Passability and a Steering Law in Singular States of Control Moment Gyros

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Singularity is a problem in attitude control of a spacecraft using control moment gyros (CMGs). In this study, the internal singular states of two six-CMG systems; the hexagonal and twin-triangular type CMGs are investigated and one of the CMG arrangement types; the twin triangular type is proven to have only passable internal singular states. By using the passability, the gimbal angle calculation that satisfies the given angular momentum of CMGs is proposed and its effectiveness is compared with an existing method. A steering law for the gimbal angular velocities based on the gimbal angle calculation is also constructed. The attitude control of the spacecraft by using the proposed steering law is examined through numerical simulations.

Key Words: Spacecraft, Attitude Control, Control Moment Gyros, Singularity

Nomenclature

\[ A \] : coefficient matrix of in torque input
\[ c_i \] : column vector of \( A \)
\[ D_k \] : \( \det([c_1, c_2, c_3]) \)
\[ e_i \] : \( h_i \cdot u \)
\[ g_i \] : gimbal axis of \( i \)-th CMG
\[ h \] : total angular momentum of CMG system
\[ h_i \] : angular momentum of \( i \)-th CMG
\[ h_0 \] : magnitude of angular momentum of CMGs
\[ J \] : inertia tensor of a spacecraft
\[ k_d \] : derivative gain
\[ k_p \] : proportional gain
\[ L_3 \] : matrix to check passability
\[ M \] : diagonal matrix with \( h_i \cdot u \)
\[ N \] : null space of \( A \)
\[ p \] : threshold of gimbal angle variation
\[ q \] : Euler parameters
\[ \Delta t \] : time interval
\[ u \] : singular vector
\[ U, S, V \] : elements of SVD of \( A \)
\[ u_i \] : column vector of \( U \)
\[ v_i \] : column vector of \( V \)
\[ W_3 \] : matrix composed of eigenvectors of \( L_3 \)
\[ \alpha \] : constant of SDA
\[ \beta \] : skew angle of CMGs
\[ \theta_i \] : gimbal angle of \( i \)-th CMG
\[ \lambda_1 \] : eigenvalue of \( L_3 \)
\[ \sigma_i \] : singular value of \( A \)
\[ \tau \] : torque generated by CMG system
\[ \omega \] : body rate of a spacecraft

Subscripts

0 : initial
c : control
r : reference
e : error

1. Introduction

Reaction wheels (RWs) and control moment gyros (CMGs) are used as actuators for attitude control of spacecraft. RWs generate reaction torques on a spacecraft by changing the rotational speed of the wheel. On the other hand, CMGs use gyro torques generated by changing the direction of the angular momentums of the wheels, which are usually larger than those of RWs. For three-axis attitude control, a system with multiple single gimbal control moment gyros (SGCMGs) is generally used. There is a wide variety of arrangements of CMGs, such as the roof type and pyramid type. The roof-type arrangement is configured so that the gimbal axes of the CMGs are arranged in parallel; Yoshikawa studied the case with four CMGs.1) In the pyramid-type arrangement, several CMGs are arranged asymmetrically, and the mounting angles of the gimbal axes (skew angles) are set at appropriate values. The typical arrangement of the pyramid type is composed of four CMGs, and many studies have been performed on this configuration.

The attitude control with CMGs has singular states where CMGs cannot generate torques in a certain direction corresponding to specific combinations of the gimbal angles. The singular states are classified into two types: passable singular states and impassable singular states. Although a number of studies have been published for four CMGs, any arrangement of four CMGs has impassable internal singular states.2) To overcome this problem, this paper proposes an arrangement of six CMGs and its control law. There are several arrangements of six CMGs, such as the hexagonal type, twin-triangular type, and roof type. Kurokawa proved that the roof-type arrangement of six CMGs has no impassable internal singular states.3) However, the roof-type arrangement has some restrictions and does not have the freedom in changing the angular momentum envelope. In contrast, the hexagonal type and twin-triangular type have the freedom in changing the angular momentum envelope, but they have not been sufficiently researched. In this paper, the hexagonal and twin-triangular types are discussed with respect to the singular states.
2. Singular States of SGCMGs

2.1. Passability of singular states

When the external torques acting on the spacecraft are neglected, the equation of motion of a rigid spacecraft with CMGs is given as follows:

\[ J\dot{\omega} + \omega \times (J\omega + h) = \tau_c, \]  

where \( J \) is the inertia tensor of the spacecraft, \( \omega \) is the body rate of the spacecraft, \( h \) is the total angular momentum of the CMG system, and \( \tau_c \) is the control torque. Here, a CMG system composed of six SGCMGs is considered. The SGCMG consists of a wheel, wheel motor, gimbal axis, and gimbal motor, as shown in Fig. 1. In Fig. 1, \( g_1 \) is a unit vector in the direction of the gimbal axis of the i-th CMG and \( h_i \) is the angular momentum of its wheel.

The control torque \( \tau_c \) is generated by the gimbal angular velocities according to the following relation:

\[ \tau_c = -h_{\omega,}\dot{A}\theta, \quad \theta = [\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6]^T, \]  

where \( A \) is a 3 × 6 matrix, which is composed of \( \partial h_i / \partial \theta_i \) and \( \dot{\theta} \) is the gimbal angle of the i-th CMG. There exists a null space of matrix \( A \) spanned by its kernel vectors. The motion of the gimbal angular velocities in the null space is called a null motion, because it does not affect the spacecraft torque \( \tau_c \). When matrix \( A \) is rank deficient, the system is in the singular state, therefore, there is a singular direction in which the control torque cannot be generated. If the component of \( \tau_c \) in the singular direction is generated by the null motion in the second order, the singular states are denoted as passable singular states. On the contrary, if the component of \( \tau_c \) in the singular direction is not generated by the null motion in the second order, the singular states are denoted as impassable singular states. In this section, the singular states and their passabilities are examined for the hexagonal and twin-triangular type CMGs.

2.2. Hexagonal type

2.2.1. Singular states

The arrangement of the hexagonal-type CMG system with six SGCMGs is shown in Fig. 2, where \( \beta \) is the skew angle of the CMGs. The total angular momentum \( h \) is expressed in the spacecraft body frame as

\[ h = \sum_{i=1}^{6} h_i = h_w \left( \frac{-s_1c_6}{c_1} + \frac{s_2c_6}{s_1c_2} + \frac{s_3c_6}{s_1s_3c_2} + \frac{s_4c_6}{s_1s_3s_2} + \frac{s_5c_6}{s_1s_3s_2s_3} + \frac{s_6c_6}{s_1s_3s_2s_3s_4} \right) \]  

where \( c_i = \cos \theta_i \), \( s_i = \sin \theta_i \), \( c_f = \cos \beta \), and \( s_f = \sin \beta \). The time derivative of \( h \) is derived as follows:

\[ \dot{h} = \sum_{i=1}^{6} \frac{\partial h_i}{\partial \theta_i} \dot{\theta}_i = h_w [c_1\dot{\theta}_1 + c_2\dot{\theta}_2 + c_3\dot{\theta}_3 + c_4\dot{\theta}_4 + c_5\dot{\theta}_5 + c_6\dot{\theta}_6] \]  

Eq. (6) is rewritten using the Cauchy-Binet formula:

\[ \det(AA^T) = \sum_k D_k^2 = 0, \]  

where a set of \( (k_1, k_2, k_3) \) is one of the combinations of three numbers from \( (1, 2, \ldots, 6) \), and \( D_k \) means the sum of all combinations. In the solution with six CMGs, the total number of combinations becomes twenty, thus, the twenty determinants should be zero simultaneously. Only four equations among the twenty are independent, and the four gimbal angles are expressed by the remaining two gimbal angles in the singular states. These expressions are rather complex, and the numerical solutions are introduced in the next section.
2.2.2. Passability

In the singular states, there is a unit vector \( u \) which is perpendicular to all \( c_i \),

\[
c_i \cdot u = 0, \quad (i = 1, \ldots, 6).
\]

Because \( g_i \) is also perpendicular to \( c_i \), the vector \( c_i \) is expressed as

\[
c_i = \begin{cases} \varepsilon_i (g_i \times u)/\|g_i \times u\| & (g_i \neq u) \\ \text{arbitrary} & (g_i = u) \end{cases},
\]

where \( \varepsilon_i = \text{sign}(h_i \cdot u) \).

The angular momentum vector \( h_i \) of the \( i \)-th CMG is given by

\[
h_i = h_u c_i \times g_i.
\]

The passabilities of the singular states are determined by using the null motion. In the singular states, the variation of the total angular momentum of the CMGs, \( \Delta h \), is expanded in a Taylor series by variations of the gimbal angles \( \Delta \theta \) as

\[
\Delta h = \sum_{i=1}^{6} \left[ \frac{\partial h_i}{\partial \theta_i} \Delta \theta_i + \frac{1}{2} \frac{\partial^2 h_i}{\partial \theta_i^2} \Delta \theta_i^2 + O(\Delta \theta_i^3) \right],
\]

and

\[
\frac{\partial^2 h_i}{\partial \theta_i^2} = -h_i.
\]

By taking the inner products of both sides of Eq. (12) with \( u \), the following equation is obtained:

\[
\Delta h \cdot u = -\frac{1}{2} \sum_{i=1}^{6} h_i \cdot u \Delta \theta_i^2.
\]

If Eq. (14) holds in the case where \( \Delta \theta \) lies in the null space of \( A \), the singular state is judged as passable. This means that the eigenvalues of the following matrix \( L_3 \) have both positive and negative signs:

\[
L_3 = N^T M N,
\]

where

\[
M = \begin{bmatrix}
  h_1 \cdot u & h_2 \cdot u & h_3 \cdot u & h_4 \cdot u & h_5 \cdot u & h_6 \cdot u
\end{bmatrix},
\]

and \( N \) is the null space of \( A \). The passabilities of the singular states of the hexagonal type can be examined by numerical calculations. Fig. 3 shows examples of the singular states of the hexagonal type. The blue, red, and green surfaces correspond to impassable external singular states, impassable internal ones, and passable internal ones, respectively. Thus, there are impassable singular states inside the angular momentum envelope of the hexagonal type.

2.3. Twin-triangular type

2.3.1. Singular states

Fig. 4 shows the twin triangular-type arrangement where two triangular arrangements of three CMGs are overlapped with the same orientation. The total angular momentum \( h \) is expressed in the spacecraft body frame as

\[
h = \sum_{i=1}^{6} h_i = h_u [c_1 \dot{\theta}_1 + c_2 \dot{\theta}_2 + c_3 \dot{\theta}_3 + c_4 \dot{\theta}_4 + c_5 \dot{\theta}_5 + c_6 \dot{\theta}_6]
\]

\[
= h_u A \dot{\theta},
\]

The time derivative of \( h \) is derived as follows:

\[
\dot{h} = \sum_{i=1}^{6} \frac{\partial h_i}{\partial \theta_i} \dot{\theta}_i = h_u [c_1 \ddot{\theta}_1 + c_2 \ddot{\theta}_2 + c_3 \ddot{\theta}_3 + c_4 \ddot{\theta}_4 + c_5 \ddot{\theta}_5 + c_6 \ddot{\theta}_6]
\]

\[
= h_u A \ddot{\theta},
\]

where \( A \) is a 3 \times 6 matrix determined by

\[
A = [c_1 c_2 c_3 c_4 c_5 c_6]
\]

\[
= \begin{bmatrix}
  -c_\theta c_\beta - c_\theta \frac{\sqrt{3}}{2} s_\beta s_\theta - \frac{\sqrt{3}}{2} s_\theta c_\beta + c_\theta \frac{\sqrt{3}}{2} c_\beta s_\theta & c_\theta c_\beta + \frac{\sqrt{3}}{2} s_\theta s_\beta & c_\theta c_\beta - \frac{\sqrt{3}}{2} c_\theta s_\beta & c_\theta c_\beta - \frac{\sqrt{3}}{2} c_\theta s_\beta & c_\theta c_\beta - \frac{\sqrt{3}}{2} c_\theta s_\beta & c_\theta c_\beta - \frac{\sqrt{3}}{2} c_\theta s_\beta
\end{bmatrix}.
\]

Fig. 3. Singular states examples of hexagonal type (left: impassable singular states, right: passable singular states).

Fig. 4. Arrangement of twin-triangular type.
Eq. (6) also holds in the singular states, where twenty equations are obtained. These singular states are classified into the following four cases in the twin-triangular type:

1. \( \tan \theta = \tan \theta_{1,3}, \ 3 \cos^2 \beta + \tan \theta_{1} = \tan \theta_{i+1} + \tan \theta_{i+2} + \tan \theta_{i} \tan \theta_{i+3} = 0 \) (i = 1, 2, 3)
2. \( \tan \theta_{1} = -\tan \theta_{2} = \tan \theta_{4} = -\tan \theta_{5} = -\sqrt{3} \cos \beta \)
3. \( \tan \theta_{3} = -\tan \theta_{4} = -\tan \theta_{5} = \sqrt{3} \cos \beta \)
4. \( \tan \theta_{2} = -\tan \theta_{3} = \tan \theta_{5} = -\tan \theta_{6} = -\sqrt{3} \cos \beta \)

2.3.2. Passability

(1) Passability of case (1)

(i) The case where the condition \( \theta_{1,3} = \theta_{1} + \pi \) holds for an arbitrary non-zero vector \( x \).

(ii) The case of \( e_{1}, e_{2}, \) and \( e_{3} \) are different, the singular states belong to the internal singular states, and the sign of \( x^{T}L_{3}x \) varies according to vector \( x \). Consequently, the eigenvalues of \( L_{3} \) take both positive and negative signs, and the singular states are passable. The singular states of case (1) are shown in Fig. 5.

\[
L_{3} = N^{T} \begin{bmatrix}
    e_{1} & e_{2} & e_{3} \\
    -e_{1} & e_{2} & e_{3}
\end{bmatrix} N
= \begin{bmatrix}
    0 & 0 & 0 & \frac{2e_{1}}{\cos \theta_{1}} & 0 & \frac{2e_{2}}{\cos \theta_{2}} & 0 & \frac{2e_{3}}{\cos \theta_{3}} & 0 \\
    0 & \frac{2e_{1}}{\cos \theta_{1}} & \frac{2e_{2}}{\cos \theta_{2}} & \frac{2e_{3}}{\cos \theta_{3}} & 0 & \frac{2e_{1}}{\cos \theta_{1}} & \frac{2e_{2}}{\cos \theta_{2}} & \frac{2e_{3}}{\cos \theta_{3}} & 0 \\
    0 & 0 & 0 & \frac{\frac{e_{1}}{\cos \theta_{1}}}{(1+\cos \beta \theta_{1})} & \frac{\frac{e_{2}}{\cos \theta_{2}}}{(1+\cos \beta \theta_{2})} & \frac{\frac{e_{3}}{\cos \theta_{3}}}{(1+\cos \beta \theta_{3})} & \frac{\frac{e_{1}}{\cos \theta_{1}}}{(1+\cos \beta \theta_{1})} & \frac{\frac{e_{2}}{\cos \theta_{2}}}{(1+\cos \beta \theta_{2})} & \frac{\frac{e_{3}}{\cos \theta_{3}}}{(1+\cos \beta \theta_{3})}
\end{bmatrix}
\]

\]

If the eigenvalues of \( L_{3} \) have both positive and negative signs, the singular states are determined as passable. In other words, the singular states are impassable when \( L_{3} \) is positive or negative definite. If \( L_{3} \) is positive definite, the following equation holds for an arbitrary non-zero vector \( x \):

\[
x^{T}L_{3}x > 0.
\]

However, because \( x_{1}^{T}L_{3}x_{1} = 0 \) in the case of \( x_{1} = [1 \ 0 \ 0 \ 0]^{T} \), \( L_{3} \) is not positive definite. Vector \( x_{1} \) is not an eigenvector corresponding to a zero eigenvalue, and the eigenvalues of \( L_{3} \) are real numbers because \( L_{3} \) is symmetric. Therefore, the eigenvalues of \( L_{3} \) have both positive and negative signs, and the singular states are passable. The same procedure can be applied to the cases where two or more combinations of \( \theta_{1,3} = \theta_{1} + \pi \) are included.
Then, by setting the variation of the angular momentum as

\[ \Delta h = \sum_{i=1}^{6} \frac{\partial h_i}{\partial \theta_j} \Delta \theta_j + \frac{1}{2} \frac{\partial^2 h_i}{\partial \theta_j^2} \Delta \theta_j^2 \, . \]  

(29)

Let the singular value decomposition (SVD) of \( A \) be

\[ A = USV^T = \sum_{i=1}^{3} \sigma_i u_i v_i^T, \quad (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0), \]  

(30)

where \( \sigma_i, u_i, \) and \( v_i \) are the singular values, a column vector of \( U \), and a column vector of \( V \), respectively. In the singular states of \( A \), the null space \( N \) is spanned as follows:

\[ N = \left[ v_3 \ v_4 \ v_5 \ v_6 \right]. \]  

(31)

In the singular states, \( \sigma_3 = 0 \), and the singular vector \( u \) becomes \( u_3 \). By taking the inner products of \( \Delta h \) with \( u_1, u_2, \) and \( u_3 \), the following equations are obtained:

\[ \Delta h \cdot u_1 = h_w \sigma_1 \Delta \theta \quad - \quad \frac{1}{2} \sum_{i=1}^{6} h_i \cdot u_1 \Delta \theta_i^2, \]  

(32)

\[ \Delta h \cdot u_2 = h_w \sigma_2 \Delta \theta \quad - \quad \frac{1}{2} \sum_{i=1}^{6} h_i \cdot u_2 \Delta \theta_i^2, \]  

(33)

\[ \Delta h \cdot u_3 = - \frac{1}{2} \sum_{i=1}^{6} h_i \cdot u_3 \Delta \theta_i^2. \]  

(34)

Because it is difficult to obtain the analytical solution satisfying Eqs. (32), (33), and (34) simultaneously, the following procedure for the numerical solution is proposed here:

- \( \Delta \theta \) satisfying Eq. (34) is obtained by a linear combination of the column vectors of \( N \). Let this solution be \( \Delta \theta_{3 \max} \). Because \( L_3 \) is a symmetric matrix, it can be decomposed as follows:

\[ L_3 = W_3 \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \Lambda_3 \end{bmatrix} W_3^T, \]  

(35)

where \( W_3 \) is an orthogonal matrix. Eigenvalues \( \lambda_i \) have both positive and negative signs, because the internal singular states are passable. Thus, the solution of Eq. (34) with the minimum norm is expressed as

\[ \Delta \theta_{3 \max} = \sqrt{-\frac{2 \Delta h \cdot u_3}{h_w \lambda_{3 \max}}} \max, \]  

(36)

where \( \lambda_{3 \max} \) is the maximum eigenvalue (in absolute values) in the eigenvalues of \( L_3 \) whose sign is opposite to the sign of \( \Delta h \cdot u_3 \), and \( \max \) is the corresponding eigenvector.

- In order to restrict the norm of \( \Delta \theta_{3 \max} \), \( \Delta \theta_{3 \max} \) is obtained from \( \Delta \theta_{3 \max} \) in Eq. (36) as follows:

\[ \Delta \theta_{3 \max} = \min \left( \frac{\Delta \theta_{3 \max}}{p} \right) \frac{\Delta \theta_{3 \max}}{p}, \]  

(37)

where \( p \) is a constant.

- By considering the effects of \( \Delta \theta_{3 \max} \) on Eqs. (32) and (33), \( \Delta \theta_{3 \max} \) is obtained as follows:

\[ \Delta \theta_{3 \max} = \frac{1}{h_w \sigma_1} \left( \Delta h \cdot u_1 + \frac{1}{2} \sum_{i=1}^{6} h_i \cdot u_1 \Delta \theta_i^2 \right) e_1, \]  

(38)

\[ + \frac{1}{h_w \sigma_2} \left( \Delta h \cdot u_2 + \frac{1}{2} \sum_{i=1}^{6} h_i \cdot u_2 \Delta \theta_i^2 \right) e_2, \]

where \( \Delta \theta_i \) denotes the components of \( \Delta \theta_{3 \max} \).
Let $\Delta \theta_{3} + \Delta \theta_{4}/2$ be the initial value of $\Delta \theta$, the solution that satisfies Eq. (29) is obtained by the Newton method.

### 3.2. Comparison with singular direction avoidance

The accuracies of the solutions for $\Delta \theta$ in Eq. (29) by the proposed method are evaluated in this subsection by the difference between the left and right sides of Eq. (29). Let this difference be $\Delta h_{k}$, then the averaged value of the norm of $\Delta h_{k}$ is calculated over the various internal singular states. Here, the solutions by the singular direction avoidance (SDA)^3 method are compared with those of the proposed method. The solutions of the gimbal angular velocities by the SDA are obtained by the SVD of matrix $A$ as follows:

$$
\dot{\theta} = [v_{1} v_{2} v_{3}] \begin{bmatrix}
\frac{1}{\sigma_{1}} & 0 & 0 \\
0 & \frac{1}{\sigma_{2}} & 0 \\
0 & 0 & \sigma_{1} + \sigma_{2}
\end{bmatrix} [u_{1} u_{2} u_{3}]^{T} \tau_{c},
$$

where $\alpha$ is a constant. In this calculation, $\Delta h$ is set in the direction of the angular momentum $h$. The calculation conditions are set as follows:

$$
\beta = 54.7 \text{ [deg]}, \quad h_{w} = 0.05 \text{ [Nms]}, \quad |\Delta h| = 5e-04 \text{ [Nms]}, \\
\alpha = 0.01, \quad p = 0.05.
$$

Table 2 shows the averaged values of $|\Delta h_{k}|$ in internal singular states.

<table>
<thead>
<tr>
<th>Proposed</th>
<th>SDA</th>
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<td>\Delta h_{k}</td>
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### 4. Control Law of a Spacecraft

The spacecraft attitude, the orientation of the body frame with respect to the inertial frame, is described by the Euler parameters $q = [q_{0} q_{1} q_{2} q_{3}]^{T}$, where $q_{w}$ is the scalar part and $[q_{1} q_{2} q_{3}]$ is the vector part.

The spacecraft is controlled to follow a reference body rate $\omega_{r}$ and reference Euler parameters $q_{r}$. Let the error of the body rate and Euler parameters with respect to the references be $\omega_{e}$ and $q_{e}$, respectively. Then, the control torque $\tau_{c}$ is given as follows:

$$
\omega_{e} = \omega - \omega_{r}, \quad (40)
$$

$$
q_{e} = q - q_{r}, \quad (41)
$$

$$
\tau_{c} = -k_{d}\omega_{e} - k_{p}^{*}q_{e}, \quad (42)
$$

where $k_{p}$ and $k_{d}$ are the proportional and derivative gains, respectively, and $q^{*}$ is the conjugate Euler parameters defined as

$$
q^{*} = [q_{0} -q_{1} -q_{2} -q_{3}]^{T}. \quad (43)
$$

The following steering law is proposed to obtain the gimbal angular velocities from the control torque $\tau_{c}$ based on the method for $\Delta \theta$ in the preceding section. When the control torque is exerted on a spacecraft, the relation between the present angular momentum $h_{(n)}$ and that after an interval $\Delta t$, $h_{(n+1)}$, becomes

$$
\begin{align*}
\Delta h &= h_{(n+1)} - h_{(n)} = \tau_{c}\Delta t. \quad (44)
\end{align*}
$$

Therefore, the following relation is obtained:

$$
\Delta h = h_{(n+1)} - h_{(n)} = -\tau_{c}\Delta t. \quad (45)
$$

When $\Delta \theta$ in Eq. (45) is obtained by using the preceding method, the gimbal angular velocities are determined as follows:

$$
\dot{\theta} = \begin{cases}
\frac{\Delta \theta}{\Delta t} & (\Delta \theta / |\Delta \theta|)\hat{\theta}_{\text{max}} \quad (|\Delta \theta| \leq \hat{\theta}_{\text{max}}\Delta t) \\
(\Delta \theta / |\Delta \theta|)\hat{\theta}_{\text{max}} & (|\Delta \theta| \geq \hat{\theta}_{\text{max}}\Delta t)
\end{cases} \quad , (46)
$$

where $\hat{\theta}_{\text{max}}$ is a threshold of the magnitude of the gimbal angular velocities.

### 5. Numerical Simulation

In this section, a numerical simulation on the attitude control of a spacecraft is described, and the usefulness of the proposed steering law is examined. Let the reference body rate and reference Euler parameters be set as shown in Fig. 7.

The simulation conditions are set as follows:

$$
J = \begin{bmatrix}
J_{x} & 0 & 0 \\
0 & J_{y} & 0 \\
0 & 0 & J_{z}
\end{bmatrix}, \quad J_{x} = J_{y} = J_{z} = 1 \text{ [kgm$^{2}$]},
$$

$$
\Delta t = 0.01 \text{ [s]}, \quad k_{p} = 6 \text{ [Nm]}, \quad k_{d} = 15 \text{ [Nms/rad]},
$$

$$
\hat{\theta}_{\text{max}} = 4 \text{ [rad/s]}, \quad \theta_{0} = \textbf{0}, \quad \textbf{q}_{0} = [1 0 0 0]^{T}
$$

where $\theta_{0}$ and $\textbf{q}_{0}$ denote the initial body rate and Euler parameters, respectively.

The proposed control law is adopted in the singular states where the condition number of $A$ is larger than 10. Except for the singular states, $\dot{\theta}$ is calculated by using the pseudo-inverse matrix.

The results of the simulation by the proposed method and SDA are shown in Figs. 8 and 9. Figs. (a), (b), (c), and (d) of both Figs. 8 and 9 show the body rate of the spacecraft with its error, Euler parameters with their errors, gimbal angles, and the condition number of $A$, respectively.

As shown in Fig. 9, by the SDA method, the body rate and Euler parameters of the spacecraft have some errors in the singular states. In contrast, by the proposed method, if the condition number of $A$ becomes larger than 10, it becomes smaller than 10 immediately, and the body rate and Euler parameters of the spacecraft almost follow the reference values (Fig. 8). Therefore, the proposed control law works more smoothly than the SDA method.
6. Conclusion

In this study, by focusing on a six-SGCMG system, the passabilities of the singular states of the hexagonal and twin triangular-type arrangements were investigated. As a result, it was shown that the internal singular states of the twin-triangular type are all passable, while the hexagonal type has some impassable internal singular states. For the twin-triangular type, a calculation method to determine the variation of the gimbal angles in the internal singular states was proposed, and the calculation accuracies were evaluated. An attitude control law with the proposed calculation method was also derived, and its usefulness was verified by a numerical simulation. Future works will focus on validating the attitude control law by ground experiments.

References