Design of Earth-Moon Cyclers Using Primer Vector Theory

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The purpose of this study is to design orbits for transporting passengers and cargo regularly from the Earth to the Moon and back. To accomplish this, so-called “cycler orbits” are used. A cycler orbit is an orbit circulating regularly between two astronomical bodies and can be achieved through gravity-assist swingbys. This study consists of two steps. The first step exploits the mechanics of the “double lunar swingby” to realize a cycler orbit without expending propellant. The second step employs the primer vector theory to design low-cost transfer orbits and calculates the minimum fuel for the mission. The proposed method of this study can realize an Earth-Moon transportation system which operates every two months with a significantly lower cost than the previous method.

Key Words: Cycler, Gravity Assist, Double Lunar Swingby, Optimal Control, Primer Vector Theory

Nomenclature

- \(a\): semimajor axis
- \(R\): radius
- \(b\): aiming radius
- \(d\): distance
- \(e\): eccentricity
- \(f\): true anomaly
- \(G\): gravitational constant
- \(h\): altitude
- \(m\): mass
- \(M\): mean anomaly
- \(n\): mean motion
- \(p\): primer vector
- \(r, r\): position vector, distance(= \(|r|\))
- \(t\): flight time
- \(u, v\): velocity vector, velocity (= \(|v|\))
- \(\Delta V\): vector of velocity change by manual impulses
- \(\Delta a\): vector of velocity change by SB
- \(x\): state vector
- \(\lambda\): adjoint vector
- \(\mu\): standard gravitational parameter

Subscripts

- \(a, p\): apogee, perigee
- \(e, m, s\): Earth, Moon, Sun
- \(i\): value when spacecraft enters SOI
- \(s\): spacecraft
- \(t\): target
- \(\text{total}\): total
- \(\text{transfer}\): transfer orbit
- \(0, f\): initial, final

1. Introduction

Why are many countries trying to explore the Moon? One of the largest reasons is that the Moon is the best celestial object for establishing a base for further space exploration. The Moon also has the following three advantages: the surface area of the Moon is large enough to store huge quantities of fuel resources; the distance between the Earth and the Moon is short enough to send and receive data transmission; and it has ideal underground conditions for fuel concentration. However, a critical problem is that it is extremely expensive to transport large quantities of equipment for Moon exploration by traditional methods. Therefore, it is necessary to devise a solution for regularly transporting humans and resources to the Moon. The purpose of this study is to design orbits for transporting passengers and cargo regularly from the Earth to the Moon and back. These orbits should be activated at a lower cost than past methods, such as the Apollo program. Such orbits could make it possible to realize a “space train system” between the Earth and the Moon in the future.

To accomplish this goal, the so-called “cycler orbits” can be used. Figure 1 illustrates an Earth-Moon cycler orbit. A cycler orbit is an orbit circulating regularly between two astronomical bodies and can be achieved through gravity-assist swingbys. For example, the Aldrin cycler1,2) is one of the most famous cycler orbits, which circulates between the planet Earth and the planet Mars in one synodic period. Typically, a Sun-spacecraft two-body model is used to design Earth-Mars cycler orbits. In comparison, the design of cycler orbits between the Earth and the Moon is much more complicated because the gravitational attraction of the two bodies must be considered.3,4) In Refs. 3) and 4), cycler orbits are found numerically since analytical solutions are no longer available in the three-body problem (3BP).

This study makes use of the analytical solution of two-body problem (2BP) instead of designing trajectory in 3BP and con-

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sists of two steps. The first step is trajectory design of cycler orbit. This step exploits the mechanics of the "double lunar swingby" to realize cycler orbit which consists of two alternating swingbys: swingby acceleration and swingby deceleration. Ideally, the cycler orbit using the double lunar swingbys is realized without expending propellant. The double lunar swingby was first used by ISEE-3 to carry out the definitive exploration of the geomagnetic tail beyond lunar orbit. The first Japanese double lunar swingby mission “Hiten” was launched in 1990 as a precursor of GEOTAIL spacecraft in which the double lunar swingby was used to keep apogees in the distant magnetotail. In contrast, this paper presents a first attempt to design Earth-Moon cycler orbits via double lunar swingby so that the spacecraft keeps rotating to encounter the Earth and the Moon repeatedly.

The second step is design of transfer trajectory to cycler orbit. This step employs the primer vector theory to design low-cost transfer orbits. When the spacecraft reaches the Earth or the Moon, an orbital maneuver is necessary to inject the spacecraft into Earth or Moon orbit for the purpose of transferring passengers and cargo. In order to reduce the cost of the maneuver, the primer vector theory is adopted. To simplify the optimization process, a novel method to design transfer orbit based on two orbital parameters is proposed and the minimum fuel for the mission is calculated.

In this study, the following assumptions are made:

- The Earth, the Moon, and the cycler orbit lie on the same plane.
- The Moon’s orbit is circular.
- The patched conics approximation is used, that is, only the dominant gravitational force acts on the spacecraft at each segment.
- Gravity-assist swingbys occur instantaneously.
- Passengers and cargo are sent from the International Space Station (ISS) to the Moon.

2. Gravity Assist Trajectory

In this section, the 3BP with the Earth, the Moon, and the spacecraft is first considered in order to calculate the SOI. Then, a formula is developed for the aiming radius and turn angle. Finally, the method of double lunar swingby is introduced.

2.1. Swingby acceleration and swingby deceleration

Using the patched conics method, the two different swingbys, “swingby acceleration” and “swingby deceleration”, are realized. The trajectory of the spacecraft inside the SOI of the planet can be considered as a hyperbolic orbit around the planet. The relative velocity vector with respect to the planet when it reaches the SOI is set to \( \mathbf{v}_{\text{in}} \). Then \( \mathbf{v}_{\text{in}} \) can be considered as a velocity at infinity. The law of conservation of energy

\[
\frac{1}{2} v^2 - \frac{\mu}{r} = \frac{\mu}{2a}
\]

at \( r^* = \infty \) yields

\[
v_{\text{in}}^2 = \frac{\mu}{a^*}
\]

The hyperbolic orbit in the polar coordinate is given by

\[
r^* = \frac{a^* (e^*^2 - 1)}{1 + e^* \cos f}
\]

Let \( \theta_0 \) be the angle between asymptote and eccentricity vector, then

\[
\cos \theta_0 = -\frac{1}{e^*}
\]

Let \( b \) be the aiming radius:

\[
b = a^* e^* \sqrt{1 - \cos^2 \theta_0} = a^* \sqrt{e^*^2 - 1}
\]

The aiming radius is defined as the lateral offset distance between the spacecraft’s velocity vector and Moon’s velocity vector when the spacecraft enters Moon’s SOI. The semimajor axis \( a^* \) and the eccentricity \( e^* \) are obtained from Eq. (2), (4) and (5) as a function of \( \mathbf{v}_{\text{in}} \) and \( b \). Moreover, the turn angle is given by

\[
\theta = 2\theta_0 - \pi
\]

It should be noticed that the turn angle here is positive (counterclockwise). In addition, the vector of velocity change by swingby is written as

\[
\Delta \mathbf{v} = 2v_{\text{in}} \frac{\sin \frac{\theta}{2}}{2}
\]

For the swingby deceleration, the same procedure is applied as above, except \( \theta \) is regarded as negative (clockwise).

2.2. Double lunar swingby

The double lunar swingby trajectory used by the first Japanese double lunar swingby mission “Hiten” has the following three notable points. First, the orbit always follows the direct motion, that is, it is always counterclockwise around the Earth. The advantage of maintaining the counterclockwise direction is that the spacecraft will have less relative velocity when it enters the Moon’s SOI, compared with the case in which the orbit is clockwise. The second point is the phase shift. In order to keep rotating around the Earth, the spacecraft has to implement swingby accelerations and swingby decelerations alternately. As a result, the spacecraft repeats a set of smaller and larger orbits continuously. However, because the size of the orbit changes after each swingby, the trajectory of “Hiten” requires a high \( \Delta V \) to maintain the rotation of the spacecraft around the Earth. The last point is the periods at which the spacecraft encounters the Earth and the Moon. The spacecraft usually encounters both celestial objects within approximately 40 days, and sometimes within approximately 100 days, between the fifth and sixth swingbys.

In this study, the double lunar swingby is employed to design Earth-Moon cycler orbits by making use of these properties. Moreover, a cycler orbit is designed without expending propellant, while “Hiten” uses \( \Delta V \) to maintain the rotation of the spacecraft around the Earth.
3. Primer Vector Theory

To obtain the fuel optimal trajectory, the cost function is introduced:

\[ J = \sum_{i=0}^{n} u_i \]

where \( u \) is the thrust acceleration due to the propulsion system. The necessary conditions for \( u \) to be a minimizing solution follow from Pontryagin’s principle\(^{13}\) and the theory developed by Lawden is called primer vector theory.\(^{3,9}\)

The equations of the motion of the spacecraft, in terms of the state vector \( x = [r^T \quad v^T]^T \), is as follows:

\[ \dot{x} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ g(r) + u \end{bmatrix} = f(x) \]

where \( g \) is the gravitational acceleration. Along the optimal trajectory, the adjoint of the state vector \( \lambda \) satisfies a differential equation:

\[ \dot{\lambda} = -\left( \begin{bmatrix} \partial f \\ \partial x \end{bmatrix} \right)^T \lambda \]

where \( \lambda = [\lambda_r^T \lambda_v^T]^T \) is the adjoint vector. The component \( \lambda_r \) of the adjoint vector is known as the "primer vector".\(^{p,9}\)

The necessary conditions for an optimal impulsive transfer can be expressed in terms of the primer vector:\(^{3,14}\)

- The primer vector and its first derivative must be continuous everywhere.
- The primer vector is a unit vector aligned in the optimal thrust direction at an impulse.
- The thrust impulses are applied in the direction of the primer vector at the times for which \( p = 1 \).
- \( p \leq 1 \) during the transfer
- For impulses which are not at the initial or final times, as a result of these conditions, \( p = 0 \)

A perturbed trajectory will have a lower cost than the reference trajectory if \( \delta J < 0 \). As a result, if the initial and final coast arcs are included in the perturbed trajectory, the difference in cost between the two trajectories will be:

\[ \delta J = -\dot{p}_0 ||\Delta V_0|| dt_0 - \dot{p}_f ||\Delta V_f|| dt_f < 0 \]

By evaluating the values of \( \dot{p}_0 \) and \( \dot{p}_f \), the initial and final coast are determined as follows:

- \( \dot{p}_0 > 0 \) and \( \dot{p}_f > 0 \) \( \Rightarrow \) Initial Coast / Late Arrival
- \( \dot{p}_0 > 0 \) and \( \dot{p}_f < 0 \) \( \Rightarrow \) Initial Coast / Final Coast
- \( \dot{p}_0 < 0 \) and \( \dot{p}_f > 0 \) \( \Rightarrow \) Early Departure / Late Arrival
- \( \dot{p}_0 < 0 \) and \( \dot{p}_f < 0 \) \( \Rightarrow \) Early Departure / Final Coast

4. Trajectory Design of Cycler Orbit

In this section, the method of designing cycler orbits is explained.
tion, and Fig. 5 shows the corresponding velocity vectors. In Fig. 4, three different orbits can be seen: hyperbolic, parabolic, and elliptical. The first two types of orbits are not suitable, because even if the spacecraft in such an orbit could cross the path of the Moon, it would reach the lunar path before the Moon reaches the encounter point. In addition, the spacecraft will never cross the lunar orbit again. Therefore, the orbit after the swingbys must be elliptical.

Elliptical orbits are chosen after swingby accelerations for this reason. However, further consideration is required regarding the orbits after swingby decelerations. Two types of elliptical orbits can be seen in Fig. 4. In the first case, the velocity vector of the spacecraft when it escapes the Moon’s SOI is directed outside the lunar orbit. Such orbits are referred to as “external orbits.” In the second case, the velocity vector of the spacecraft when it escapes the Moon’s SOI is directed inside the lunar orbit. Such orbits are called “internal orbits.” Accordingly, there are two options for continuing cycler orbits after swingby decelerations.

Although the spacecraft will encounter the Moon earlier along an internal orbit than on an external one, the time difference between the two types of orbits is only a few days. Furthermore, the external orbits have an advantage with respect to the design of cycler orbits; the successive swingbys can be realized by repeating the two aiming radius $b_1$ and $b_2$ for deceleration and acceleration, respectively. Therefore, the aiming radius for successive swingbys would be $b_{2k} \approx b_2$ and $b_{2k+1} \approx b_1$. In general, external orbits require larger turn angles, and thus smaller aiming radii $b_2$. However, it is found that the feasible solution $b_2$ exists for the Earth-Moon system. Therefore, external orbits are used in this study, which is the main advantage of the proposed method.

Figure 6 shows the possible orbit for $r_p = 3000 \text{ km}$, $r_w = 5 \times 10^5 \text{ km}$, $b_1 = 20254 \text{ km}$, and $b_2 = 4101 \text{ km}$. This condition is used in the next section.

**4.3. Encounter condition for successive swingbys**

By considering the geometry of the encounter as discussed in Sec. 4.2, the encounter condition is derived. The period for the Moon to encounter the spacecraft is

$$t_m = \frac{4\pi - 2M_{m,1}}{n_m}, \frac{6\pi - 2M_{m,2}}{n_m}, \frac{8\pi - 2M_{m,3}}{n_m}, \cdots$$  \hspace{1cm} (16)

where $M_{m,i}$ is the difference of mean anomalies from where the spacecraft does swingby with the Moon to where it does the next swingby with the Moon after the Moon rotates around the Earth for $i$ times after the previous swingby.

Thus, the shortest period for the spacecraft to encounter the Moon is $t_m = \frac{4\pi - 2M_{m,1}}{n_m}$. Therefore, the encounter condition is chosen as $t_m = \frac{4\pi - 2M_{m,1}}{n_m}$. If there is no solution under this condition, it is replaced by the next condition, obtained by adding $2\pi$ to the denominator of $t_m$. However, the solution is always found in the first condition in this study.

Let the semimajor axis, mean motion and mean anomaly of the spacecraft after the swingby be $a'$, $n'$ and $M'$, respectively. Then the period for the spacecraft to encounter the Moon is

$$t_{sc} = \frac{2\pi - 2M'}{n'}$$  \hspace{1cm} (17)

where $M$ is the difference of mean anomalies from where the spacecraft does swingby with the Moon to where it does the next swingby with the Moon. From Eqs. (16) and (17), the encounter condition is obtained:

$$\frac{2\pi - 2M'}{n'} = \frac{4\pi - 2M_{m}}{n_m}$$  \hspace{1cm} (18)

The aiming radius $b_1$ which satisfies Eq. (18) is sought since $M_{m,i}$ is a function of $v_{m,i}$ and $b_i$.

For swingby decelerations, the period for the Moon to encounter is exactly one period:

$$t_m = \frac{2\pi}{n_m}$$  \hspace{1cm} (19)

On the other hand, the period for the spacecraft is given by

$$t_{sc} = \frac{4\pi}{n'}$$  \hspace{1cm} (20)
From Eqs. (19) and (20), the encounter condition for the swingby decelerations is obtained by

\[
\frac{4\pi}{n'} = \frac{2\pi}{n_m}
\]  

(21)

The aiming radius \( b_i \) which satisfies Eq. (18) is sought for the swingby decelerations.

The cycler orbit designed by the proposed method is shown in Fig. 7. In Fig. 7, the trajectory of the spacecraft from the initial orbit to the orbit before the ninth swingby is shown by the blue lines, and swingby accelerations and swingby decelerations are shown by the red dots. The aiming radius is \( b_2 = 4,101 \) for each even-numbered swingby and \( b_3 = 20,254 \) km for each odd-numbered swingby. Therefore, the orbits keep alternating with the same phase shift between the even-numbered and odd-numbered swingbys after the eighth one as well. The design parameters are set as \( r_p = 30,000 \) km, \( r_a = 500,000 \) km, and \( b = 25,381 \) km. In addition, this orbit always follows the direct motion, so that the relative velocity with respect to the Moon is smaller than that of the retrograde motion when it passes the Moon. The detailed data of the gravity assists are shown in Table 1.

Furthermore, the period from an even-numbered swingby to an odd-numbered one is always approximately 27 days and 12 hours, whereas that from an odd-numbered swingby to an even-numbered one is always approximately 35 days and 17 hours. Table 2 summarizes the period and status of spacecraft. By using this cycler orbit, the transportation system can be operated every two months from Table 2.

Table 2. Required times for Earth–Moon transportation system.

<table>
<thead>
<tr>
<th>Direction</th>
<th>Required time [day]</th>
<th>Status of spacecraft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth → Moon</td>
<td>3.42</td>
<td>operating</td>
</tr>
<tr>
<td>Moon → Moon</td>
<td>35.71</td>
<td>waiting</td>
</tr>
<tr>
<td>Moon → Earth</td>
<td>10.33</td>
<td>operating</td>
</tr>
<tr>
<td>Earth → Earth</td>
<td>13.75</td>
<td>waiting</td>
</tr>
</tbody>
</table>

Fig. 8. Geometry of transfer orbit.

5. Design of Transfer Trajectory to Cycler Orbit

In this section, the transfer trajectory to the cycler orbit from the orbit around the Earth is considered and the minimum cost of transfer is evaluated by applying the primer vector theory. It is assumed that the initial orbit of spacecraft is a circular orbit with the orbit altitude 400 km, which is the orbit of the ISS.

5.1. Design of nominal transfer trajectory

The nominal transfer trajectory is designed as follows:
1. Set the apogee altitude of the transfer trajectory, \( r_{a,\text{transfer}} \) [km]. (Its perigee is fixed as 400 km.)
2. Compute the true anomaly of the transfer trajectory and then subtract it from the true anomaly of the initial orbit of the cycler to obtain \( \Delta f \).
3. Rotate the transfer orbit clockwise by a degree of \( \Delta f \) with respect to the center of the Earth in order to make the transfer orbit cross the cycler orbit at the target altitude \( h_t \).

where the apogee altitude \( r_{a,\text{transfer}} \) and a target altitude \( h_t \) of the transfer trajectory are introduced as shown in Fig. 8. Thus, the transfer orbit is determined by two parameters \( r_{a,\text{transfer}} \) and \( h_t \), and perigee altitude is fixed as 400 km. Also, because \( \Delta f \) depends on \( h_t \), the maneuver locations are altered when the apogee altitude is changed and the rotational angle is the same as \( \Delta f \). Thus, the location of the first maneuver is rotated clockwise along the orbit of the ISS. In short, the higher the apogee altitude, the earlier the first maneuver, and vice versa.

Table 1. The detailed data of the gravity assists.

<table>
<thead>
<tr>
<th>Data of GA</th>
<th>Initial GA</th>
<th>Even-numbered GA</th>
<th>Odd-numbered GA</th>
</tr>
</thead>
<tbody>
<tr>
<td>The velocity of the S/C before GA [km/s]</td>
<td>[0.6436, 0.3901]</td>
<td>[-0.8578, 0.7491]</td>
<td>[0.5689, 0.3217]</td>
</tr>
<tr>
<td>The velocity of the S/C after GA [km/s]</td>
<td>[0.8578, 0.7491]</td>
<td>[0.5689, 0.3217]</td>
<td>[0.8578, 0.7491]</td>
</tr>
<tr>
<td>The velocity of the Moon [km/s]</td>
<td>[0, 1.0175]</td>
<td>[0, 1.0175]</td>
<td>[0, 1.0175]</td>
</tr>
<tr>
<td>The relative velocity before GA [km/s]</td>
<td>[0.6436, -0.6274]</td>
<td>[-0.8578, -0.2684]</td>
<td>[0.5689, -0.6958]</td>
</tr>
<tr>
<td>The relative velocity after GA [km/s]</td>
<td>[0.8578, -0.2684]</td>
<td>[0.5689, -0.6958]</td>
<td>[0.8578, -0.2684]</td>
</tr>
<tr>
<td>Turn angle [deg]</td>
<td>26.8908</td>
<td>111.8940</td>
<td>33.3552</td>
</tr>
<tr>
<td>Aiming radius [km]</td>
<td>25381</td>
<td>4101</td>
<td>20254</td>
</tr>
</tbody>
</table>
5.2. Optimization by primer vector theory

As an example, the transfer orbit is considered for \( h_t = 3 \times 10^5 \) km and \( r_{a,\text{transfer}} = 5 \times 10^5 \) km, as shown by the red line in Fig. 7. The histories of the corresponding primer vector magnitude \( p \) and the derivative of the primer vector magnitude, \( \dot{p} \), are shown in Fig. 10. The total \( \Delta V \) is

\[
\Delta V_{\text{total}} = 3.2760 \text{ km/s}
\]  

(22)

From Fig. 10, the intermediate impulse is not required because \( p \leq 1 \) during the transfer. In addition, because \( p_0 < 0 \) and \( p_f > 0 \), the transfer orbit must have an early departure and late arrival.

In the next step, the two parameters are changed to \( h_t = 3.85 \times 10^5 \) km and \( r_{a,\text{transfer}} = 5 \times 10^5 \) km according to Eq. (14). This trajectory is shown by the pink line in Fig. 9. As a result,

\[
\Delta V_{\text{total}} = 3.2744 \text{ km/s}
\]  

(23)

It can be seen from Fig. 11 that the intermediate impulse is not required and early departure and late arrival is required. By continuing this minimization process, the local optimal transfer orbit is found. The local minimum of the total \( \Delta V \) is

\[
\Delta V_{\text{total}} = 3.2076 \text{ km/s}
\]  

(24)

where \( h_t = 3.85 \times 10^5 \) km and \( r_{a,\text{transfer}} = 4.65 \times 10^5 \) km. This orbit is shown as the blue line in Fig. 9. The plots of \( p \) and \( \dot{p} \) are shown in Fig. 12.

6. Conclusion

In this paper, the trajectory design of cycler orbits to realize an Earth-Moon transportation system is considered. The method relies on three theories: patched conics method, double lunar swingby, and primer vector theory. As a result, it is possible to design cycler orbits that encounter the Earth and the Moon regularly without fuel and to minimize \( \Delta V \) to transfer to the cycler orbit from the initial orbit. Specifically, there are three important parameters for designing cycler orbits: aiming radius, apogee, and perigee of the initial orbit. Moreover, in order to apply the primer vector theory, the nominal transfer orbit is designed by two parameters: apogee altitude of the transfer orbit and target altitude. The proposed method can realize the Earth-Moon transportation system which can be operated every two months with low cost.

Practically, the gravitational influence of the Moon should also be taken into account. This will cause deviations of the orbit from the ideal cycler orbit and a correction maneuver is necessary along the cycler orbit. These issues will be discussed in the future work.

References

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