Spacecraft Attitude Control with RWs via LPV Control Theory: Comparison of Two Different Methods in One Framework

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Satellite dynamics is described by a nonlinear differential equation. Most of recent studies about attitude control have used nonlinear controllers. However, with these controllers, control performance is ignored in most cases. To overcome these problems, we applied Linear Parameter-Varying (LPV) control theory to attitude control problem. To avoid difficulties coming from the nonlinearity in satellite dynamics, we modeled dynamics of spacecraft as an LPV system and applied a Gain-Scheduled (GS) controller to this model using Linear Matrix Inequalities (LMIs). In this paper, by using two methods, GS controllers are designed to guarantee overall stability and to achieve \(\mathcal{H}_2\) performance with distinct Lyapunov solutions. By using these controllers, 3-axis attitude control of a spacecraft with Reaction Wheels (RWs) shall be achieved. To examine how the proposed approach improves the control performance, two proposed methods shall be compared with each other.

Key Words: Attitude Control, RW, LPV, Gain-Scheduled Control

Nomenclature

- \(J\) : inertia matrix of the spacecraft
- \(I_{ws}\) : inertia matrices of wheel spin axis
- \(q\) : quaternion (Euler Parameters)
- \(\omega\) : angular velocity vector of the spacecraft
- \(\Omega\) : wheel spin rate vector
- \(G_s\) : spin axis matrix
- \(F_B\) : body fixed-frame
- \(\rho\) : scheduling parameter vector

Subscripts

- \(0\) : initial
- \(e\) : error
- \(d\) : desired

Superscripts

- \(\times\) : skew-symmetric
- \(\dagger\) : conjugate

1. Introduction

In recent years, attitude control of spacecraft has been studied extensively. They deal with several kinds of actuators such as Momentum Exchange Devices (MEDs) and external torque generators (e. g. gas jets or magnetic torquers). External torque generators have disadvantages such as limited resources or small torque. MEDs have been used for complete attitude control of spacecraft as actuators, in which they do not require fuel. Since Reaction Wheels (RWs) can generate control torque precisely, RWs in MEDs are often used for attitude control of spacecraft. Therefore, in this paper, we mainly focus on RWs to realize complete attitude control.

Most of recent studies use Lyapunov function-based controllers to realize attitude control.\(^1)\)\(^-\)\(^3)\) With Lyapunov function-based controllers, overall stability of attitude control is always guaranteed, however, control performance is hard to evaluate. To overcome this problem, few studies\(^4)\)\(^-\)\(^7)\) are attained complete attitude control with Linear Parameter-Varying (LPV) control theory. In LPV control theory, we modeled the dynamics of a spacecraft as an LPV system and applied a Gain-Scheduled (GS) controller to this model using Linear Matrix Inequalities (LMIs).\(^8)\)\(^-\)\(^9)\) In this paper, to avoid the difficulty from excess of the number of scheduling parameters in the LPV system, we shall introduce two methods.

The first one is the Parameter-Dependent Coordinate Transformation (PDCT) method, which introduces a virtual state variable together with a parameter-dependent coordinate transformation.\(^6)\) With this method, the original plant (Dynamics + Kinematics) can be transformed into a simple LPV model, which is covered with fewer vertices, compared to the original plant. The controller to this simple LPV model can be designed much easily, since the number of vertices considerably decreases. By getting back the coordinate transformation, a controller to the original plant can be obtained.

The second one is the approximation of the final error quaternion value (AQ) method.\(^7)\) With this method, the original plant can be approximated into a simple LPV model by using the approximation of the final error quaternion value in kinematics of the original plant.

After that, GS controllers are designed to guarantee overall stability and to achieve \(\mathcal{H}_2\) performance with distinct Lyapunov solutions. Through some numerical simulations, we show the differences of control performance between the PDCT method and the AQ one.

The rest of this paper is organized as follows. In Section 2, we show a brief overview of the dynamics and the kinematics of a spacecraft. In Section 3, the generalized LPV model of spacecraft dynamics with multi-MEDs shall be established. Section 4 presents two methods (PDCT and AQ) to design the GS controller easily. In Section 5, numerical simulations us-
ing two proposed methods are given to show the effectiveness and comparison between them. Finally, Section 6 concludes the paper.

2. Spacecraft Model

In this paper, we deal with a spacecraft with RWs as shown in Fig. 1. In this section, a dynamics equation of a spacecraft with RWs is presented. After that, a kinematics equation based on the quaternion is described.

2.1. Dynamics

The spacecraft considered in this paper is assumed to be a rigid body. The body fixed-frame $\mathcal{F}_B$ is represented by a set of unit vectors $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$. We assume that all the RWs have the same moment of inertia $I_{sw}$. Using the definition in Nomenclature, the total angular momentum of a spacecraft with RWs is given as follows:

$$H = J_\omega + G_{sw} \Omega.$$  \hfill (1)

Under the assumption that the spacecraft is not applied by any external disturbance torque, the dynamics is given by

$$\dot{H} + \omega \times H = 0.$$  \hfill (2)

Substituting Eq. (1) into the first term of Eq. (2), we have

$$J_\omega + G_{sw} \dot{\Omega} + \omega \times H = 0$$  \hfill (3)

where the notation $\omega$ denotes the following skew-symmetric matrix:

$$\omega := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad \dot{x} = [x_1, x_2, x_3]^T. \hfill (4)$$

This is the dynamics of a spacecraft with multi-RWs.

2.2. Kinematics

The quaternion consists of the vector part and the scalar one. Given the principal rotation axis $\hat{a} = [a_x, a_y, a_z]^T$ with $\hat{a} \cdot \hat{a} = 1$ and the rotation angle $\Theta$, the quaternion (Euler Parameter) is defined by

$$q = \begin{bmatrix} q_0 \\ q_4 \end{bmatrix} := \frac{\hat{a} \sin \frac{\Theta}{2}}{\cos \frac{\Theta}{2}}.$$  \hfill (5)

with the constraint:

$$q^T q = \hat{a}^T \sin^2 \frac{\Theta}{2} + \cos^2 \frac{\Theta}{2} = 1.$$ \hfill (6)

To formulate the attitude tracking problem of a spacecraft, we need the error quaternion $q_e = q_d - q$, where $q$ denotes the current quaternion and $q_d$ denotes the desired quaternion. The kinematics equation is given by

$$\begin{bmatrix} \dot{q}_e \\ \dot{q}_e \end{bmatrix} = G(q_e) \omega, \quad G(q_e) := \frac{1}{2} \begin{bmatrix} q_{4e} I_3 + \vec{q}_e \\ -\vec{q}_e \end{bmatrix}.$$ \hfill (7)

3. LPV Model

Based on the dynamics of Eq. (3), we develop an easy-to-use LPV model, in which we deal with 3-axis attitude control of a spacecraft equipped with RWs.

The Jacobian linearization of Eq. (3) around the equilibrium point $(\omega_0 = 0, \Omega_0 = 0)$ leads to the LPV system of spacecraft dynamics. The spacecraft dynamics with RWs and the kinematic equation based on quaternions are given as follows:

$$\begin{bmatrix} \dot{\omega} \\ \dot{q}_e \end{bmatrix} = A(\rho) \omega + B u,$$

where the scheduling parameter vector is $\rho = G_\Omega \in \mathbb{R}^3$, and the control input vector is $u = \Omega$ together with

$$A(\rho) = J^{-1}(I_{sw}G_\Omega)^\gamma,$$

$$B = -I_{sw}J^{-1}G_x,$$

where

$$G(x) := \begin{bmatrix} 0_{3x3} \\ \frac{\hat{a}}{2} \vec{G}(\vec{q}_e) \omega \\ 0_{3x3} \end{bmatrix} + \begin{bmatrix} B \\ 0_{3x4} \end{bmatrix} u.$$ \hfill (11)

or simply described as

$$\dot{x} = A(\rho, \vec{q}_e)x + Bu$$ \hfill (12)

where

$$A(\rho, \vec{q}_e) := \begin{bmatrix} A(\rho) & 0_{3x3} \\ \frac{1}{2} \vec{G}(\vec{q}_e) & 0_{3x3} \end{bmatrix},$$ \hfill (13)

$$B_e := \begin{bmatrix} B \\ 0_{3x4} \end{bmatrix}.$$ \hfill (14)

Replacing $\vec{G}(\vec{q}_e)$ by a convex hull with a scheduling parameter $\vec{q}_e$, Eq. (11) can be modeled as an LPV system. In this case, the total number of scheduling parameter is $3 + 3$ and the total number of vertices of the convex hull is $64(= 2^6)$. Hence, it is difficult to design a GS controller due to excesses of the number of scheduling parameters.

4. Controller Synthesis for LPV Plant

To avoid the difficulty from excess of the number of scheduling parameters, in this section, we shall introduce two methods, which are the PDCT method and the AQ method.
4.1. Controller synthesis with PDCT

We shall eliminate the scheduling parameter vector $\vec{q}_e$ of Eq. (11) from the coefficient matrix to reduce the number of scheduling parameters. By using the Parameter-Dependent Coordinate Transformation (PDCT) matrix:

$$T := \begin{bmatrix} I_{3x3} & 0_{3x3} \end{bmatrix} \tilde{G}(\vec{q}_e)^{-1},$$

(15)

while introducing a virtual state $\zeta$ with $\zeta := \tilde{G}(\vec{q}_e)^{-1}\vec{q}_e$, we obtain the following simple LPV model which is easy to use for control design. By using $T$, Eq. (11) can be expressed as follows:

$$\begin{bmatrix} \dot{\omega} \\
\tilde{G}(\vec{q}_e)^{-1}\dot{\vec{q}}_e 
\end{bmatrix} = \begin{bmatrix} A(\rho) \\
\frac{1}{2} \tilde{G}(\vec{q}_e) \end{bmatrix} \begin{bmatrix} \omega \\
\vec{q}_e \end{bmatrix} + \begin{bmatrix} B \\
0_{3x4} \end{bmatrix} u$$

(16)

$$\begin{bmatrix} \dot{\omega} \\
\dot{\zeta} 
\end{bmatrix} = \begin{bmatrix} A(\rho) \\
\frac{1}{2} I_{3x3} \end{bmatrix} \begin{bmatrix} \omega \\
\zeta \end{bmatrix} + \begin{bmatrix} B \\
0_{3x4} \end{bmatrix} u$$

(17)

$$\Leftrightarrow \dot{x} = \tilde{A}_i(\rho)\dot{x} + B_e u,$$

(18)

where $x := [\omega^T \zeta^T]^T$ and

$$\tilde{A}_i(\rho) := \begin{bmatrix} A(\rho) \\
\frac{1}{2} \tilde{G}(\vec{q}_e) \end{bmatrix}$$

(19)

$$B_e := \begin{bmatrix} B \\
0_{3x4} \end{bmatrix}.$$ 

(20)

Note that $\zeta \neq \tilde{G}(\vec{q}_e)^{-1}\vec{q}_e$ (since $\tilde{G}(\vec{q}_e)^{-1}$ is not constant) in the lower part of the state variable in Eq. (16) has been replaced by $\zeta$ in Eq. (17). This can be realized since this part is eliminated by the premultiplied matrix $\tilde{A}_i(\rho)$. (Note that two blocks of the right half of $\tilde{A}_i(\rho)$ are zero entries.) The GS controller designed to this simple LPV model guarantees overall stability and control performance for the original plant as in Eq. (18). This controller is given by

$$u = -\tilde{K}(\rho)\dot{x}.$$

(21)

Figure 2 shows the relationship of two plants in Eq. (12) and Eq. (18). After this operation, the number of vertices has been reduced into 8($= 2^3$). Setting $p = 8$ as the number of the vertices, the LPV system can be expressed by the following polytopic representation:

$$\tilde{A}_i(\rho) = \left\{ \sum_{i=1}^{p} \lambda_i(\rho) \tilde{A}_i : \lambda_i(\rho) \geq 0, \sum_{i=1}^{p} \lambda_i(\rho) = 1 \right\}.$$ 

(22)

Then, the GS controller to the simple LPV model in Eq. (18) is constructed by

$$\tilde{K}(\rho) = \left\{ \sum_{i=1}^{p} \lambda_i(\rho) \tilde{K}_i : \lambda_i(\rho) \geq 0, \sum_{i=1}^{p} \lambda_i(\rho) = 1 \right\}.$$ 

(23)

Figure 3 shows the diagram of polytopic system in Eqs. (22) and (23) (in case of $p = 8$).

Now, we consider a GS controller $\tilde{K}(\rho)$ that guarantees overall stability and achieves $\mathcal{H}_2$ performance for the simple LPV model as in Eq. (18). First, we introduce the generalized plant for Eq. (18) defined as follows:

$$\begin{bmatrix} \dot{\tilde{X}} \\
\dot{\tilde{W}} \end{bmatrix} = \tilde{A}_i(\rho)\tilde{X} + B_e u + E\vec{w}$$

(24)

$$z = CX + Du$$

where the coefficient matrix set $(C, D)$ is normally selected such that they normally satisfy the condition $C^T D = 0$, $C^T D > 0$, and where $\vec{w}$ and $z$ are the disturbance input vector and the performance output vector for the simple LPV model in Eq. (18), respectively. Then, we consider the following LMI problem:

$$\inf_{\tilde{X}, \tilde{W}, \tilde{F}} \text{Trace} (\tilde{Z})$$

subject to

$$\begin{bmatrix} (\tilde{A}_i\tilde{X} - B_e\tilde{W}) + (\bullet)^T & \bullet \\ CX - D\tilde{W} & -I \end{bmatrix} < 0,$$

(25)

$$\begin{bmatrix} \tilde{X}^T & \tilde{Z} \end{bmatrix} > 0, \quad \text{for all } 1 \leq i \leq p.$$ 

(26)

Using the optimal solution sets $\tilde{X}$, $\tilde{W}$ to the problem in Eqs. (25) and (26), we have the extreme controllers:

$$\tilde{K}_i = \Psi_i \tilde{X}_i^{-1}, \quad 1 \leq i \leq p.$$ 

(27)

Then, the GS controller to the simple LPV model in Eq. (18) is constructed by substituting Eq. (27) into Eq. (23).

Note that the common Lyapunov solution $\tilde{X} > 0$ was used in the past GS controller design and resulted in conservatism. As
an alternative, we use another method, in which the distinct Lyapunov solutions $\tilde{X}_i > 0$ are adopted. Then, we have
\[
\inf_{\tilde{w} \in X_{\tilde{w}}} \left[ \text{Trace} \left( \dot{\tilde{Z}} \right) \right] \quad \text{subject to}
\begin{bmatrix}
(\tilde{A}_{ci} - B_i \tilde{K}_i) \tilde{X}_i + (\bullet)^T \ast
\end{bmatrix} < 0,
\]
\[
\tilde{X}_i \ast \tilde{Z}_i > 0, \quad \text{for each } 1 \leq i \leq p.
\]

Using the optimal solution sets $(\tilde{X}_i, \tilde{W}_i)$, less conservative extreme controllers can be obtained. Using these extreme controllers $\tilde{K}_i = \tilde{W}_i \tilde{X}_i^{-1}$, $1 \leq i \leq p$, a GS controller is again constructed. In order to guarantee overall stability and control performance in a whole operation range, we seek a common Lyapunov solution $\tilde{X}_c > 0$ that satisfies the following LMIs
\[
\inf_{\tilde{w} \in X_{\tilde{w}}} \left[ \text{Trace} \left( \dot{\tilde{Z}} \right) \right] \quad \text{subject to}
\begin{bmatrix}
(\tilde{A}_{ci} - B_i \tilde{K}_i) \tilde{X}_i + (\bullet)^T \ast
\end{bmatrix} < 0,
\]
\[
\tilde{X}_i \ast \tilde{Z}_i > 0, \quad \text{for all } 1 \leq i \leq p.
\]

for a set of previously designed extreme controllers $\tilde{K}_i$, $1 \leq i \leq p$. Getting back the coordinate transformation $x := T^{-1}(\tilde{q}_c)x$, this GS controller can be transformed into the controller $\tilde{K}(\rho, \tilde{q}_c)$ corresponding to the original plant in Eq. (12) as follows:
\[
\tilde{u} = -\tilde{K}(\rho) \tilde{x} = -\tilde{K}(\rho) T(\tilde{q}_c)^{-1} \tilde{x} = -K(\rho, \tilde{q}_c) x
\]

where
\[
\tilde{K}(\rho, \tilde{q}_c) := \tilde{K}(\rho) T(\tilde{q}_c).
\]

When the LMIs in Eqs. (30) and (31) are infeasible, we can check just the overall stability of the closed-loop system in a whole operation range, while replacing them by the following inequalities:
\[
\tilde{X}_c > 0, \quad (\tilde{A}_{cl} - B_i \tilde{K}_i) \tilde{X}_c + \tilde{X}_c (\tilde{A}_{cl} - B_i \tilde{K}_i)^T < 0,
\]

for all $1 \leq i \leq p$.

4.2. Controller synthesis with AQ

we shall eliminate the scheduling parameter vector $\tilde{q}_c$ of Eq. (11) by using approximation of the final error quaternion value (AQ): 7)
\[
\tilde{q}_c \approx 0, \quad q_{cl} \approx 1.
\]

Then, kinematics Eq. (8) is approximated by
\[
\tilde{G}(\tilde{q}_c) = (q_{cl} I_3 + \tilde{q}_c) \approx I_{3x3}.
\]

As a result, the state-space representation of Eq. (11) can be reduced into
\[
\begin{bmatrix}
\dot{\tilde{w}}, \tilde{q}_c
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_{\tilde{w}} \\
0_{3x3} & 0_{3x3}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}, \tilde{q}_c
\end{bmatrix} +
\begin{bmatrix}
B \\
0_{3x3}
\end{bmatrix} u
\]

\[
\Leftrightarrow \dot{\tilde{x}} = \tilde{A}(\rho) x + B_c u,
\]

where
\[
\tilde{A}_c(\rho) := \begin{bmatrix}
A(\rho) & 0_{3x3} \\
0_{3x3} & 0_{3x3}
\end{bmatrix}, \quad B_c := \begin{bmatrix}
B \\
0_{3x3}
\end{bmatrix}.
\]

The GS controller designed to this simple LPV model is given by
\[
u = -\tilde{K}(\rho)x.
\]

To design the GS controller, we use the same method as PDCT one in Subsection 4.1., in which the distinct Lyapunov solutions $\tilde{X}_i > 0$ are adopted. Then, we have
\[
\inf_{\tilde{w} \in X_{\tilde{w}}} \left[ \text{Trace} \left( \dot{\tilde{Z}} \right) \right] \quad \text{subject to}
\begin{bmatrix}
(\tilde{A}_{cl} - B_i \tilde{K}_i) \tilde{X}_i + (\bullet)^T \ast
\end{bmatrix} < 0,
\]
\[
\tilde{X}_i \ast \tilde{Z}_i > 0, \quad \text{for each } 1 \leq i \leq p.
\]

Using the optimal solution sets $(\tilde{X}_i, \tilde{W}_i)$, less conservative extreme controllers can be obtained. Using these extreme controllers $\tilde{K}_i = \tilde{W}_i \tilde{X}_i^{-1}$, $1 \leq i \leq p$, a GS controller is again constructed. We seek a common Lyapunov solution $\tilde{X}_c > 0$ that satisfies the following LMIs
\[
\inf_{\tilde{w} \in X_{\tilde{w}}} \left[ \text{Trace} \left( \dot{\tilde{Z}} \right) \right] \quad \text{subject to}
\begin{bmatrix}
(\tilde{A}_{cl} - B_i \tilde{K}_i) \tilde{X}_c + (\bullet)^T \ast
\end{bmatrix} < 0,
\]
\[
\tilde{X}_c \ast \tilde{Z}_c > 0, \quad \text{for all } 1 \leq i \leq p.
\]

for a set of previously designed extreme controllers $\tilde{K}_i$, $1 \leq i \leq p$.

Therefore, we can get the GS controller $\tilde{K}(\rho)$ constructed by polytopic system in Eq. (23).

5. Numerical Simulation

We present numerical simulations with the PDCT method and the AQ method described before. The simulation parameters, the initial condition, and the desired quaternions are given in Table 1. The controller design parameters $C$, $D$ and $E$ are given by
\[
C = \begin{bmatrix}
15 \times I_3 & 0_{3x3} \\
0_{3x3} & I_3
\end{bmatrix}, \quad D = \begin{bmatrix}
0_{6x4} & 20 \times I_4
\end{bmatrix}, \quad E = \begin{bmatrix}
I_3 \\
0_{3x3}
\end{bmatrix}.
\]

The obtained controller gain of the extreme controllers $\tilde{K}_i, 1 \leq i \leq 8$ are presented in Appendix.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$</td>
<td>diag[10 10 8]</td>
<td>[kgm²]</td>
</tr>
<tr>
<td>$I_{os}$</td>
<td>0.002</td>
<td>[kgm²]</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>[0.03 – 0.02 0.04]^T</td>
<td>[rad/s]</td>
</tr>
<tr>
<td>$q_0$</td>
<td>[0 0 0 1]^T</td>
<td>–</td>
</tr>
<tr>
<td>$q_d$ (Case 1)</td>
<td>[-0.2 0.2 0.75 0.6]^T</td>
<td>–</td>
</tr>
<tr>
<td>$q_d$ (Case 2)</td>
<td>[0.6 – 0.4 0.66 0.2]^T</td>
<td>–</td>
</tr>
</tbody>
</table>
Fig. 4. Angular velocity with PDCT (Case 1).

Fig. 5. Quaternion with PDCT (Case 1).

Fig. 6. Angular velocity with PDCT (Case 2).

Fig. 7. Quaternion with PDCT (Case 2).

Fig. 8. Angular velocity with AQ (Case 1).

Fig. 9. Quaternion with AQ (Case 1).

Fig. 10. Angular velocity with AQ (Case 2).

Fig. 11. Quaternion with AQ (Case 2).
Complete attitude control of the spacecraft has been achieved by the GS controller with the PDCT method as shown in Figs. 4–7. And complete attitude control of the spacecraft has been achieved by the GS controller with the AQ method as shown in Figs. 8–11.

In the small attitude maneuver (Case 1) as shown in Figs. 4–5, 8–9, these controllers attain complete attitude control at almost the same control performance. On the other hand, in the large attitude maneuver (Case 2) as shown in Figs. 6–7, 10–11, the control performance by the GS controller with the PDCT method has been improved, compared with the control performance by the GS controller with the AQ method. This is because the AQ method includes approximation in the modeling of the LPV plant.

6. Conclusion

In this paper, we have established an LPV model for complete attitude control of a spacecraft with RWs. Based on this LPV model, we have developed an easy-to-use LPV model by using two proposed methods (PDCT and AQ). These two methods overcame the difficulty from excess of the number of scheduling parameters. Through a numerical example, we demonstrated the efficiency of the PDCT method to compare with the AQ method. As a result, the PDCT method has improved control performance in the case of large attitude maneuver.

Acknowledgments

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References


Appendix: Obtained extreme controllers

We show the obtained gain of the extreme controllers $\hat{K}_i, 1 \leq i \leq 8$ by using LMIs Eqs. (28) and (29) as follows:

$$\hat{K}_1 = \begin{bmatrix} \begin{smallmatrix} -0.7776 & 0.0572 & 0.0512 & -0.0214 & 0.0206 & -0.0346 \\ 0.0572 & -0.7890 & 0.0461 & -0.0345 & -0.0217 & 0.0205 \\ 0.0639 & 0.0576 & -0.7708 & 0.0210 & -0.0344 & -0.0214 \\ -0.3790 & -0.3892 & -0.3889 & -0.0202 & -0.0205 & -0.0205 \end{smallmatrix} \end{bmatrix}$$

(45)

$$\hat{K}_2 = \begin{bmatrix} \begin{smallmatrix} -0.8204 & 0.1128 & 0.0893 & -0.0250 & 0.0363 & -0.0117 \\ 0.1128 & -0.7308 & -0.0451 & -0.0160 & -0.0317 & -0.0286 \\ -0.1116 & -0.0564 & -0.8069 & 0.0402 & 0.0101 & -0.0191 \\ -0.3440 & -0.3893 & -0.4403 & -0.0005 & 0.0085 & -0.0343 \end{smallmatrix} \end{bmatrix}$$

(46)

$$\hat{K}_3 = \begin{bmatrix} \begin{smallmatrix} -0.8444 & 0.1067 & -0.0406 & -0.0195 & 0.0399 & 0.0106 \\ 0.1067 & -0.8312 & 0.0911 & -0.0110 & -0.0254 & 0.0363 \\ -0.0507 & 0.1139 & -0.7081 & -0.0288 & -0.0162 & -0.0315 \\ -0.4552 & -0.3525 & -0.3796 & -0.0342 & -0.0010 & 0.0088 \end{smallmatrix} \end{bmatrix}$$

(47)

$$\hat{K}_4 = \begin{bmatrix} \begin{smallmatrix} -0.8461 & -0.0559 & 0.0904 & -0.0197 & 0.0099 & 0.0400 \\ -0.0559 & -0.7290 & 0.9001 & -0.0286 & -0.0314 & -0.0167 \\ 0.1130 & 0.1127 & -0.7884 & -0.0111 & 0.0366 & -0.0249 \\ -0.4556 & -0.3881 & -0.3510 & -0.0343 & 0.0087 & -0.0010 \end{smallmatrix} \end{bmatrix}$$

(48)

$$\hat{K}_5 = \begin{bmatrix} \begin{smallmatrix} -0.7290 & -0.0559 & 0.9001 & -0.0314 & -0.0286 & 0.1600 \\ 0.1128 & -0.8204 & 0.0893 & 0.0363 & -0.0250 & -0.0117 \\ -0.0564 & 0.1116 & -0.8069 & 0.0101 & 0.0402 & -0.0191 \\ -0.3893 & -0.3440 & -0.4403 & 0.0085 & -0.0005 & -0.0343 \end{smallmatrix} \end{bmatrix}$$

(49)

$$\hat{K}_6 = \begin{bmatrix} \begin{smallmatrix} -0.8312 & 0.1067 & 0.0911 & -0.0254 & -0.0110 & 0.0363 \\ 0.0167 & -0.8444 & -0.0406 & 0.0399 & -0.0195 & 0.0106 \\ 0.1139 & -0.0507 & -0.7081 & -0.0162 & -0.0288 & -0.0315 \\ -0.3525 & -0.4552 & -0.3796 & -0.0010 & 0.0342 & 0.0088 \end{smallmatrix} \end{bmatrix}$$

(50)

$$\hat{K}_7 = \begin{bmatrix} \begin{smallmatrix} -0.7890 & 0.0572 & 0.0461 & -0.0217 & -0.0345 & 0.0205 \\ 0.0572 & -0.7776 & 0.0512 & 0.0206 & -0.0214 & -0.0346 \\ 0.0576 & 0.0639 & -0.7708 & -0.0344 & 0.0210 & -0.0214 \\ -0.3892 & -0.3790 & -0.3889 & -0.0205 & -0.0202 & -0.0205 \end{smallmatrix} \end{bmatrix}$$

(51)

From these extreme controllers Eqs. (45)–(52) and Eq. (23), we can obtain the GS controller $\hat{K}(\rho)$. 

Pd_20