Density Measurements in Cylindrical Nozzle Jets by Rainbow Schlieren Deflectometry*

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A rainbow schlieren system with a spatial resolution of 20 μm is employed to determine the hue and density profiles in an axisymmetric domain with a field of view of 100 mm in diameter. The rainbow schlieren method is applied for the underexpanded sonic jet from a cylindrical nozzle with larger length-to-diameter ratios. Experiments have been performed in a range of nozzle operating pressure ratios from 2.0 to 5.0. As a result, rainbow schlieren images show jet boundaries, expansion waves, and oblique shock waves in the shock-cell structure by continuous color gradation. Density profiles at any cross-section of the jet flows can be obtained from the Abel inversion. The schlieren images and density contour plots for high nozzle pressure ratios show the archetypal characteristics of shock cell structures appearing repeatedly in the flowfield. Also, an analytical model to predict the flow properties at the exit plane of the cylindrical nozzle is proposed. It is shown that the analytical results are in good quantitative agreement with the experimental ones from the rainbow schlieren.

Key Words: Rainbow Schlieren Deflectometry, Quantitative Measurements, Cylindrical Nozzle Jets, Flow Model

Nomenclature

- A: cross-sectional area (m²)
- Cc: contraction coefficient (-)
- D: inner diameter (m)
- d: transverse ray displacement (m)
- Hue: color on rainbow filter (deg)
- M: Mach number (-)
- p: pressure (Pa)
- r: radius (m)
- T: temperature (K)
- u: velocity (m/s)
- x,y,z: rectangular coordinate system in Fig. 1 (m)
- γ: specific heat ratio (-)
- θ,φ: angles defined in Fig. 9 (rad)
- ρ: density (kg/m³)

Superscript

* : choking state

Subscripts

- ac: incoming convergent radial flows just ahead of nozzle inlet
- b: value at surrounding air
- e: nozzle exit plane
- os: value at plenum chamber
- r: reattachment point

1. Introduction

A clear understanding of the qualitative and quantitative properties of complex shock containing supersonic jet flows requires high-resolution, optical measurements across the whole flowfield. Schlieren methods1) are widely used as optical tools to study the characteristics of compressible flows qualitatively. One of the most popular applications is in the experimental study of supersonic flows in a duct or supersonic jets from a nozzle because of its simple optical arrangement with a high degree of resolution and ability to easily observe such structures as shock waves, Prandtl-Meyer compression and expansion fans in supersonic jet flows. Furthermore, this method for flow visualization does not require the introduction of additives into the flowfield and is capable of providing useful qualitative information on the variations in fluid density, temperature, and static pressure. Although the monochrome schlieren technique with a knife-edge in the filter plane or the cutoff plane is commonly employed because of a simple optical configuration, it generally allows only qualitative flow visualization. On the contrary, there are several advantages of adding color to the schlieren image.

In a simple color schlieren system, the knife edge in the cutoff plane is replaced by a tricolor filter with a background color center and two colors separated by the central band. The direction of the filter can usually be adjusted so that the density gradients in different...
directions can be examined. Sometimes, Prandtl-Meyer compression fans and shock waves, and Prandtl-Meyer expansion fans in a supersonic flow are visualized in a color schlieren system with a tricolor filter; for example, as red and blue color images, respectively. However, it is impossible to discriminate the degree of the compression or the expansion from the schlieren image; i.e., the shock waves are visualized as the same color as the Prandtl-Meyer compression fans regardless of the degree of compression.

The conventional schlieren techniques mentioned above have been extensively employed for qualitative flow visualization. The simplest method for visualizing axisymmetric flowfields with varying refractive index quantitatively is probably the rainbow schlieren deflectometry technique. A rainbow filter with a continuous color spectrum was first proposed by Howes. He fabricated it by projecting the spectral output of a microjets discharged from an orifice injector of 500 μm in diameter. Multiple shock-cell structures with expansion fans, freejet boundary, incident, normal, and reflected shock waves, subsonic slip stream, and outer shear layer in the underexpanded jet were optically observed with continuous hue distributions to infer the density field. They estimated pressure, temperature, and Mach number profiles along the jet centerline using the density data inferred from the rainbow schlieren measurements. However, isentropic relations were assumed for regions across a strong shock wave with a Mach disk to obtain flow properties along the jet centerline. Therefore, there is no conclusive proof that the inferred values on the pressure, temperature, and Mach number in the region near the Mach disk are correct.

Yamamoto et al. 6) applied the rainbow schlieren deflectometry for correctly expanded supersonic jets from a convergent-divergent axisymmetric nozzle and the experimental data from the schlieren were compared with a numerical simulation as well as the measured values using a Pitot probe. They showed that the density distribution at a cross-section normal with respect to the jet axis agrees quantitatively with those from the simulation and experiment.

In recent years, the design and fabrication of a micro-electro mechanical system (MEMS) has increased the need for a detailed understanding of fluid flow in microscale devices including micro orifices, micro nozzles, micro valves, and so on. A microscale nozzle frequently has an axial length larger compared to the inner diameter of the nozzle, and fabrication of a nozzle with sufficient accuracy is difficult. Therefore, a cylindrical nozzle with sharp upstream edges can be expected for microscale mass flow meters under choked conditions because of its simple nozzle shape and it is easy to manufacture. The flow characteristics through such nozzles have been discussed by some investigators 5, 8) for ones with smaller length-to-diameter ratios, but little is known in current literature about compressible flow properties through ones with larger length-to-diameter ratios. Therefore, in the present study, the rainbow schlieren deflectometry is applied for a compressible jet from a cylindrical nozzle with sharp upstream edges where its axial length is larger compared to the inner diameter of the nozzle. Also, a physical model to predict the flow properties at the exit plane of the nozzle is proposed using the control volume approach and the analytical results are compared with the experimental data from the rainbow schlieren deflectometry. As a result, a mechanism for the flow loss in a cylindrical nozzle with sharp upstream edges is clarified.

2. Experimental Procedure

2.1. Experimental apparatus

Figure 1 shows a schematic drawing of a blowdown type supersonic wind tunnel with simple optical configurations for the schlieren system. The rectangular Cartesian coordinate system with x, y, and z axes is also shown in Fig. 1. The high-pressure dry air supplied by a compressor to an air storage tank is discharged into the atmosphere through a solenoid valve, a plenum chamber, and a cylindrical nozzle. The air storage tank provides a total capacity of an approximate 2 m³ and is capable of supplying air at a maximum storage pressure of 1 MPa.
The solenoid valve is used to regulate the pressure in the plenum chamber, which can be held constant to within ±0.5 kPa automatically as the plenum pressure varies. The temperature in the plenum chamber \( T_0 \) is measured using a thermocouple, and it maintains constant at room temperature (approximately 294 K ± 0.1 K) during tests. The ambient density \( \rho_a \) is 1.2 ± 0.05 kg/m\(^3\) and back pressure \( p_b \) is 101 ± 1 kPa. The operating pressure ratio defined as the ratio of plenum pressure to back pressure is varied from 2.0 to 5.0, within the range of the underexpanded sonic jet. However, only a limited selection is presented in the present paper. For each operating pressure ratio, 100 data were collected in 0.1 s at a sampling rate of 1 kHz.

As shown in Fig. 2, the nozzle used in the present experiment is cylindrical in shape and has the corner of the right angle at the nozzle inlet with an 8 mm inner diameter and is 85 mm long. The operating pressure ratio defined as the ratio of plenum pressure to back pressure is 2.0 to 5.0, within the range of the underexpanded sonic jet.

The rainbow schlieren system shown in Fig. 1 consists of rail-mounted optical components including a 50 micron-diameter pinhole, two 100-mm-diameter, 500 mm focal length achromatic lenses, a computer-generated 35-mm-wide slide with color gradations in a 1.4-mm-wide strip, and a digital camera with variable focal length lens. A continuous 250 W metal halide light source connected to a 50-micron-diameter fiberoptic cable provides the light input at the pinhole through a 16.56 mm focal length objective lens. The camera output in the RGB format is digitized by a personal computer with a 24-bit color frame grabber.

The dashed line in Fig. 1 (b) shows that a light ray from the collimating lens is deflected while passing through the underexpanded sonic jet with refractive index gradients. The second lens decollimates the deflected ray to form a displaced image of the source at the cutoff plane. The camera lens is then used to image the jet flow onto the recording medium of the digital camera.

**2.2. Rainbow filter**

The rainbow filter used in the present experiments is shown in Fig. 3. The filter was fabricated using computer software and then printed digitally on a high resolution 35 mm color film recorder. It has continuous hue variation from \( \text{Hue} = 0 \) to 314 deg in a 1.4-mm-wide strip and the origin \( y = 0 \) corresponds to \( \text{Hue} = 185 \) deg. The changes in color are arranged in the direction in which the density gradients are to be observed.

The characteristics of the rainbow filter were performed by traversing the filter in intervals of 0.1 mm in the \( y \) direction at the cutoff plane of the rainbow schlieren apparatus in Fig. 1 before starting experiments.

The calibration result of the rainbow filter is shown as open symbols in Fig. 4. The abscissa is \( \text{Hue} \) [deg] and the ordinate is the transverse ray displacement \( d \) [mm] from the \( x \) axis at the cutoff plane. The solid line indicates a least squares regression of the experimental data. A detailed description of the filter calibration method is contained in the paper by Yamamoto et al. \(^6\). The method to determine the density field in an axisymmetric jet from the schlieren images with the horizontal rainbow filter was reported by Al-Amar et al. \(^4\).
3. Results and Discussion

3.1. Rainbow schlieren pictures

Although the present experiments were performed in a range of nozzle pressure ratios between 2.0 and 5.0, only the results for three nozzle pressure ratios of 2.0, 3.0, and 4.0 are considered for flow visualization. The results for a nozzle pressure ratio of 5.0 will be reported in another paper. In the present experiments, the back pressure $p_b$ was the same as the atmospheric pressure, and plenum pressure $p_{os}$ was only varied by adjusting the solenoid valve. The rainbow filter shown in Fig. 3 was used to visualize the jet flow where rainbow schlieren pictures were taken by filters oriented vertically and horizontally with respect to the $z$ axis, respectively. The rainbow schlieren pictures obtained in the present experiments have a high resolution of 20 $\mu$m.

The rainbow schlieren results of the jets from the cylindrical nozzle used in the present study are presented in Figs. 5 and 6 with flow from left to right. The filter orientation is shown on the upper right corner in these pictures and the location of the background hue is represented as the dashed line on the filter. The abscissa is the distance $z$ [mm] from the nozzle exit plane and the ordinate is the vertical distance from the center of the nozzle exit plane. These pictures correspond to the vertical and horizontal knife edge, respectively, in the conventional schlieren system with a knife edge.

In the case of $p_{os}/p_b = 2.0$, no distinct features in the jet structure can be seen in Fig. 5(a) because there are almost no density gradients in the flow direction. Figure 6(a), with the horizontal rainbow filter for the same nozzle pressure ratio, shows color gradation in the vertical direction near the jet boundaries at $y = \pm 4$ mm. The upper and lower colors at their boundaries are different.
from each other. Because the asymmetric rainbow filter shown in Fig. 3 is used in the present schlieren system, the rays passing through the upper and lower jet boundaries have downward and upward angular deflection toward the opposite boundaries, respectively.

The rainbow schlieren pictures of Figs. 5(b) and 6(b) for $p_{os}/p_b = 3.0$ indicate that the expansion inside the cylindrical nozzle is insufficient to reach the back pressure because a series of expansion waves originating at the lip of the nozzle are clearly visible by the continuous decrease in hue. Additionally, they show the archetypal characteristics of shock-cell structures appearing repeatedly in the jet flowfield. Uniform or background color can be seen vertically far away from the jet centerline. In the rainbow schlieren picture of Fig. 6(b), showing the density gradient in the $y$ direction, the same color as the background hue is also observed at the jet centerline where the transverse component of the density gradient is, and hence, the transverse light ray displacement passing through the jet centerline is consequently zero.

When the nozzle pressure ratio increases from 3.0 to 4.0 as shown in Figs. 5(c) and 6(c), distinct oblique shock waves from the jet boundaries and their reflected oblique shock waves at the jet centerline appear in the first shock-cell. A Mach disk still does not appear in the first shock cell. As a result, the overall shock-cell structure is similar to the result for $p_{os}/p_b = 3.0$. Moreover, Fig. 6(c) with the horizontal rainbow filter, shows that the strength of the reflected oblique shock waves at the jet centerline of the oblique shock waves in the first shock-cell does not hold a constant value but changes gradually depending on the location, because they are displayed by the color gradation in the schlieren picture. Such representation cannot be observed in the conventional color schlieren because of the discontinuous color stripes of the filter. The spacing between respective shock-cells increases with increasing nozzle pressure ratio.

### 3.2. Density contour plots

The schlieren pictures of Figs. 5 and 6 give a good qualitative indication of the shape of the various features of the jet structure. A more quantitative visualization of the jet flowfield is given in the density contour plots of Fig. 7. The plots were determined from the Abel inversion of the hue data of the schlieren pictures with the horizontal rainbow filter for each nozzle pressure ratio and show the densities on the cross-section including the jet centerline ($y = 0 \text{ mm}$). Furthermore, the results are presented with the color bar showing the density ranges from a maximum value of 2.6 kg/m$^3$ to a minimum value of 0.5 kg/m$^3$. The solid lines with a numeral in Fig. 7 show that densities are isopycnic. For all the plots in Fig. 7, a uniform color such as green or background hue is observed far away vertically from the jet centerline since there are no transverse density gradients there.

Figure 7(a) for $p_{os}/p_b = 2.0$ shows that the jet from the nozzle exit plane is nearly correctly expanded. Consequently, the density in the jet is almost constant over all the regions within the field of view and no significant shock wave can be observed in the flowfield. When $p_{os}/p_b$ equals 3.0, the regular and quasi-periodic shock-cell structure appears in the jet flowfield. When $p_{os}/p_b$ increases to 4.0, as shown in Fig. 7(c), all of the shock-cells become extremely strong. Additionally, light refraction is strongest through the jet boundaries, oblique shocks, reflected oblique shocks, Prandtl-Meyer compression and expansion regions where the lateral and longitudinal density gradients are strongest. Each hue can be associated with a specific refraction displacement at the position of the rainbow filter. It is impossible to obtain such quantitative information by conventional schlieren measurements.

Figure 7 gives a good quantitative indication of the shock shock-cell structures. Therefore, the density contour plot by the rainbow schlieren deflectometry is valuable in gaining qualitative and quantitative knowledge of the shock containing flow.

### 3.3. Density along jet centerline

The density distributions along the jet centerline ($y = 0 \text{ mm}$) are depicted in Fig. 8. The abscissa is the distance $z$
As shown in Fig. 8(a), for \( p_{\text{in}}/p_b = 2.0 \), the density from the nozzle exit plane has almost the same value of 1.35 kg/m\(^3\) over the far downstream distance. When it is assumed that the nozzle flow obeys the isentropic process and it is just choked at the nozzle exit plane, the density at the exit becomes 1.52 kg/m\(^3\) for the present experimental conditions. The same calculation procedure was repeated for the other two nozzle pressure ratios. The results are indicated as the leftward pointing arrow on the ordinate in Figs. 8(a), (b), and (c). The theoretical value for the exit density is higher than the experimental one shown in Fig. 8(c).

As shown in Fig. 8(b), when the nozzle pressure ratio increases to \( p_{\text{in}}/p_b = 3.0 \), the jet from the nozzle is underexpanded with a wavy form due to the quasi-periodic shock-cell structure, and several positive and negative peaks can be seen on the centerline density distribution. The locations of each shock in the shock-cell structure are indicated by a steep rise in density on the centerline. However, the positive and negative peaks of density decrease gradually toward downstream because of successive reductions in the strength of each shock in the jet \(^9\) and finally approach the ambient density \( \rho_b = 1.2 \) kg/m\(^3\). As shown in Fig. 8(c) for \( p_{\text{in}}/p_b = 4.0 \), the jet density decreases abruptly from the density at the nozzle exit toward the local minimum below the ambient density before it increases rapidly by the oblique shocks in the first shock-cell. A similar tendency for density variation across consecutive downstream shock-cells can be clearly observed in Fig. 8(c).

What needs to be emphasized from Fig. 8 is that the density values at the nozzle exit plane for the present experiments are always lower than the theoretical ones, and the difference between experiment and theory increases as the nozzle pressure ratio increases. This means that losses such as wall frictions or shock waves exist inside the nozzle because the flow in a constant-area duct with sudden contraction at the inlet may be choked at the vena contracta within the duct.\(^{8,10}\) Kolhe and Agrawal\(^5\) suggested that, for a microjet issuing from a thick orifice with sharp upstream edges, the flow is choked downstream of the vena contracta formed within the orifice and a shock wave occurs in the vicinity of the orifice exit. The mechanism of flow loss through a cylindrical nozzle with sharp upstream edges will be clarified in Section 3.4.

### 3.4. Physical model for compressible flow through cylindrical nozzle

The control-volume approach is utilized to obtain the flow properties at the nozzle exit theoretically. The flow through a cylindrical nozzle with inner diameter \( D \) is shown in Fig. 9. The dashed line in Fig. 9 indicates a control volume enclosing the incoming convergent radial flow expressed as an arc, the wall surface, and the location where the flow reattaches after separation due to the flow contraction at the inlet. Taking into account the assumptions that the incoming flows from the location of the arc are choked just at the vena contracta, all losses in the control volume are negligible, the flow is incompressible from the reservoir far upstream of the nozzle to the location of the arc, and the flow at the vena contracta or the throat is uniform over the duct cross-section. Accordingly, the continuity condition for the adiabatic steady flows through the control volume can be expressed as

\[
\rho_u u_w \int_0^\theta 2\pi r^2 \sin \theta \, d\theta = \rho^* u^* A^*
\]

where \( \rho_u = \rho_{\text{in}} \) and \( u_w \) indicate the density and velocity at the arc, respectively, and \( r \) is the radius of
curvature for the arc, \( A = \pi D^2 / 4 \) the cross-sectional area of the nozzle, \( \theta \) the angle from the nozzle axis, \( \varphi \) the half angle of the circular cone where the cone point is not usually identical with the center of the vena contracta. Additionally, the asterisk means the choking state and the subscript "ac" is the flow properties at the arc.

The momentum conservation equation can be derived between the nozzle inlet and vena contracta as follows:

\[
p_0 A + \int_0^\varphi 2 \pi \mu_0 r^2 \sin \theta \cos \theta d\theta = p^* A + p^* A^* u^* \quad (2)
\]

If \( \varphi = \pi / 2 \) and \( r = D / 2 \), combined with Eqs. (1) and (2) as well as the state equation for air with the specific heat ratio \( \gamma \), the contraction coefficient \( C_c = A^*/A \) satisfies the relation

\[
C_c = 1 - \left( 1 - \frac{p^*}{p_{in}} \right)^{\gamma / \gamma} \left( 1 - \frac{p^*}{p_{in}} \right) = 0.721 \quad (3)
\]

for \( \gamma = 1.4 \) and \( p^*/p_{in} = 0.528 \).

According to Streeter et al., 11) the contraction coefficient for water determined by Weisbach is \( C_c = 0.624 \) for sudden contraction in a pipeline with a square-edged opening from a reservoir. Ward Smith 12) showed that \( C_c \) is somewhere between the two approximate values of 0.56 and 0.64 for thick orifices with sharp upstream edges widely studied under incompressible, high-Reynolds-number flow conditions.

Furthermore, the flow Mach number \( M_{ac} \) at the location of the arc can be estimated from combination of Eqs. (1) ~ (3) for \( C_c = 0.721 \) and \( \gamma = 1.4 \) as follows:

\[
M_{ac} = \frac{1}{2} \left( \frac{p^*}{p_0} \right)^{\gamma / 2} C_c = 0.209 \quad (4)
\]

This means that the assumption of incompressible flow from the reservoir to the arc is acceptable, because air flow can be considered incompressible for Mach numbers below 0.3. 13)

The flow Mach number \( M_r \) of the flow reattachment point can be correlated with the duct area \( A = \pi D^2 / 4 \) and the flow area \( A^* = C_c A \) at the vena contracta by the one-dimensional isentropic relation if all loses in the control volume can be negligible; i.e.,

\[
\frac{A}{A^*} = \frac{1}{M_r^2} \left[ 2 \left( 1 + \frac{\gamma - 1}{2} M_r^2 \right) \right]^{\gamma / (\gamma - 1)} \quad (5)
\]

It is assumed that a normal shock wave stands at any location downstream of the reattachment point and the flow is choked at the nozzle exit plane due to the wall friction; i.e., the Mach number of unity at the exit. As a result, the flow properties at the nozzle exit plane such as density and pressure can be calculated using Eqs. (3) and (5) in addition to the normal shock relation.

As shown in Fig. 10, the result from the present flow model is compared with that from the rainbow schlieren technique. The solid and dashed lines indicate the values calculated by the present flow model and those calculated based upon the assumption of the isentropic flow from the reservoir upstream of the nozzle to the nozzle exit, respectively. The open symbols are the present experimental values from the rainbow schlieren technique. The analytical results from the present flow model agree very favorably with the experimental ones. This means that the flow loss in a cylindrical nozzle with a larger length-to-diameter ratio attributes significantly to the shock wave formed downstream of the vena contracta near the nozzle inlet.

4. Conclusion

We have applied the rainbow schlieren deflectometry for cylindrical nozzle jets. Experimental schlieren images
were employed to determine the density profiles in axisymmetric jets with a field of 100 mm in diameter and at a spatial resolution of 20 μm. Several important conclusions obtained from the present study are summarized as follows.

Rainbow schlieren images show jet boundaries, expansion waves, and oblique shock waves by color gradation. The density profile at any cross-section of the jet flow for each nozzle pressure ratio can be obtained from the Abel inversion of the schlieren image using the horizontal rainbow filter. As a result for $p_{\text{os}}/p_{b}=2.0$, the jet density contour plot shows a nearly constant value that is slightly higher than the ambient density. However, for the other nozzle pressure ratios, they indicate several shock cells quantitatively. Additionally, densities along the jet centerline have regular and cyclic positive and negative peaks in the downstream direction, and finally approach the ambient density. A new flow model for compressible flow through a cylindrical nozzle with sharp upstream edges was proposed based upon the control volume approach and the analytical results were compared with those from the rainbow schlieren technique. The present analytical model can quantitatively predict the flow properties at the nozzle exit plane. A mechanism for the flow loss in a cylindrical nozzle with larger length-to-diameter ratios was clarified.

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