Spacecraft Attitude Control Using a Control Moment Gyro with Multiple Degrees of Freedom

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In this study, the performance of rate damping control from a control moment gyro is investigated. Two types of wheel angular momentum (variable and constant) and two quantities of gimbal axes (one or two) are considered. When the angular momentum of the spacecraft body is fully absorbed by the wheel, the rate damping of the spacecraft can be achieved. In contrast, for cases of with constant wheel speed, the rate damping of the spacecraft body cannot be achieved. The steady state analysis of the rate damping control in these cases shows the stationary value of the angular velocity of the spacecraft. Numerical simulations verify the analytical results.

Key Words: Spacecraft, Rate Damping, Control Moment Gyro, Stability Analysis

Nomenclature

\[ J \] \quad \text{: inertia of spacecraft}
\[ \omega \] \quad \text{: angular velocity of spacecraft}
\[ h_t \] \quad \text{: total angular momentum of spacecraft}
\[ h_w \] \quad \text{: angular momentum of wheel}
\[ h_{\theta x} \] \quad \text{: unit vector along angular momentum of wheel}
\[ h_{\theta y} \] \quad \text{: magnitude of angular momentum of wheel}
\[ \theta_x \] \quad \text{: angle of outer gimbal}
\[ \theta_y \] \quad \text{: angle of inner gimbal}

1. Introduction

Control moment gyros (CMGs) are attitude control devices that apply a torque to a spacecraft by tilting or changing the magnitude of the wheel's angular momentum. The tilting is achieved via one or two gimbals. In this study, the rate damping of a spacecraft using one CMG is considered. CMGs are classified by their variability of the wheel speed and the number of the gimbal angles. These options create four types of CMGs: a single-gimbal CMG with constant wheel speed (SGCMG), a double-gimbal CMG with constant wheel speed (DGCMG), a variable-speed single-gimbal CMG (VSSGCMG), and a variable-speed double-gimbal CMG (VSDGCMG).1–4) The rate damping performance of each type of CMG is considered in this study, based on a simple control law derived from the Lyapunov function. Rate damping is a basic function of attitude control actuators, thus it is important to know how well various types of CMGs perform this function.5–7)

To consider the rate damping performance with each type of CMG, the first consideration is whether the rate damping can be achieved. In the case of the variable-speed CMGs (VSDGCMG or VSSGCMG), the rate damping of the spacecraft can be achieved because the angular momentum of the spacecraft body can be fully absorbed by the wheel. In contrast, with constant-speed CMGs (DGCMG or SGCMG), the rate damping cannot be achieved. The second issue is the steady state performance of the rate damping using a DGCMG or an SGCMG. Using the steady state analysis of the rate damping control, the above candidates were focused. The stability conditions of the steady states were obtained by conducting a stability analysis from the linearized equations around the steady states. Finally, numerical simulations were conducted in order to examine the steady states.

2. Equations of Motion

Consider a VSDGCMG system that consists of two gimbal axes and a variable speed wheel, as shown in Fig. 1. The angular momentums of the gimbal rotation are much smaller than that of the wheel, and are therefore neglected. The total angular momentum of the spacecraft, \[ h_t = J \omega + h_w. \] (1)

The wheel axis of the VSDGCMG is set along the \( z \)-axis of the body frame \( F_B \) in the case of \( \theta_x = \theta_y = 0 \). The outer gimbal first rotates around the \( x \)-axis of the body frame, \( F_B \). The inner gimbal then rotates around the \( y \)-axis. A unit vector for the angular momentum of the wheel is given by

\[
\hat{h}_w = \begin{bmatrix}
\sin \theta_y \\
-\sin \theta_x \cos \theta_y \\
\cos \theta_x \cos \theta_y 
\end{bmatrix}.
\] (2)

The equations of motion of the spacecraft are expressed as follows:

\[
J \ddot{\omega} + \omega \times h_w = Bu,
\] (3)

where \( B \) and \( u \) are given by

\[
B = \begin{bmatrix}
0 & -\cos \theta_y & -\sin \theta_y \\
\cos \theta_x \cos \theta_y & -\sin \theta_x & \sin \theta_x \cos \theta_y \\
\sin \theta_x \cos \theta_y & \cos \theta_x & -\cos \theta_x \cos \theta_y 
\end{bmatrix},
\]

\[
u = \begin{bmatrix}
h_{\theta x} \\
h_{\theta y} \\
h_w
\end{bmatrix}.
\]
3. Rate Damping

3.1. Control law

The kinetic energy of the spacecraft body, $T$, is expressed as follows:

$$ T = \frac{1}{2} \omega^T J \omega. $$

(4)

The following control law is considered here in order to decrease the kinetic energy $T$:

$$ u = -KB^T \omega, $$

(5)

where $K > 0$ is a control gain matrix. The kinetic energy $T$ does not increase from this control law, because $\dot{T}$ is expressed as

$$ \dot{T} = \omega^T J \dot{\omega} = \omega^T [-\omega \times h_t + Bu] = -\omega^T BKB^T \omega \leq 0. $$

(6)

3.2. VSCMG

In the case of a VSCMG, the spacecraft body stops its angular motion if all the angular momentum of the spacecraft body is absorbed by the wheel. In the following analysis, therefore, the steady state of the spacecraft body is considered for a VSS-GCMG and VSDGCMG. The magnitude of the wheel angular momentum is not restricted in these cases.

3.2.1. VSSGCMG

In the case of a VSSGCMG, the gimbal angle $\theta_y$ of the VSDGCMG is fixed at $\theta_y = 0$. In this case, the stationary value of $\omega$, which is $\omega_\infty$, satisfies the following equation:

$$ BKB^T \omega_\infty = 0. $$

(7)

Therefore, $\omega_\infty$ is expressed as a vector that spans the null space of $BKB^T$ (the null space of $B^T$) as follows:

$$ \omega_\infty = \omega_\infty \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \omega_\infty \hat{\theta}_y. $$

(8)

where $\hat{\theta}_y$ is a unit vector parallel with the gimbal axis. Then, $h_t$ is expressed as

$$ h_t = J\omega_\infty + \dot{\omega}_w = \omega_\infty J\hat{\theta}_y + \dot{\omega}_w \hat{\theta}_y. $$

(9)

From the above equations of motion, the following equation results when $\omega = 0$ and $u = 0$ in the steady state condition:

$$ \omega_\infty \times h_t = 0. $$

(10)

We define the $(i, j)$-component of $J$ in $F_g$ to be $J_{ij}$. The following equations result for the case of $\omega_\infty \neq 0$ from above:

$$ J_{12}\omega_\infty - h_w \sin \theta_y = 0, \quad J_{13}\omega_\infty + h_w \cos \theta_y = 0 $$

(11)

If the gimbal axis is assumed to be one of the principal axes, as in section 3.3.2., there is no gimbal angle $\theta_y$ that satisfies the above equations, because $J_{12} = J_{13} = 0$. Therefore, $\omega_\infty$ becomes 0.

3.2.2. VSDGCMG

In the case of a VSDGCMG, if the matrix $B$ is non-singular, $\omega_\infty$ becomes 0 because there is no null space in $B^T$. The matrix $B$ becomes singular when $\cos \theta_y = 0$. In this case, $\omega_\infty$ is expressed as follows:

$$ \omega_\infty = \omega_\infty \begin{bmatrix} 0 \\ \cos \theta_y \\ \sin \theta_y \end{bmatrix} = \omega_\infty \hat{\theta}_y. $$

(12)

From $\cos \theta_y = 0$, $\dot{h}_w$ is expressed as

$$ \dot{h}_w = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}. $$

(13)

Then, $h_t$ is expressed as follows:

$$ h_t = \begin{bmatrix} (J_{12} \cos \theta_y + J_{13} \sin \theta_y) \omega_\infty \pm \dot{h}_w \\ (J_{22} \cos \theta_y + J_{23} \sin \theta_y) \omega_\infty \\ (J_{32} \cos \theta_y + J_{33} \sin \theta_y) \omega_\infty \end{bmatrix}. $$

(14)

Because $h_t$ becomes parallel with $\omega_\infty$ in the case of $\omega_\infty \neq 0$, the following equations hold:

$$ (J_{12} \cos \theta_y + J_{13} \sin \theta_y) \omega_\infty \pm \dot{h}_w = 0, $$

(15)

$$ (J_{22} \cos \theta_y + J_{23} \sin \theta_y) \cos \theta_y $$

and

$$ (J_{32} \cos \theta_y + J_{33} \sin \theta_y) \sin \theta_y = 0. $$

(16)

From Eq. (16) above,

$$ \tan 2\theta_y = \frac{2J_{32}}{J_{22} - J_{33}}. $$

(17)

Although the steady state of $\omega_\infty \neq 0$ can exist when the gimbal angles satisfy the above relations, this steady state is not stable because the kinetic energy $T$ is reduced when $\theta_x$ or $\theta_y$ deflects from the steady states. In conclusion, the angular velocity $\omega$ becomes 0 in the steady state.

3.3. SGCMG

The cases for a restricted wheel angular momentum are next considered. When the wheel angular momentum is saturated in VSCMGs, these situations are the same as those with an SGCMG and a DGCMG. Therefore, the cases of an SGCMG and a DGCMG are considered here.

3.3.1. Steady state

Rate damping cannot be achieved by an SGCMG. However, the kinetic energy of the spacecraft body can be decreased by the control law. In this case, $B$ is expressed as follows:

$$ B = \begin{bmatrix} 0 \\ \cos \theta_y \\ \sin \theta_y \end{bmatrix}, \quad u = \dot{h}_w \hat{\theta}_y. $$

(18)
The stationary value of $\omega$, which is $\omega_{\infty}$, is expressed as a vector that spans the null space of $BB^T$ (the null space of $B^T$) as follows:

$$\omega_{\infty} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ -\sin \theta \xi \\ \cos \theta \xi \end{bmatrix} = b_1 \hat{g}_x + b_2 \hat{h}_w,$$

(19)

where $b_1$ and $b_2$ are arbitrary constants. On the other hand, when $\omega = 0$ and $u = 0$ in the steady state condition, the following relation holds:

$$\omega_{\infty} \times h = 0.$$

(20)

Therefore, if $\omega_{\infty} \neq 0$ is in the steady state, $\omega_{\infty}$ becomes parallel with $h$. The following equation then holds in the steady state for a certain constant, $\alpha$:

$$\alpha \left( b_1 \hat{g}_x + b_2 \hat{h}_w \right) = b_1 J \hat{g}_x + b_2 J \hat{h}_w + h_w \hat{h}_w.$$

(21)

By taking an inner product of both sides of the above equation with $\hat{g}_x$, $\hat{h}_w$, and $\tau = \hat{g}_x \times h$, respectively, the following equations are obtained:

$$ab_1 = b_1 \hat{g}_x^T J \hat{g}_x + b_2 \hat{g}_x^T J \hat{h}_w,$$

(22)

$$ab_2 = b_1 \hat{h}_w^T J \hat{g}_x + b_2 \hat{h}_w^T J \hat{h}_w + h_w,$$

(23)

$$0 = b_1 \hat{g}_x^T J \hat{g}_x + b_2 \hat{g}_x^T J \hat{h}_w.$$

(24)

Furthermore, the magnitude of the angular momentum of the whole spacecraft, $h$, satisfies the following equation:

$$a^2 \left( b_1^2 + b_2^2 \right) = h^2.$$  

(25)

3.3.2. Case where the gimbal axis coincides with the principal axis

From these equations, the solutions for unknowns $\alpha$, $b_1$, $b_2$, and $\theta \xi$ are obtained. However, it is difficult to obtain the solutions analytically from these simultaneous equations because $\theta \xi$ is included in $\hat{h}_w$. Therefore, the following assumption is taken in the analysis:

$\hat{g}_x$ is one of the principal axes of $J$

This assumption means $J_{12} = J_{13} = 0$. Then, the following equations hold:

$$\hat{g}_x^T J \hat{w} = 0, \quad \hat{g}_x^T J \hat{g}_x = 0.$$

(26)

Substitution of the above equations into Eqs. (22)–(24) yields

$$ab_1 = b_1 J_{11},$$

(27)

$$ab_2 = b_2 \hat{h}_w^T J \hat{h}_w + h_w,$$

(28)

$$0 = b_1 \hat{g}_x^T J \hat{h}_w.$$

(29)

From Eq. (27), $b_1 = 0$ or $\alpha = J_{11}$ holds. From Eq. (28),

$$ab_2 \hat{h}_w = b_2 J \hat{h}_w + h_w \hat{h}_w, \quad b_2 \neq 0.$$

(30)

Then,

$$J \hat{h}_w = \left( \frac{h_w}{b_2} \right) \hat{h}_w.$$

(31)

Therefore, $\hat{h}_w$ becomes an eigenvector of $J$. Because $b_2 \neq 0$, the following relation holds from Eq. (29):

$$\hat{g}_x^T J \hat{h}_w = 0.$$  

(32)

This equation is expressed explicitly as

$$(J_{33} - J_{22}) \sin \theta \xi \cos \theta \xi + J_{33} (2 \cos^2 \theta \xi - 1) = 0.$$  

(33)

The gimbal angle $\theta \xi$ satisfies the same following equation with Eq. (17):

$$\tan 2 \theta \xi = \frac{2 J_{33}}{J_{22} - J_{33}}.$$  

(34)

3.3.3. Stability in the steady state

The stability of the steady state is examined by substituting the following input, $u$, into the equations of motion:

$$u = h_w \hat{\theta} \xi = k h_w \hat{\omega}^T \omega.$$  

(35)

where $k$ is a positive scalar gain of the rate damping. This substitution yields

$$\hat{\omega} = J^{-1} \left( -k h_w \hat{\omega}^T \omega + \hat{h} \right) \omega.$$  

(36)

By combining the above equation and $\theta \xi$, the following equations are obtained:

$$\dot{x} = f(x), \quad x = \begin{bmatrix} \omega \\ \theta \xi \end{bmatrix},$$

(37)

$$f(x) = \begin{bmatrix} J^{-1} \left( -k h_w \hat{\omega}^T \omega + \hat{h} \right) \omega \\ k \hat{\omega}^T \omega \end{bmatrix}.$$  

(38)

In order to examine the stability of the above equation in the vicinity of the steady state condition, $f(x)$ is linearized around the steady state and the characteristic equation of the linearized system is derived. By setting the steady state as $x = x_{\infty}$, the deviation from the steady state, $\Delta x$, follows

$$\Delta x = A \Delta x, \quad A = \frac{\partial f}{\partial x} \bigg|_{x=x_{\infty}}.$$  

(39)

In the above equation, the $3 \times 3$ top-left components of the matrix $A$ are expressed as

$$J^{-1} \left( -k h_w \hat{\omega}^T \omega + \hat{h} - \hat{\omega} J \right).$$  

(40)

In order to diagonalize the inertia matrix in the body frame, let us introduce the body frame $F_B$ whose $x$-axis and $z$-axis are aligned with $\hat{g}_x$ and $\hat{h}_w$, respectively. The coordinate transformation matrix $U_{BF}$ from the body frame $F_B$ to the nominal body frame $F_B$ is given by

$$U_{BF} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta \xi & 0 \\ 0 & \sin \theta \xi & \cos \theta \xi \end{bmatrix}.$$  

(41)

By expressing the inertia matrix in the body frame $F_B$, the following equation is obtained:

$$U_{BB}^T J U_{BB} = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22}' & J_{23}' \\ 0 & J_{23}' & J_{33}' \end{bmatrix},$$  

(42)

$$J_{22}' = J_{22} \cos^2 \theta \xi - 2 J_{23} \sin \theta \xi \cos \theta \xi + J_{33} \sin^2 \theta \xi,$$

(43)

$$J_{23}' = (J_{33} - J_{22}) \sin \theta \xi \cos \theta \xi + J_{33} (2 \cos^2 \theta \xi - 1),$$

(44)

$$J_{33}' = J_{22} \sin^2 \theta \xi - 2 J_{23} \sin \theta \xi \cos \theta \xi + J_{33} \cos^2 \theta \xi.$$  

(45)

By substituting the stationary value of $\tan 2 \theta \xi$ in Eq. (34) into the above equation, $J_{23}'$ becomes 0 and the matrix is diagonalized.
In the steady state, $\alpha - h_{w}/b_2$ becomes an eigenvalue of the inertia matrix where the corresponding eigenvector is $\hat{h}_w$. By expressing this eigenvalue as $J_h$, $J_h$ becomes

$$J_h = J'_h.$$  \hspace{1cm} (43)

Because the eigenvalue $J_p$ is also the eigenvalue of the 2 $\times$ 2 lower-right side of the matrix $J$, the following relations hold:

$$J_p = \frac{J_{22} + J_{33} + \sqrt{(J_{22} - J_{33})^2 + 4J_{23}^2}}{2},$$

$$J_m = \frac{J_{22} + J_{33} - \sqrt{(J_{22} - J_{33})^2 + 4J_{23}^2}}{2}.$$  \hspace{1cm} (44)

Whether $J_h$ becomes $J_p$ or $J_m$ depends on the stability of the steady state condition, as described below.

(1) Stability of the steady state in the case of $b_1 = 0$

We define the characteristic equation of $A$ with $b_1 = 0$ to be

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0.$$  \hspace{1cm} (45)

The coefficients of the characteristic equation in the case of $J_h = J_p$ are expressed as follows:

$$a_3 = \frac{k(h_J h_m + h_a(J_p - J_m))}{J_p J_m},$$

$$a_2 = -\frac{\left[J_{11}(h_t - h_w) - J_p h_J \right] \left[h_t(J_p - J_m) + h_a J_m \right]}{J_{11}J_p J_m},$$

$$a_1 = -\frac{k(h_t - h_w)^2(J_p - J_m) - J_p h_J}{J_{11}J_p J_m},$$

$$a_0 = 0,$$

$$a_2a_3 - a_1 = \frac{kh_J h_a J_p J_m}{J_{11} J_p J_m}.$$  \hspace{1cm} (50)

One of the eigenvalues of $A$ becomes 0 because $a_0 = 0$. In this case, the steady state condition is judged to be stable when all the real parts of the remaining eigenvalues are negative. The stability condition then becomes $a_3 > 0$, $a_2 > 0$, $a_1 > 0$, and $a_2a_3 - a_1 > 0$. Because $J_{11} - J_p + J_m > 0$ from the triangle inequality, the stability condition in the case of $\alpha = J_{11}$ and $J_h = J_p$ is given by

$$J_p > 1 - \frac{h_w}{h_t}.$$  \hspace{1cm} (51)

On the other hand, the coefficients of the characteristic equation in the case of $J_h = J_m$ become

$$a_3 = \frac{k(h_J h_p - h_a(J_p - J_m))}{J_p J_m},$$

$$a_2 = -\frac{\left[J_{11}(h_t - h_w) - J_m h_J \right] \left[h_t(J_p - J_m) + h_a J_p \right]}{J_{11}J_p J_m},$$

$$a_1 = -\frac{k(h_t - h_w)^2(J_p - J_m) - J_m h_J}{J_{11}J_p J_m},$$

$$a_0 = 0,$$

$$a_2a_3 - a_1 = \frac{kh_J h_a J_p J_m}{J_{11} J_p J_m}.$$  \hspace{1cm} (56)

In this case, the conditions $a_1 > 0$ and $a_2a_3 - a_1 > 0$ do not hold simultaneously. Therefore, the steady state condition is unstable in the case of $b_1 = 0$ and $J_h = J_m$.

(2) Stability of the steady state in the case of $\alpha = J_{11}$

Let the characteristic equation of $A$ with $\alpha = J_{11}$ be

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0.$$  \hspace{1cm} (57)

The coefficients of the characteristic equation in the case of $J_h = J_p$ are expressed as follows:

$$a_3 = \frac{kh_a(J_{11} - J_p + J_m)}{(J_{11} - J_p) J_m},$$

$$a_2 = \frac{(J_{11} - J_m) \left[J_{11}(h_t + h_m) - h_t J_p \right]}{J_{11} J_p J_m (J_{11} - J_p)},$$

$$a_1 = \frac{kh_a(J_p - J_m) J_p J_m}{(J_{11} - J_p) J_{11} J_m},$$

$$a_0 = 0,$$

$$a_2a_3 - a_1 = \frac{kh_a J_{11} J_p J_m}{J_{11} J_p J_m}.$$  \hspace{1cm} (62)

One of the eigenvalues of $A$ becomes 0 because $a_0 = 0$. Similarly, as in the previous section, the steady state condition is judged to be stable when all the real parts of the remaining eigenvalues are negative. The stability condition then becomes $a_3 > 0$, $a_2 > 0$, $a_1 > 0$, and $a_2a_3 - a_1 > 0$. Because $J_{11} - J_p + J_m > 0$ from the triangle inequality, the stability condition in the case of $\alpha = J_{11}$ and $J_h = J_p$ is given by

$$J_p > 1 - \frac{h_w}{h_t}.$$  \hspace{1cm} (63)

On the other hand, the coefficients of the characteristic equation in the case of $J_h = J_m$ become

$$a_3 = \frac{kh_a(J_{11} + J_p - J_m)}{(J_{11} - J_m) J_p},$$

$$a_2 = \frac{(J_{11} - J_p) \left[J_{11}(h_t + h_m) - h_t J_m \right]}{J_{11} J_p J_m (J_{11} - J_p)},$$

$$a_1 = \frac{kh_a(J_p - J_m) J_p J_m}{(J_{11} - J_p) J_{11} J_m},$$

$$a_0 = 0,$$

$$a_2a_3 - a_1 = \frac{kh_a J_{11} J_p J_m}{J_{11} J_p J_m}.$$  \hspace{1cm} (68)

In this case, the stability conditions do not hold simultaneously. Therefore, the steady state is unstable in the case of $\alpha = J_{11}$ and $J_h = J_m$.

(3) Summary of the stability of the steady state condition

Let us summarize the stability conditions in the cases of $b_1 = 0$ and $\alpha = J_{11}$ as follows: The steady states in both cases are stable when

$$\alpha - \frac{h_w}{b_2} = J_p.$$  \hspace{1cm} (69)

Therefore, $J'_h$ corresponds to the maximum eigenvalue $J_p$. Because the following equation holds for the angle $\theta_t$ that satisfies
Eq. (34)

$$\frac{dJ_p}{d\theta_x} = 0, \quad \frac{d^2J_p}{d\theta_x^2} = \frac{2(J_{22} - J_{33})}{\cos 2\theta_x},$$

(70)

the sign of \(\cos 2\theta_x\) is opposite to that of \(J_{22} - J_{33}\). For the two steady states in the cases of \(b_1 = 0\) and \(\alpha = J_{11}\), the stability condition becomes

$$\frac{J_p}{J_{11}} > 1 - \frac{h_w}{\hat{h}_t} : \text{the steady state at } b_1 = 0 \text{ is stable}$$
$$\frac{J_p}{J_{11}} < 1 - \frac{h_w}{\hat{h}_t} : \text{the steady state at } \alpha = J_{11} \text{ is stable}$$

From Eq. (19), the kinetic energy of the spacecraft body, \(T_s\), in the steady state condition becomes

$$T_s = \frac{1}{2} \left(J_{11}b_1^2 + J_pb_2^2\right).$$

(71)

The kinetic energies at the two steady state conditions are calculated as

$$T_s = \frac{(h_t - h_w)^2}{2J_p} : b_1 = 0,$$

$$T_s = \frac{1}{2} \left(\frac{h^2_t}{J_{11}} - \frac{h^2_w}{J_{11}}\right) : \alpha = J_{11}.$$  

These two kinetic energies are shown in Fig. 2, where \(T_0 = h_t^2/(2J_{11})\) and the stable states are plotted in solid lines. Because \(J_p\) is the larger moment of inertia between \(J_p\) and \(J_w\), \(J_p/J_{11} > 0.5\) holds from the triangle inequality. As shown in this figure, only the steady state condition that corresponds to the minimum kinetic energy is stable.

Based on the assumption of the gimbal axis coincides with the principal axis, the kinetic energy of the rate damping using the SGCMG \(T_{SG}\) is summarized in Table 1, where \(J_{max}, J_{med}\), and \(J_{min}\) denote the maximum, medium, and minimum eigenvalue of the inertia matrix of the spacecraft, respectively.

<table>
<thead>
<tr>
<th>(J_{11})</th>
<th>(J_p)</th>
<th>(T_{SG})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(J_{max})</td>
<td>(J_{med})</td>
<td>(\frac{J_{max}}{2J_{med}}) or (\frac{1}{2}(\frac{h^2_t}{J_{11}} - \frac{h^2_w}{J_{11}}))</td>
</tr>
<tr>
<td>(J_{med})</td>
<td>(J_{max})</td>
<td>(\frac{1}{2}(\frac{h^2_t}{J_{11}} - \frac{h^2_w}{J_{11}}))</td>
</tr>
<tr>
<td>(J_{min})</td>
<td>(J_{max})</td>
<td>(\frac{1}{2}(\frac{h^2_t}{J_{11}} - \frac{h^2_w}{J_{11}}))</td>
</tr>
</tbody>
</table>

3.4. DGCMG

3.4.1. Steady state

The rate damping cannot be achieved by one DGCMG, similar to the case of an SGCMG. In the case of a DGCMG, \(B\) is expressed as follows:

$$B = \begin{bmatrix} 0 & \cos \theta_y & -\cos \theta_y \\ \sin \theta_y & \sin \theta_y & \cos \theta_y \end{bmatrix}, \quad u = \begin{bmatrix} h_w \theta_x \\ \hat{h}_w \theta_y \end{bmatrix}.$$  

(72)

The stationary value of \(\omega\), which is \(\omega_\infty\), satisfies the following equation:

$$BKB^T \omega_\infty = 0.$$  

(73)

Then, \(\omega_\infty\) is expressed as a vector that spans the null space of \(B^T\) as follows:

$$\omega_\infty = b_3 \begin{bmatrix} \sin \theta_y \\ \cos \theta_y \cos \theta_y \end{bmatrix} = b_3 \hat{h}_w,$$

(74)

where \(b_3\) is a constant. The above equation means that \(\omega_\infty\) becomes parallel with the angular momentum vector of the wheel. Therefore, the total angular momentum of the spacecraft, \(h_t\), becomes

$$h_t = J\omega_\infty + h_w \hat{h}_w = b_3 J \hat{h}_w + h_w \hat{h}_w.$$  

(75)

The total angular momentum \(h_t\) becomes constant in the body frame while in the steady state condition. On the other hand, the total angular momentum \(h_t\) is conserved in the inertial space, resulting in the following equation:

$$\omega_\infty \times h_t = 0.$$  

(76)

Therefore, \(\omega_\infty\) is also parallel with \(h_t\) and the following equation is satisfied:

$$J\omega_\infty = b_4 \omega_\infty,$$

(77)

where \(b_4\) is a positive constant. From this equation, \(b_4\) is an eigenvalue of \(J\) where the corresponding eigenvector is \(\omega_\infty\).

The total angular momentum of the spacecraft then becomes

$$h_t = (b_3 b_4 + h_w) \hat{h}_w.$$  

(78)

The constant \(b_3\) is obtained by using the magnitude of the total angular momentum, \(h_t\), as

$$b_3 = \frac{h_t - h_w}{b_4}.$$  

(79)

The kinetic energy of the spacecraft body in the steady state, \(T_{\omega_\infty}\), is expressed as

$$T_{\omega_\infty} = \frac{1}{2} \omega^T_{\omega_\infty} J \omega_{\omega_\infty} = \frac{1}{2} b_4 \omega^2_{\omega_\infty} = \frac{1}{2} b_4^2 b_4 = \frac{(h_t - h_w)^2}{2b_4^2}.$$  

(80)
3.4.2. Stability in the steady state condition

Stability in the steady state condition of the DGCMG is judged by the linearized equations in the same manner as with the SGCMG; the dynamics including the control system are expressed in the form of Eq. (37). The linearized equation is expressed in the form of Eq. (39). The state vector \( \mathbf{x} \) is expressed as

\[
\mathbf{x} = \begin{bmatrix} \mathbf{a}^T \theta_x \theta_y \end{bmatrix}^T. \tag{81}
\]

Although Eq. (39) is expressed in the body frame \( \mathcal{F}_B \), the matrix becomes a complex form in this case. The following coordinate transformation is therefore performed. The coordinate \( \mathcal{F}_B \), with basis vectors \( \hat{e}_1, \hat{e}_2 \), and \( \hat{e}_3 \), are expressed in the body frame \( \mathcal{F}_b \) as

\[
\hat{e}_1 = \begin{bmatrix} 0 \\ \cos \theta_y \\ \sin \theta_y \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} -\cos \theta_y \\ -\sin \theta_y \cos \theta_y \\ \cos \theta_y \sin \theta_y \end{bmatrix}, \quad \hat{e}_3 = \begin{bmatrix} \sin \theta_y \\ \cos \theta_y \cos \theta_y \\ \cos \theta_y \sin \theta_y \end{bmatrix},
\]

where \( \hat{e}_3 = \hat{h}_w \) and \( \hat{e}_3 = \hat{h}_w \). Because the above coordinate system depends on the gimbal angles \( \theta_x \) and \( \theta_y \), it becomes a time-variant system. In the steady state condition, however, \( \theta_x = \theta_y = 0 \) holds, thus the time-variant coefficient matrix of Eq. (39) can be treated as constant. In this case, the vector \( \hat{h}_w \) and the matrix \( \mathbf{B} \) become simplified. Let \( \mathbf{J} \) in the steady state condition in \( \mathcal{F}_B \) be expressed as

\[
\mathbf{J} = \begin{bmatrix} J_{1}^f & J_{12}^f & 0 \\ J_{12}^f & J_{2}^f & 0 \\ 0 & 0 & J_{3}^f \end{bmatrix}.
\]

Since \( \hat{e}_3 = \hat{h}_w \) becomes an eigenvector of \( \mathbf{J} \), the (1, 3) and (2, 3) components of \( \mathbf{J} \) in \( \mathcal{F}_B \) become 0.

When the matrix \( \mathbf{A} \) in Eq. (39) is expressed in \( \mathcal{F}_B \), all components in the third column and the third row become 0. Therefore, the matrix \( \mathbf{A} \) has a zero eigenvalue whose eigenvector is \( \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \). The component of the spacecraft angular velocity corresponding to \( \hat{e}_3 \) can take an arbitrary value from this characteristic, but its value is uniquely determined by the angular momentum conservation of the whole spacecraft.

The characteristic equation of the matrix \( \mathbf{A} \) is given as follows:

\[
s^5 + c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s = 0. \tag{82}
\]

The system is judged to be stable when the real parts of the solutions in the above equation except for 0 are negative. The coefficients \( c_4 \sim c_1 \) are found below. The control gain \( \mathbf{K} \) is assumed to be a scalar \( K \) for simplicity:

\[
c_4 = \frac{k \cos^2 \theta_y J_{1}^f J_{12}^f h_w + J_{1} h_w}{J_{1} J_{12}^f} + \frac{k J_{1} J_{12}^f h_w + J_{1} h_w}{J_{1} J_{12}^f}, \tag{83}
\]

\[
c_3 = \frac{k^2 \cos^2 \theta_y J_{1} h_w + (J_{1} + J_{12}^f) J_{1} h_w + J_{1}^2 h_w}{J_{1}^2 J_{12}^f}, \tag{84}
\]

\[
c_2 = \frac{k \cos^2 \theta_y J_{1}^f J_{12}^f h_w + J_{1}^2 J_{1} h_w + J_{1}^2 J_{12}^f h_w}{J_{1} J_{12}^f J_{1}^f}, \tag{85}
\]

\[
c_1 = \frac{k^2 \cos^2 \theta_y J_{1}^f J_{12}^f h_w}{J_{1}^2 J_{1}^f J_{12}^f}, \tag{86}
\]

where

\[
J_{12}^f = h_w = h_i - h_w, \tag{87}
\]

\[
I_{1} = J_{1} J_{12}^f - J_{1}^2, \tag{88}
\]

\[
I_{2} = J_{1}^2 - (J_{1}^f + J_{12}^f) J_{1} + J_{1} J_{12}^f - J_{1}^2. \tag{89}
\]

It is necessary for the system stability that \( c_4 \sim c_1 \) are all positive. Because \( I_{1} > 0 \), the necessary condition for stability becomes

\[
I_{2} > 0. \tag{90}
\]

The eigenvalues of the matrix of \( \mathbf{J} \) expressed in \( \mathcal{F}_B \) consist of \( J_{1}^{f} \) and the solutions of the following equation:

\[
x^2 - (J_{1}^{f} + J_{1}^{f}) x + J_{1} J_{12}^f - J_{1}^2 = 0. \tag{91}
\]

Therefore, \( I_{2} > 0 \) means that the right-hand side of the above equation is positive when \( x = J_{1}^{f} \) is substituted into the equation. This also means that \( J_{1}^{f} \) is the maximum or minimum eigenvalue of \( \mathbf{J} \). Because the kinetic energy of the system becomes the maximum when \( J_{1}^{f} \) is the minimum eigenvalue, the system is considered to be stable when \( J_{1}^{f} \) is the maximum eigenvalue. The stable steady state condition then becomes the rotation around the axis of the maximum moment of inertia. The stationary values of \( \omega_{\infty}, h_i, \) and \( T_{\infty} \) satisfy the following relations, respectively:

\[
\omega_{\infty} = \frac{h_i - h_w}{J_{\max}} \hat{h}_w, \quad h_i = h_i \hat{h}_w, \quad T_{\infty} = \frac{(h_i - h_w)^2}{2 J_{\max}}. \tag{92}
\]

The kinetic energy is the minimum value which satisfies the conservation of the total angular momentum.

4. Kinetic Energy of a Spacecraft in Steady State Condition with SGCMG or DGCMG

The differences of the kinetic energies in the steady state conditions between the DGCMG and the SGCMG are considered in this section. First, the kinetic energy of rate damping using the DGCMG (\( T_{DG} \)) is uniquely determined from Eq. (92). The kinetic energy of the rate damping using the SGCMG (\( T_{SG} \)) is summarized in Table 1. By considering Eq. (92) and Table 1, the relationship between \( T_{DG} \) and \( T_{SG} \) is summarized into the following three cases:

\[
\text{case 1:} \quad \frac{T_{SG}}{T_{DG}} = 1 \quad \text{(93)}
\]

\[
\text{case 2:} \quad \frac{T_{SG}}{T_{DG}} = \frac{J_{\max}}{J_{med}} \left( 1 - \frac{h_w}{h_i} \right) \quad \text{(94)}
\]
In the first case, the kinetic energy of the SGCMG and DGCMG are the same, when the gimbal axis of the SGCMG coincides with the eigenvector of the minimum or medium eigenvalue. In the second and third cases, the kinetic energies are different between the SGCMG and DGCMG. Here, the gimbal axis of the SGCMG coincides with the eigenvector corresponding to the maximum eigenvalue of \( J \). In these cases, the kinetic energy of the DGCMG, \( T_{DG} \), is smaller than that of SGCMG, \( T_{SG} \).

5. Numerical Examples

In this section, numerical examples are shown in the case of an SGCMG and a DGCMG. The inertia matrix \( J \) in the body frame \( F_B \), the initial angular velocity of the spacecraft body, \( \omega_0 \), in the body frame \( F_B \), and the initial gimbal angles \( \theta_{0x} \) and \( \theta_{0y} \); and the magnitude of the wheel angular momentum \( h_w \) are given as

\[
J = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0.5 \\ 0 & 0.5 & 5 \end{bmatrix} \text{ [kgm}^2\text{]},
\]

\[
\omega_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix} \text{ [rad/s]}, \quad \begin{bmatrix} \theta_{0x} \\ \theta_{0y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad h_w = 0.282 \text{ [Nms]}. \quad (96)
\]

The control gain \( k \) of the rate damping is set as

\[
k = 0.2 \text{ [Nms/rad]}. \]

5.1. SGCMG

Simulation results of the rate damping with an SGCMG are shown in Fig. 3, where the spacecraft attitude, angular velocity, gimbal angle \( (\theta_i) \), and kinetic energy of the spacecraft are plotted. The attitude consists of Euler parameters with respect to the inertial frame; the scalar part \( q_s \) and the vector part \( q_{sv} \), \( q_{sy} \), and \( q_{sz} \). In this case, the values of \( 1 - h_w/h_i \) and \( J_p/J_{11} \) are given as

\[
1 - \frac{h_w}{h_i} = 0.8518, \quad \frac{J_p}{J_{11}} = 0.7439. \quad (97)
\]

Therefore, the case of \( \alpha = J_{11} \) is stable. From the steady state analysis, \( \omega_\infty, \theta_\infty \), and \( T_\infty \) are predicted as

\[
\omega_\infty = \begin{bmatrix} 0.2660 \\ 0.0602 \end{bmatrix} \text{ [rad/s]}, \quad \theta_\infty = -22.5 \text{ [deg]}, \quad T_\infty = 0.3121 \text{ [kgm}^2\text{/s}^2\text{]}, \quad (98)
\]

and \( T_\infty \) in the steady state condition become

\[
\omega_\infty = \begin{bmatrix} 0.2660 \\ 0.0602 \end{bmatrix} \text{ [rad/s]}, \quad \theta_\infty = -22.5 \text{ [deg]}, \quad T_\infty = 0.3121 \text{ [kgm}^2\text{/s}^2\text{]} . \quad (99)
\]

From these results, the stationary values in the simulation are exactly identical as the analytical values.

5.2. DGCMG

Simulation results of the rate damping with a DGCMG are shown in Fig. 4. The spacecraft attitude (Euler parameters with respect to the inertial frame), angular velocity, gimbal angles \( (\theta_i) \), and kinetic energy of the spacecraft are plotted. Since the x-axis corresponds to the maximum moment of inertia, \( b_2 = 7 \text{ [kgm}^2\text{]} \). From the steady state analysis, \( \omega_\infty, \theta_\infty \), and \( T_\infty \) are predicted as

\[
\omega_\infty = \begin{bmatrix} 0.2688 \\ 0 \end{bmatrix} \text{ [rad/s]}, \quad \theta_\infty = 90 \text{ [deg]}, \quad T_\infty = 0.2528 \text{ [kgm}^2\text{/s}^2\text{]}. \quad (100)
\]

In Fig. 4, the simulated variables at 300 [s] are considered to be in the steady state condition. The values of \( \omega_\infty, \theta_\infty \), and \( T_\infty \) become

\[
\omega_\infty = \begin{bmatrix} 0.2688 \\ 0 \end{bmatrix} \text{ [rad/s]}, \quad \theta_\infty = 90 \text{ [deg]}, \quad T_\infty = 0.2528 \text{ [kgm}^2\text{/s}^2\text{]}. \quad (101)
\]

From these results, the stationary values in the simulation are exactly identical as the analytical values.

5.3. Comparison of kinetic energy of a spacecraft in steady state

In these numerical examples, the gimbal axis of the SGCMG coincides with the eigenvector corresponding to the maximum eigenvalue of \( J \) in Eq. (96). The relationship between the angular moments \( \omega_\infty, \theta_\infty \), and the moments of inertia \( (J_{11}, J_p) \) is shown in Eq. (97). The analytical solution of \( T_{SG}/T_{DG} \) is

\[
\frac{T_{SG}}{T_{DG}} = 1 + \frac{h_w(J_{max} h_i + 2 J_{med} h_i (1 - h_w/h_i - \frac{J_{med}}{J_{max}}))}{(J_{max} - J_{med})(h_i - h_w)^2} \]

\[
= 1.2346. \quad (102)
\]

The result of the numerical simulation for \( T_{SG}/T_{DG} \) becomes

\[
\frac{T_{SG}}{T_{DG}} = 1.2346. \quad (103)
\]

The result agrees well with the analytical solution, where the kinetic energy of the SGCMG \( (T_{SG} = 0.3121 \text{ [kgm}^2\text{/s}^2\text{]} \) in the steady state is larger than that of the DGCMG \( (T_{SG} = 0.2528 \text{ [kgm}^2\text{/s}^2\text{]} \). Therefore, the relationship between \( T_{SG} \) and \( T_{DG} \) corresponding to the third case Eq. (95) in Section 4.
Fig. 3. Simulation results of an SGCMG.

Fig. 4. Simulation results of a DGCMG.
6. Conclusion

In this paper, the rate damping control of one CMG is considered. A feedback control law for the rate damping is designed based on Lyapunov analysis. From the steady-state analysis of the control law, a VSDGCMG and a VSSGCMG can achieve the rate damping of a spacecraft when the initial angular momentum of the spacecraft can be absorbed by the variation of the wheel angular momentum. On the other hand, a DGCMG and a SGCMG cannot achieve the rate damping, because they cannot change the wheel angular momentum, and therefore, they cannot fully absorb the spacecraft angular momentum. However, the rate damping is partially achieved by the gimbal motion in both cases. In the case of a DGCMG, there is a unique steady state where the spacecraft has the minimum kinetic energy and the angular momentum of the wheel becomes parallel with the eigenvector corresponding to the maximum eigenvalue of the inertia matrix of the spacecraft body. In the case of a SGCMG, there are two steady states and it is shown that only the steady state corresponding to the minimum kinetic energy is stable. These analytical results are verified by the numerical simulations.

References