correction method for the estimated rate is proposed.

The outline of the paper is as follows: Section 2 gives expressions of both attitude and attitude rate, where the unknown parameters are explicitly described for both problems. Section 3 presents the attitude rate estimation algorithm and the implementation issues of the numerical differentiation. Section 4 presents the attitude estimation algorithm and discusses the differences between the problem formulations. Section 5 illustrates the effectiveness of the proposed algorithms by numerical simulations. Conclusions are summarized in Section 6.

2. Expressions of Attitude and Attitude Rate

2.1. Attitude expression

Let $\hat{\mathbf{m}}^I$, $\hat{\mathbf{m}}^B$, and $\mathbf{q}_{IB}$ denote the normalized magnetic field vector in the inertial frame, that in the body frame, and the Euler parameters representing the spacecraft attitude in the inertial frame, respectively. Then, the following relation holds:

$$\hat{\mathbf{m}}^I = \mathbf{q}_{IB} * \hat{\mathbf{m}}^B * \mathbf{q}_{IB}^\dagger \quad (1)$$

Let us consider the attitude expression using $\hat{\mathbf{m}}^I$ and $\hat{\mathbf{m}}^B$. Firstly, an additional frame M is introduced where $\mathbf{q}_{IM}$ and $\mathbf{q}_{MB}$ represent the rotation along the axis $\hat{\mathbf{m}}^I$ and that along the axis perpendicular to both $\hat{\mathbf{m}}^I$ and $\hat{\mathbf{m}}^B$, respectively. $\mathbf{q}_{IB}$ is factored as follows:

$$\mathbf{q}_{IB} = \mathbf{q}_{IM} * \mathbf{q}_{MB}, \quad (2)$$

where $\mathbf{q}_{IM}$ is given by

$$V(\mathbf{q}_{IM}) = \hat{\mathbf{m}}^I \sin \frac{\phi_m}{2}, \quad S(\mathbf{q}_{IM}) = \cos \frac{\phi_m}{2} \quad (3)$$

using a certain rotation angle ϕ_m . It then follows

$$\hat{\mathbf{m}}^I = \mathbf{q}_{IM} * \hat{\mathbf{m}}^I * \mathbf{q}_{IM}^\dagger \quad (4)$$

for arbitrary ϕ_m . Let $\mathbf{q}_{MB} = V(\mathbf{q}_{MB})$, $q_{MB} = S(\mathbf{q}_{MB})$ for notational simplicity. By its construction, $\mathbf{q}_{MB}$ satisfies the following relations:

$$\mathbf{q}_{MB} \cdot \hat{\mathbf{m}}^I = 0, \quad \mathbf{q}_{MB} \cdot \hat{\mathbf{m}}^B = 0 \quad (5)$$

From Eqs. (1), (2), and (4), $\mathbf{q}_{MB}$ satisfies the following relation:

$$\hat{\mathbf{m}}^I = \mathbf{q}_{MB} * \hat{\mathbf{m}}^B * \mathbf{q}_{MB}^\dagger \quad (6)$$

By substituting $\mathbf{q}_{MB}$ and q_{MB} into the above equation, $\hat{\mathbf{m}}^I$ is expressed as

$$\hat{\mathbf{m}}^I = -\hat{\mathbf{m}}^B + 2q_{MB}^2 \hat{\mathbf{m}}^B + 2q_{MB} (\mathbf{q}_{MB} \times \hat{\mathbf{m}}^B) \quad (7)$$

By taking the inner product and the vector product with $\hat{\mathbf{m}}^B$, the following relations are obtained:

$$\hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B = -1 + 2q_{MB}^2 \quad (8)$$

$$\hat{\mathbf{m}}^B \times \hat{\mathbf{m}}^I = 2q_{MB} \mathbf{q}_{MB} \quad (9)$$

If q_{MB} is chosen as $q_{MB} \geq 0$, $\mathbf{q}_{MB}$ and q_{MB} are given by

$$q_{MB} = \frac{\hat{\mathbf{m}}^B \times \hat{\mathbf{m}}^I}{2q_{MB}}, \quad q_{MB} = \sqrt{\frac{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B}{2}} \quad (10)$$

In summary, $\mathbf{q}_{MB}$ is uniquely determined, whereas $\mathbf{q}_{IM}$ is not uniquely determined due to the unknown free parameter ϕ_m . In the subsequent of this paper, the attitude $\mathbf{q}_{IB}$ is obtained by estimating the unknown variable ϕ_m .

2.2. Attitude rate expression

Let us investigate the relation between the time derivative of the unknown variable $\dot{\phi}_m$ and the angular velocity in the body frame ω^B .

The time derivative of $\mathbf{q}_{IB}$ is given by

$$\dot{\mathbf{q}}_{IB} = \frac{1}{2} \mathbf{q}_{IB} * \omega^B \quad (11)$$

By multiplying $\mathbf{q}_{IB}^\dagger$ from the left, ω^B is obtained as

$$\omega^B = 2\mathbf{q}_{IB}^\dagger * \dot{\mathbf{q}}_{IB} \quad (12)$$

Here, $\hat{\mathbf{m}}^I$ is supposed to be stationary. In practice, this assumption is admissible when $\hat{\mathbf{m}}^B$ is observed in a short period of time.

By using this assumption, the time derivative of $\hat{\mathbf{m}}^B$ is computed as follows:

$$\dot{\hat{\mathbf{m}}}^B = -\omega^B \times \hat{\mathbf{m}}^B \quad (13)$$

By substituting Eqs. (2), (10), and (13) into Eq. (12) and rearranging the result, $\dot{\phi}_m$ is obtained as follows:

$$\dot{\phi}_m = \frac{(\hat{\mathbf{m}}^I + \hat{\mathbf{m}}^B) \cdot \omega^B}{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B} \quad (14)$$

Let ω_{MB} and ω_{IM} denote the angular velocity of the body frame to the additional frame M and that of the additional frame M to the inertial frame, respectively. Then, ω satisfies the following relation:

$$\omega = \omega_{MB} + \omega_{IM} \quad (15)$$

By using the time derivatives of $\mathbf{q}_{MB}$ and $\mathbf{q}_{IM}$, ω_{MB} and ω_{IM} are represented by the Euler parameters as follows:

$$\omega_{MB}^B = 2\mathbf{q}_{MB}^\dagger * \dot{\mathbf{q}}_{MB}, \quad \omega_{IM}^I = 2\mathbf{q}_{IM}^\dagger * \dot{\mathbf{q}}_{IM} \quad (16)$$

By substituting Eqs. (3) and (10) into the above equations, ω_{MB}^B and ω_{IM}^I are obtained as

$$\omega_{MB}^B = \frac{\dot{\hat{\mathbf{m}}}^B \times (\hat{\mathbf{m}}^I + \hat{\mathbf{m}}^B)}{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B}, \quad \omega_{IM}^I = \hat{\mathbf{m}}^I \dot{\phi}_m \quad (17)$$

Therefore, ω^B is given by

$$\omega^B = \frac{\dot{\hat{\mathbf{m}}}^B \times (\hat{\mathbf{m}}^I + \hat{\mathbf{m}}^B)}{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B} + \hat{\mathbf{m}}^B \dot{\phi}_m \quad (18)$$

If $\dot{\hat{\mathbf{m}}}^B$ is approximated from the numerical differentiation of the measured value $\hat{\mathbf{m}}^B$, the unknown variable in the right hand side is $\dot{\phi}_m$. In the subsequence of this paper, the angular velocity vector ω^B is obtained by estimating the unknown variable $\dot{\phi}_m$ from the past observations of $\hat{\mathbf{m}}^B$ in a short period of time.

3. Attitude Rate Estimation

3.1. Attitude rate estimation from the conservation of angular momentum

In this section, the attitude rate estimation problem is considered where $\dot{\mathbf{m}}^I$ and $\mathbf{h}^I$ are unknown but supposed to be stationary in a short period of time. The inertia matrix $\mathbf{J}^B$ is supposed to be known.

The angular momentum $\mathbf{h}$ is given by

$$\mathbf{h}_i^B = \mathbf{J}^B \boldsymbol{\omega}_i^B = \mathbf{J}^B \boldsymbol{\omega}_{MBi}^B + \mathbf{J}^B \dot{\mathbf{m}}_i^B \phi_{mi} \quad (19)$$

where the subscript i denotes the time t_i in the short period of observation. Let α denote the inner product of $\mathbf{h}$ and $\dot{\mathbf{m}}$. By the assumption that $\mathbf{h}^I$ and $\dot{\mathbf{m}}^I$ are stationary, α also becomes stationary, i.e.,

$$\alpha = \mathbf{h}_i^B \cdot \dot{\mathbf{m}}_i^B = \left(\mathbf{J}^B \boldsymbol{\omega}_{MBi}^B + \mathbf{J}^B \dot{\mathbf{m}}_i^B \phi_{mi} \right) \cdot \dot{\mathbf{m}}_i^B \quad (20)$$

does not depend on i . It follows that

$$\dot{\phi}_{mi} = a_i \alpha + b_i, \quad (21)$$

where a_i and b_i are given by

$$a_i = \frac{1}{\dot{\mathbf{m}}_i^B \cdot \mathbf{J}^B \dot{\mathbf{m}}_i^B}, \quad b_i = -\frac{\dot{\mathbf{m}}_i^B \cdot \mathbf{J}^B \boldsymbol{\omega}_{MBi}^B}{\dot{\mathbf{m}}_i^B \cdot \mathbf{J}^B \dot{\mathbf{m}}_i^B}$$

By substituting Eq. (21) into Eq. (19), $\mathbf{h}_i^B$ is expressed as

$$\mathbf{h}_i^B = \mathbf{c}_i \alpha + \mathbf{d}_i, \quad (22)$$

where $\mathbf{c}_i$ and $\mathbf{d}_i$ are given by

$$\mathbf{c}_i = a_i \mathbf{J}^B \dot{\mathbf{m}}_i^B, \quad \mathbf{d}_i = \mathbf{J}^B \boldsymbol{\omega}_{MBi}^B + b_i \mathbf{J}^B \dot{\mathbf{m}}_i^B$$

Let H denote the square of the norm of the angular momentum $\mathbf{h}$ given by

$$(\mathbf{c}_i \alpha + \mathbf{d}_i) \cdot (\mathbf{c}_i \alpha + \mathbf{d}_i) = H, \quad (23)$$

where H is constant from the conservation of angular momentum.

Here, although a specific noise model is not considered in the magnetometer, $\dot{\mathbf{m}}^B$ is supposed to be corrupted by sensor noise and disturbances. Then, Eq. (23) is not always satisfied in a practical sense. In order to estimate $\dot{\phi}_m$, the following cost function is introduced:

$$\begin{aligned} J_r &= \frac{1}{2} \sum_i [(\mathbf{c}_i \alpha + \mathbf{d}_i) \cdot (\mathbf{c}_i \alpha + \mathbf{d}_i) - H]^2 \\ &= \frac{1}{2} \sum_i (e_i \alpha^2 + f_i \alpha + g_i - H)^2, \end{aligned} \quad (24)$$

where e_i , f_i , and g_i are given by

$$e_i = \mathbf{c}_i \cdot \mathbf{c}_i, \quad f_i = 2\mathbf{c}_i \cdot \mathbf{d}_i, \quad g_i = \mathbf{d}_i \cdot \mathbf{d}_i$$

We estimate the unknown constants α and H by minimizing J_r from the known variables $\mathbf{c}_i$ and $\mathbf{d}_i$. The necessary

conditions of J_r for the minimum with respect to α and H are given by

$$\frac{\partial J_r}{\partial \alpha} = 0, \quad \frac{\partial J_r}{\partial H} = 0 \quad (25)$$

Substitution of J_r into the above equations becomes

$$\begin{aligned} \frac{\partial J_r}{\partial \alpha} &= \sum_i (e_i \alpha^2 + f_i \alpha + g_i)(2e_i \alpha + f_i) \\ &\quad - \sum_i (2e_i \alpha + f_i)H = 0, \end{aligned} \quad (26)$$

$$\frac{\partial J_r}{\partial H} = - \sum_i (e_i \alpha^2 + f_i \alpha + g_i) + nH = 0, \quad (27)$$

where n denotes the number of observation data. It follows from Eq. (27) that H is given by

$$H = \frac{1}{n} \sum_i (e_i \alpha^2 + f_i \alpha + g_i) \quad (28)$$

By substituting this equation into Eq. (26), Eq. (26) can be rewritten as

$$\begin{aligned} &2(\bar{e}^2 - \bar{e}^2)\alpha^3 + 3(\bar{e}\bar{f} - \bar{e}\bar{f})\alpha^2 \\ &+ (2(\bar{e}\bar{g} - \bar{e}\bar{g}) + \bar{f}^2 - \bar{f}^2)\alpha + (\bar{f}\bar{g} - \bar{f}\bar{g}) = 0, \end{aligned} \quad (29)$$

where $\bar{\cdot}$ denotes the average operation, i.e.,

$$\bar{f} = \frac{1}{n} \sum_i f_i$$

and $\bar{e}^2 - \bar{e}^2$ denotes the variance of e_i , i.e.,

$$\frac{1}{n} \sum_i (e_i - \bar{e})^2 = \bar{e}^2 - \bar{e}^2 \geq 0$$

Equation (29) is a cubic equation for α . Among the three solutions of Eq. (29), the real solution which minimizes J_r becomes an estimate of α . This gives the best estimate of H in the Euclidean norm.

In summary, an estimate of $\dot{\phi}_m$ is given by substituting this estimated α into Eq. (21) and an estimate of $\boldsymbol{\omega}^B$ is given by substituting this estimated $\dot{\phi}_m$ into Eq. (18). Moreover, an estimate of H is also given by substituting this estimated α into Eq. (28). Then, the estimate of the magnitude of $\mathbf{h}$ is given by

$$|\mathbf{h}| = \sqrt{H} \quad (30)$$

3.2. Numerical differentiation of the earth magnetic field vector

In the proposed attitude rate estimation algorithm, the time derivative $\dot{\mathbf{m}}_i^B$ is used in Eq. (18). However, the second time derivative $\ddot{\mathbf{m}}_i^B$ is not required. If the conservation of angular momentum is represented by $\dot{\mathbf{h}}^I = 0$, the second time derivative $\ddot{\mathbf{m}}_i^B$ will be required for the rate estimation. Therefore, the proposed algorithm has the advantage being relatively robust to the measurement noise.

Next, we focus on the implementation of the proposed algorithm by approximating $\dot{\hat{\mathbf{m}}}_i^B$ from the numerical differentiation of the measured value $\hat{\mathbf{m}}_i^B$. If $\hat{\mathbf{m}}_i^B$ is simply approximated from the difference of $\hat{\mathbf{m}}_i^B$, this approximation is fragile for the measurement noise. Hence, $\dot{\hat{\mathbf{m}}}_i^B$ is approximated from filtering $\hat{\mathbf{m}}_i^B$ by the multiplication of the second-order low pass filter and the time derivative filter, i.e.,

$$\mathcal{L}(\dot{\hat{\mathbf{m}}}^B) \simeq \frac{\omega_f^2 s}{s^2 + 2\zeta_f \omega_f s + \omega_f^2} \mathcal{L}(\hat{\mathbf{m}}^B) \quad (31)$$

where $\mathcal{L}$ denotes the Laplace transformation, and ζ_f and ω_f are filter parameters. The filter parameters are selected so that the cutoff frequency is sufficiently higher than the rough estimate of the norm of the angular velocity ω^B . Since the filter in the right hand side of Eq. (31) is strictly proper, the effect of the measurement noise in $\hat{\mathbf{m}}_i^B$ is expected to be reduced. However, the second-order low pass filter generates the phase delay from the actual $\dot{\hat{\mathbf{m}}}_i^B$. This phase delay is crucial for the conservation of angular momentum, which is the basis of the proposed algorithm. In order to overcome this problem, $\dot{\hat{\mathbf{m}}}^B$ is substituted by the filtered $\dot{\hat{\mathbf{m}}}^B$ by applying the same filter in the right hand side of Eq. (31), i.e.,

$$\mathcal{L}(\dot{\hat{\mathbf{m}}}^B) \simeq \frac{\omega_f^2}{s^2 + 2\zeta_f \omega_f s + \omega_f^2} \mathcal{L}(\hat{\mathbf{m}}^B) \quad (32)$$

Then, because both $\hat{\mathbf{m}}^B$ and $\dot{\hat{\mathbf{m}}}_i^B$ are similarly delayed, the conservation of the angular momentum is approximately recovered. Hence, the proposed algorithm are implemented by using Eqs. (31) and (32). In this case, the estimates of $\dot{\phi}_{mi}$ and ω^B are relatively robust for the measurement noise.

4. Attitude Estimation

4.1. Attitude estimation with rate gyros

In this subsection, we consider the attitude estimation problem with rate gyros, i.e., ω^B is supposed to be available for the attitude estimation. This means that the attitude estimation algorithm in this subsection is independent from the previous attitude rate estimation algorithm. In contrast to the previous attitude estimation algorithm, $\hat{\mathbf{m}}^I$ is supposed to be known and time varying. Then, the attitude $\mathbf{q}_{IB}$ is estimated from $\hat{\mathbf{m}}^I$, $\hat{\mathbf{m}}^B$, and ω^B in a long period of time.

Let $\hat{\mathbf{m}}_i^I$ denote the value of $\hat{\mathbf{m}}^I$ at time t_i . Then, the following relation holds:

$$\hat{\mathbf{m}}_i^I = \mathbf{q}_{IBi} * \hat{\mathbf{m}}_i^B * \mathbf{q}_{IBi}^\dagger, \quad (33)$$

where $\mathbf{q}_{IBi}$ denotes the spacecraft attitude at time t_i .

The additional frame M_j at time t_j ($t_j > t_i$) is introduced in a similar way to Eq. (10). Then, $\mathbf{q}_{IBi}$ is factored as

$$\mathbf{q}_{IBi} = \mathbf{q}_{IMj} * \mathbf{q}_{MjBi} \quad (34)$$

By its construction, $\mathbf{q}_{IMj}$ is determined as follows:

$$V(\mathbf{q}_{IMj}) = \hat{\mathbf{m}}_j^I \sin \frac{\phi_{mj}}{2}, \quad S(\mathbf{q}_{IMj}) = \cos \frac{\phi_{mj}}{2} \quad (35)$$

where ϕ_{mj} denotes a unknown variable to be estimated. $\mathbf{q}_{MjBi}$ can be computed from $\mathbf{q}_{MjBj}$ by factorizing

$$\mathbf{q}_{MjBi} = \mathbf{q}_{MjBj} * \mathbf{q}_{BjB(j-1)} * \cdots * \mathbf{q}_{B(i+1)Bi} \quad (36)$$

where $\mathbf{q}_{BkB(k+1)} (= \mathbf{q}_{B(k+1)Bk}^\dagger)$ is approximately computed as follows:

$$\begin{aligned} V(\mathbf{q}_{BkB(k+1)}) &= \hat{\omega}_k^B \sin \frac{\omega_k \Delta t}{2}, \\ S(\mathbf{q}_{BkB(k+1)}) &= \cos \frac{\omega_k \Delta t}{2}, \\ \hat{\omega}_k^B &= \frac{\omega_k^B}{\omega_k}, \quad \omega_k = |\omega_k|, \end{aligned}$$

for each $k = i, \dots, j-1$.

Substitution of Eq. (34) into Eq. (33) yields

$$\hat{\mathbf{m}}_i^I = \mathbf{q}_{IMj} * \hat{\mathbf{m}}_i^{Mj} * \mathbf{q}_{IMj}^\dagger, \quad (37)$$

where $\hat{\mathbf{m}}_i^{Mj}$ is given by

$$\hat{\mathbf{m}}_i^{Mj} = \mathbf{q}_{MjBi} * \hat{\mathbf{m}}_i^B * \mathbf{q}_{MjBi}^\dagger$$

By substituting Eq. (35) into Eq. (37), the following equation is obtained for $k = i, \dots, j-1$:

$$\mathbf{A}_{kj} \mathbf{s}_j = \mathbf{0} \quad (38)$$

where $\mathbf{A}_{kj}$ and $\mathbf{s}_j$ are given by

$$\mathbf{A}_{kj} = \begin{bmatrix} -\hat{\mathbf{m}}_j^I \times (\hat{\mathbf{m}}_k^I + \hat{\mathbf{m}}_k^{Mj}) & \hat{\mathbf{m}}_k^I - \hat{\mathbf{m}}_k^{Mj} \\ -\hat{\mathbf{m}}_j^I \cdot (\hat{\mathbf{m}}_k^I - \hat{\mathbf{m}}_k^{Mj}) & 0 \end{bmatrix} \quad (39)$$

$$\mathbf{s}_j = \begin{bmatrix} \sin \frac{\phi_{mj}}{2} \\ \cos \frac{\phi_{mj}}{2} \end{bmatrix} \quad (40)$$

In order to estimate ϕ_{mj} , the following cost function is introduced:

$$J_a = \sum_{k=i}^{j-1} (\mathbf{A}_{kj} \mathbf{s}_j)^T (\mathbf{A}_{kj} \mathbf{s}_j) + \lambda_j (1 - \mathbf{s}_j^T \mathbf{s}_j), \quad (41)$$

where λ_j denotes the Lagrange multiplier. In order to minimize J_a with respect to $\mathbf{s}_j$ under the constraint of $\|\mathbf{s}_j\| = 1$, the necessary conditions are given by

$$\frac{\partial J_a}{\partial \mathbf{s}_j} = 0, \quad \frac{\partial J_a}{\partial \lambda_j} = 0$$

It follows that

$$\mathbf{A}_j \mathbf{s}_j = \lambda_j \mathbf{s}_j, \quad \|\mathbf{s}_j\| = 1, \quad (42)$$

where

$$\mathbf{A}_j = \sum_{k=i}^{j-1} \mathbf{A}_{kj}^T \mathbf{A}_{kj} \quad (43)$$

Then, J_a becomes

$$J_a = \mathbf{s}_j^T \mathbf{A}_j \mathbf{s}_j = \lambda_j \quad (44)$$

Since the eigenvalues of the nonnegative symmetric matrix are real and nonnegative, all eigenvalues of $\mathbf{A}_j$ are real and nonnegative. Therefore, the minimum eigenvalue of $\mathbf{A}_j$ is chosen as λ_j between two nonnegative eigenvalues of $\mathbf{A}_j$. Then, ϕ_{mj} is determined from the corresponding eigenvector $\mathbf{s}_j$.

In summary, the attitude $\mathbf{q}_{IBj}$ is estimated from $\hat{\mathbf{m}}_k^I, \hat{\mathbf{m}}_k^B$, and ω_k^B ($k = i, \dots, j-1$) as follows: Firstly, Eq. (42) is solved and the unit eigenvector $\mathbf{s}_j$ is obtained which corresponds to the minimum eigenvalue of $\mathbf{A}_j$. Then, ϕ_{mj} is obtained from Eq. (40). Finally, $\mathbf{q}_{IBj}$ is obtained by substituting Eqs. (10) and (35) into Eq. (2).

4.2. Attitude estimation without rate gyros

In this subsection, we consider the attitude estimation problem with the estimated rate by magnetometers, i.e., ω^B is not supposed to be available and its estimate in Section 3 is utilized for the attitude estimation. In contrast with the attitude rate estimation problem in the previous section, $\hat{\mathbf{m}}^I$ is supposed to be known from the magnetic field model and time varying in a long period of time.

When $\hat{\mathbf{m}}^I$ is time varying, Eq. (13) is modified into

$$\dot{\hat{\mathbf{m}}}^I = \mathbf{q}_{IB} * (\dot{\hat{\mathbf{m}}}^B + \omega^B \times \hat{\mathbf{m}}^B) * \mathbf{q}_{IB}^\dagger \quad (45)$$

Then, Eq. (18) is modified as follows:

$$\omega^B = \omega_{\text{org}}^B + \mathbf{q}_{IB}^\dagger * \omega_{\text{cor}}^I * \mathbf{q}_{IB} \quad (46)$$

where

$$\omega_{\text{org}}^B = \frac{\dot{\hat{\mathbf{m}}}^B \times (\hat{\mathbf{m}}^I + \hat{\mathbf{m}}^B)}{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B} + \hat{\mathbf{m}}^B \dot{\phi}_m \quad (47)$$

$$\omega_{\text{cor}}^I = \frac{\dot{\hat{\mathbf{m}}}^I \times (\hat{\mathbf{m}}^I + \hat{\mathbf{m}}^B)}{1 + \hat{\mathbf{m}}^I \cdot \hat{\mathbf{m}}^B} \quad (48)$$

The first term of Eq. (46), ω_{org}^B , corresponds to the original term in Eq. (18). Although $\hat{\mathbf{m}}^I$ is time varying, it can be regarded as quasi stationary in a short period of time; therefore, it can be computed from the proposed attitude rate estimation algorithm in the previous section. The second term of Eq. (46) is the correction term.

Note that $\mathbf{q}_{IB}$ is necessary to compute the correction term, and that the estimate of $\mathbf{q}_{IB}$ is not immediately obtained after the attitude estimation is carried out. For the time period while the estimated $\mathbf{q}_{IB}$ is not obtained, the estimates of ω^B is accumulated by substituting Eq. (46) for

$$\omega^B \simeq \omega_{\text{org}}^B + \mathbf{q}_{MB}^\dagger * \omega_{\text{cor}}^I * \mathbf{q}_{MB} \quad (49)$$

where $\mathbf{q}_{MB}$ is uniquely determined from $\hat{\mathbf{m}}^I$ and $\hat{\mathbf{m}}^B$.

The reminder of the attitude estimation is analogous to Section 4.1. In the beginning, $\mathbf{q}_{IBj}$ is computed by replacing ω_k^B in Eq. (36) with Eq. (49) for $k = i, \dots, j-1$ at each time t_j . Once $\mathbf{q}_{IBj}$ is sufficiently accumulated, $\mathbf{q}_{IBj}$ is computed by replacing ω_k^B in Eq. (36) with Eq. (46) for $k = i, \dots, j-1$ and then ω_j^B is computed by substituting $\mathbf{q}_{IBj}$ into Eq. (46) at each time t_j .

4.3. Discussions

In the attitude rate estimation problem, $\hat{\mathbf{m}}^I$ is supposed to be stationary in a short period of time. In contrast, in the attitude estimation problem, $\hat{\mathbf{m}}^I$ is supposed to be time-varying in a long period of time. If the angular velocity vector ω^B is measurable by using rate gyros, both problems are independent. However, if the angular velocity vector ω^B is estimated by using magnetometers, the different assumptions on $\hat{\mathbf{m}}^I$ look contradictory. Therefore, the correction term in Eq. (46) becomes necessary.

It is also possible to establish the validity of those assumptions by taking different sizes for observation periods. That is, $\hat{\mathbf{m}}^I$ is varying in a long period time, whereas it can be regarded stationary in a short period time.

Now, we make a remark on the necessity of time varying $\hat{\mathbf{m}}^I$ for the attitude estimation problem. When $\hat{\mathbf{m}}^I$ is constant, i.e., $\hat{\mathbf{m}}_k^I = \hat{\mathbf{m}}_j^I$ for $k = i, \dots, j-1$, $\hat{\mathbf{m}}_k^{M_j}$ becomes from Eqs. (4) and (37)

$$\begin{aligned} \hat{\mathbf{m}}_k^{M_j} &= \mathbf{q}_{IMj}^\dagger * \hat{\mathbf{m}}_k^I * \mathbf{q}_{IMj} \\ &= \mathbf{q}_{IMj}^\dagger * \hat{\mathbf{m}}_j^I * \mathbf{q}_{IMj} = \hat{\mathbf{m}}_j^I \end{aligned} \quad (50)$$

It then follows that $\mathbf{A}_j = \mathbf{0}$; therefore, the minimum eigenvalue problem in Eq. (42) does not give an estimate of $\mathbf{q}_{IBj}$. Hence, the time-varying characteristic of $\hat{\mathbf{m}}^I$ is crucial for the proposed attitude estimation algorithm.

5. Numerical Examples

5.1. Attitude rate estimation

We evaluate the proposed attitude rate estimation algorithm by numerical simulations. Let $\mathbf{J}^B$, $\mathbf{h}^I$, and $\mathbf{q}_{IB}(0)$ be given by

$$\begin{aligned} \mathbf{J}^B &= \begin{bmatrix} 10 & & \\ & 15 & \\ & & 20 \end{bmatrix} [\text{kgm}^2], \\ \mathbf{h}^I &= \begin{bmatrix} 0.383 \\ 0.702 \\ 0.600 \end{bmatrix} [\text{Nms}], \\ \mathbf{q}_{IB}(0) &= \begin{bmatrix} -0.253 \\ -0.052 \\ -0.113 \\ 0.960 \end{bmatrix}, \end{aligned}$$

respectively, where $\mathbf{h}^I$ and $\mathbf{q}_{IB}(0)$ are randomly selected vectors. $\mathbf{h}^I$ is supposed to be constant during the simulations. The Gaussian white noise with the standard deviation of 0.005 is added to each component of the measured $\hat{\mathbf{m}}^B$. The filter parameters are chosen as $\omega_f = 1 [\text{rad/s}]$ and

Table 1. Simulation conditions.

Altitude	500 [km]
Eccentricity	0
Inclination	30 [deg]
Longitude of ascending node	0 [deg]
Initial argument of latitude	0 [deg]
Geomagnetic field model	Dipole model ⁸⁾

$\zeta_f = 0.7$. The other simulation conditions are summarized in Table 1.

Figure 1 shows the time history of $\hat{\mathbf{m}}^I$ which is computed based on the dipole model of the geomagnetic field model⁸⁾ (x , y , and z denote x -, y -, and z -component, respectively). Figure 2 shows the time history of $\hat{\mathbf{m}}^B$ which is measured in the body frame.

In the proposed attitude rate estimation algorithm, the angular velocity ω^B is estimated by computing the unknown variable $\dot{\phi}_m$ from the past observation of $\hat{\mathbf{m}}^B$ in a short period of time, where 500 observation data are used with the sampling period $\Delta t = 0.1$ [sec]. Firstly, the observation data is accumulated for 50[sec], and then, the attitude rate estimation is carried out after 50[sec].

Figure 3 shows the time history of the true value of $\dot{\phi}_m$ (blue line) and its estimate (green line). Figure 4 shows the time histories of the true value of ω^B (thick lines) and its estimate (thin lines). Figure 5 shows the time histories of the attitude rate estimation error which equals the true attitude rate minus the estimated attitude rate. These figures show that $\dot{\phi}_m$ and ω^B are relatively accurate.

Fig. 1. Earth magnetic field vector $\hat{\mathbf{m}}^I$ in the inertial frame.

5.2. Attitude estimation

In this subsection, we evaluate the proposed attitude estimation algorithm by numerical simulations. In the proposed estimation algorithm without rate gyros, we estimate both the angular velocity ω^B and the attitude $\mathbf{q}_{IB}$ by computing the unknown variables $\dot{\phi}_m$ and ϕ_m , respectively, from the past observation of $\hat{\mathbf{m}}^B$. The same simulation

Fig. 2. Measured earth magnetic field vector $\hat{\mathbf{m}}^B$ in the body frame. Fig. 3. True value of $\dot{\phi}_m$ (blue line) and its estimate (green line). Fig. 4. True value of ω^B (thick lines) and its estimate (thin lines).

Fig. 5. Attitude rate estimation error (True attitude rate - Estimated attitude rate).

condition in the attitude rate estimation is assumed, whereas the number of the observation data is increased to 3000 points, i.e., 300[sec] for the attitude estimation. We firstly accumulate the observation data for 400[sec]; then, the attitude rate estimation is carried out after 400[sec].

Figure 6 shows the time histories of the true attitude, $\mathbf{q}_{IB}$ (thick lines; $q_1 \sim q_3$ denote the vector part and q_4 denote the scalar part) and its estimate, $\hat{\mathbf{q}}_{IB}$ (thin lines). Figure 7 shows the time histories of the attitude estimation error which is evaluated by the norm of $V(\hat{\mathbf{q}}_{IB}^\dagger * \mathbf{q}_{IB})$. Although the accuracy of the estimation is not high enough, the estimated attitude follows the true attitude.

Fig. 6. True value of $\mathbf{q}_{IB}$ (thick lines) and its estimate (thin lines).

6. Conclusions

We have proposed new attitude and attitude rate estimation algorithms for a spacecraft using magnetometers. An attitude rate estimation algorithm presumes that the earth magnetic field vector in the inertial frame is quasi stationary in a short period of time. This algorithm requires the approximate computation of the time derivative of the magnet filed vector in the body frame, whereas it does not require

Fig. 7. Attitude estimation error $|V(\hat{\mathbf{q}}_{IB}^\dagger * \mathbf{q}_{IB})|$.

the second time derivative; therefore, this algorithm is relatively robust for the measurement noise. The estimate of the angular velocity vector is given by solving a cubic equation. In contrast, an attitude estimation algorithm utilizes the changes of the earth magnetic field vector in the inertial frame in a long period of time. This attitude estimation algorithm can be combined with the result of the attitude rate estimation algorithm and estimates the spacecraft attitude in terms of Euler parameters by solving a minimum eigenvalue problem. The effectiveness of the proposed algorithm is demonstrated by numerical simulations.

One of the important future topics is a real-time implementation. To this end, an extension of the proposed batch algorithms to the recursive ones will be helpful to reduce the computational load.

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