Spacecraft Attitude Control Using a Double-Gimbal Control Moment Gyro

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This paper presents a novel attitude control scheme for a spacecraft. It involves the use of a double-gimbal control moment gyro (CMG), wherein the rotational speed of the momentum wheel is kept constant. Tilting the spin axis of the momentum wheel around either an inner or an outer gimbal axis can generate attitude control torques in one rotational direction. In order to compensate for the absence of control torque in the other rotational direction, a combination of rotations of the inner and outer axes is designed to generate a cyclic motion about the spin axis of the momentum wheel resulting in three-axis attitude control. Analytical and numerical algorithms are developed to compute the control inputs, i.e., the tilt angles of the inner and outer gimbal axes, which are responsible for the desired change in attitude. The effectiveness of the proposed control scheme is verified by numerical simulations.

Key Words: Spacecraft, Attitude, Control, Control Moment Gyro, Double-Gimbal Control Moment Gyro

1. Introduction

Control moment gyros (CMGs) are momentum exchange devices used in spacecraft for attitude control. They can generate large control torques in a spacecraft by tilting the rotational axis of a momentum wheel. CMGs have been successfully employed mainly for large space vehicles, such as the International Space Station. Nowadays, the possibilities of installing CMGs in small satellites are being explored because the next-generation science and observation satellites will require precise and agile rotational maneuverability for their missions\textsuperscript{1). In order to compensate for the breakdown of a component of the CMGs, it is essential to devise fault-tolerant control systems to maintain attitude control. Thus, it is desirable to develop an attitude control scheme for a CMG with one wheel.

The attitude control problem with a single-gimbal variable-speed CMG has been studied\textsuperscript{2-4}. These studies show the possibility of line-of-sight control; however, according to the law of conservation of angular momentum, a single-gimbal variable-speed CMG is not capable of achieving three-axis control.

When a double-gimbal CMG is used, the spin axis of a momentum wheel can be tilted around two axes of the inner and outer gimbals; then, the control torques are generated around those two axes that are perpendicular to the spin axis. However, three-axis control is rather complicated. The attitude control problem with a variable-speed double-gimbal CMG, in which a reaction torque is added by changing the rotational speed of the wheel, has been studied\textsuperscript{5}. Although three-axis control is achieved, the torque generated by changing the rotational speed of the momentum wheel is considerably weaker than other torques. In addition, the possibility of a decrease in gyroscopic torques occurs when the rotational speed decreases.

In this study, we propose a novel three-axis attitude control scheme for a spacecraft. Here, a double-gimbal CMG in which the rotational speed of the momentum wheel is kept constant is used. The time trajectories of the inner and outer gimbal angles are designed such that the spin axis of the momentum wheel can undergo cyclic motion in inertial space. Thereafter, the spacecraft body is tilted to the desired attitude.

The outline of this paper is as follows: system equations of motion are given in Section 2. Suitably designed time trajectories of the inner and outer gimbal angles that generate the desirable cyclic motion of the wheel axis are described in Section 3. Analytical and numerical algorithms developed to compute the trajectories of the cyclic motion of the wheel axis, which causes the spacecraft body to tilt to the desired attitude, are described in Section 4. Numerical simulations carried out to demonstrate the effectiveness of the proposed control scheme are described in Section 5. The conclusions are summarized in Section 6.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Nomenclature} & \\
\hline
$\mathcal{F}_I$ & inertial coordinates \\
$s^T$ & representation of vector or matrix in terms of $\mathcal{F}_I$ \\
$\mathcal{F}_B$ & spacecraft body fixed coordinates \\
$s^B$ & representation of vector or matrix in terms of $\mathcal{F}_B$ \\
$U_{IB}$ & coordinates trans. matrix from $\mathcal{F}_B$ to $\mathcal{F}_I$ \\
$q_{1B}$ & vector part of the Euler parameters from $\mathcal{F}_B$ to $\mathcal{F}_I$ \\
$q_{1B}$ & scalar part of the Euler parameters from $\mathcal{F}_B$ to $\mathcal{F}_I$ \\
$J$ & inertia tensor of spacecraft body around the center of mass \\
$\omega$ & angular velocity of $\mathcal{F}_B$ with respect to $\mathcal{F}_I$ \\
h_{sw} & angular momentum of wheel \\
z_{sw} & unit vector of h_{sw} \\
h_{sw} & magnitude of h_{sw} \\
h_{tot} & angular momentum of entire spacecraft \\
z_{tot} & unit vector of h_{tot} \\
$\theta_x$ & rotation angle of outer gimbal for spacecraft body \\
$\theta_y$ & rotation angle of inner gimbal for outer gimbal \\
$I_n$ & $n \times n$ identity matrix \\
0 & zero vector or matrix with appropriate dimensions \\
s & time derivative of components of vector or matrix \\
$s^T$ & transpose of vector or matrix \\
$\tilde{a}$ & vector product operation, i.e., \\
\begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
& for $a = \begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix}$ \\
\hline
\end{tabular}
\caption{Nomenclature}
\end{table}
2. System Equations of Motion

Consider a rigid spacecraft that uses a double-gimbal CMG to provide internal torques. A model of a double-gimbal CMG is shown in Fig. 1. In order to eliminate translational motion, it is assumed that the center of mass of the wheel coincides with those of the inner and outer gimbals. The origin of the spacecraft body frame \( F_B \) is located at the center of mass of the entire spacecraft. When \( \theta_x = \theta_y = 0 \), the wheel axis is set along the \( z \) axis of \( F_B \) and the outer gimbal axis is set along the \( x \) axis of \( F_B \).

The total angular momentum of the entire spacecraft can be expressed in terms of \( F_B \) as

\[
\mathbf{h}^B_{\text{tot}} = \mathbf{J}^B \omega^B + \mathbf{h}^B_w, \tag{1}
\]

where the angular momentum induced by the gimbal motion is supposed to be neglected. The angular momentum of the wheel is expressed in terms of \( F_B \) as

\[
\mathbf{h}^B_w = \mathbf{h}_w z^B_w, \tag{2}
\]

and the angular momentum of the entire spacecraft is expressed in terms of \( F_I \) as

\[
\mathbf{h}^I_{\text{tot}} = \mathbf{h}_w z^I_{\text{tot}}. \tag{3}
\]

The kinematic equations of the entire spacecraft in terms of the Euler parameters are given by

\[
\begin{align*}
q^B_1 &= \frac{1}{2} \dot{q}^B_1 \omega^B + \frac{1}{2} q^B_1 \omega^B, \\
\dot{q}^B_1 &= -\frac{1}{2} q^B_1 T \omega^B. \tag{4}
\end{align*}
\]

Here, it should be noted that, if any other kinematic description is used, the conclusion will remain.

3. Attitude Change with Nutation

3.1. Nutational motion

The wheel axes in terms of \( F_B \) and \( F_I \) are related by

\[
z^B_w = U_{B I} z^I_w. \tag{5}
\]

Using the definitions of \( \theta_x \) and \( \theta_y \), \( z^B_w \) is represented as

\[
z^B_w = \begin{bmatrix} \sin \theta_y & -\sin \theta_x \cos \theta_y & \cos \theta_x \cos \theta_y \end{bmatrix}^T. \tag{6}
\]

Hence, at each time instant, \( z^B_w \) can be arbitrarily specified by selecting appropriate values of \( \theta_x \) and \( \theta_y \). Taking the time derivative of \( z^B_w \) in Eq. (5), we have

\[
\dot{B} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{bmatrix} = U_{B I} (\dot{z}^I_w - \dot{\omega}^I z^I_w),
\]

where \( B \) denotes the partial derivative of \( z^B_w \) with respect to \( \theta_x \) and \( \theta_y \):

\[
B = \begin{bmatrix} 0 & \cos \theta_x \cos \theta_y & -\cos \theta_y \\ \cos \theta_x \cos \theta_y & -\sin \theta_x \sin \theta_y & \sin \theta_y \\ \sin \theta_x \cos \theta_y & \cos \theta_y \sin \theta_y & \cos \theta_y \end{bmatrix}.
\]

It can be shown that \( B \) is the full column rank for \( \cos \theta_y \neq 0 \). Consequently, if \( z^I_w \) is described by a smooth function of time and if it does not pass the singular points given by \( \cos \theta_y = 0 \), \( z^B_w \) can be generated by selecting appropriate initial angles and by selecting the time derivative of the gimbal angles as follows:

\[
\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{bmatrix} = B^I U_{B I} (\dot{z}^I_w - \dot{\omega}^I z^I_w). \tag{6}
\]

where \( B^I \) denotes the pseudoinverse of \( B \).

According to the law of conservation of angular momentum, \( z^I_{\text{tot}} \) is conserved. Let us consider a cyclic movement of \( z^I_{\text{tot}} \) with \( z^I_{\text{tot}}(t) = z^I_{\text{tot}}(0) \) at the initial time \( t = 0 \) and the final time \( t = T \). The angular velocity \( \omega^B \) becomes cyclic and it satisfies the condition \( \omega^B = 0 \) at the initial and final times. In contrast, the attitude of the spacecraft body \((q_{1B}, q_{1B})\) is not cyclic: the spacecraft body nutates due to the cyclic movement of \( z^I_{\text{tot}} \). In general, it is difficult to compute this attitude change precisely. In a subsequent study, we consider a small cyclic movement of \( z^I_w \) around the neighborhood of \( z^I_{\text{tot}} \) and then approximate the attitude change of the spacecraft body in order to propose an attitude control scheme.

3.2. Approximate change in the attitude of spacecraft body

Let \( F_B \) denote the coordinates of the spacecraft body, and let the pair \((q_{1B}, q_{1B})\) denote the attitude of the spacecraft body in terms of the Euler parameters during the small cyclic movement at time \( t \). Here, we introduce the subscript \( t \) in order to explicitly represent the correspondence to time \( t \). In particular, \( F_{B_0} \) denote the initial coordinates of the spacecraft body and the pair \((q_{1B_1}, q_{1B_1})\)

Fig. 1. Double gimbal CMG.
denotes the initial attitude of the spacecraft body in terms of
the Euler parameters at \( t = 0 \). \( \mathcal{F}_{B_1} \) denote the final co-
ordinates of the spacecraft body and the pair \((q_{1B_1}, q_{1B_1})\) denotes
the final attitude of the spacecraft body in terms of the Euler
parameters at \( t = T \).

The attitude change in the spacecraft body is represented
by the pair \((q_{3B_1}, q_{3B_1})\) in terms of the Euler parameters:

\[
q_{3B_1} = \mathcal{F}_{B_1} q_{1B_1} + q_{1B_1} \mathcal{F}_{B_1} - q_{1B_1} q_{1B_1}
\]

\[
q_{3B_1} = q_{1B_1} q_{1B_1} + q_{1B_1} q_{1B_1},
\]

(7)

In particular, the pair \((q_{3B_1}, q_{3B_1})\) denotes the initial
attitude change

\[
q_{3B_1} = q_{3B_1} q_{3B_1} \Big|_{t=0} = 0, \quad q_{3B_1} = q_{3B_1} q_{3B_1} \Big|_{t=T} = 1
\]

and the pair \((q_{3B_1}, q_{3B_1})\) denotes the final attitude change

\[
q_{3B_1} = q_{3B_1} q_{3B_1} \Big|_{t=T}, \quad q_{3B_1} = q_{3B_1} q_{3B_1} \Big|_{t=T}.
\]

Substituting \( \mathcal{F}_{B_1} \) in \( \mathcal{F}_{B} \) and Eqs. (2) and (3) in Eq. (1),
we have

\[
h_w z_{tot} = J_{B1} \omega_{B1} + h_w z_{tot}.
\]

Therefore,

\[
\omega_{B1} = (J_{B1})^{-1} U_{B1} h_w (z_{tot} - z_{tot})
\]

where \( U_{B1} \) denotes the coordinate transformation matrix
for the transformation from \( \mathcal{F}_{B} \) to \( \mathcal{F}_{B_1} \). Substituting
Eqs. (4) and (8) in the time derivative of Eq. (7), we obtain the
kinematic equations of attitude change:

\[
\dot{q}_{3B_1} = \dot{q}_{3B_1} G(z_{tot} - z_{tot}) + q_{3B_1} \dot{q}_{3B_1} G(z_{tot} - z_{tot})
\]

\[
\dot{q}_{3B_1} = -\dot{q}_{3B_1} G(z_{tot} - z_{tot})
\]

(9)

where

\[
G = \frac{h_w}{2} (J_{B1})^{-1} U_{B1}.
\]

By integrating Eq. (9) with the following initial condition

\[
q_{3B_1} = 0, \quad q_{3B_1} = 1,
\]

the attitude change \((q_{3B_1}, q_{3B_1})\) can be calculated.
The effects of the small cyclic movement of \( z_{tot} \) on
Eq. (9) are incorporated in \( \mathcal{F}_{B_1} \). For simplicity,
assume that the total angular momentum of the entire
spacecraft is along the z-axis for \( \mathcal{F}_{B} \) as

\[
z_{tot} = [0 \quad 0 \quad 1]^T.
\]

Then, \( z_{tot} \) in the neighborhood of \( z_{tot} \) can be represented by

\[
z_{tot} = [z_1 \quad z_2 \quad \sqrt{1 - z_1^2} - z_2^2]^T
\]

(11)

for some sufficiently small \( z_1 \) and \( z_2 \), i.e., \(|z_1|,|z_2| \ll 1\).
Considering the second-order approximation of \( z_{tot} - z_{tot} \),
we have

\[
z_{tot} - z_{tot} \simeq [ -z_1 - z_2 \quad \frac{1}{2}(z_1^2 + z_2^2)]^T
\]

(12)

The coordinate transformation matrix \( U_{B1} \) is factored as

\[
U_{B1} = U_{B1} U_{B1} U_{B1}
\]

(13)

where \( U_{B1} \) denotes the coordinate transformation matrix
for the transformation from \( \mathcal{F}_{B} \) to \( \mathcal{F}_{B_1} \) and \( U_{B1} \) denotes
the coordinate transformation matrix for the transformation
from \( \mathcal{F}_{B} \) to \( \mathcal{F}_{B_1} \). \( U_{B1} \) is independent of \( z_{tot} \). In contrast,
\( U_{B_1} \) depends on \( z_{tot} \). Considering the first-order approximation
of \( U_{B_1} \), in the neighborhood of \( U_{B_1} = \mathcal{I} \), we have
the following approximation:

\[
U_{B1} - I_{x} = -2 \dot{q}_{B1} q_{B1}.
\]

(14)

Thus, by substituting Eqs. (13)–(14) into Eq. (9), the second
order approximation of Eq. (9) can be written in the matrix
form as follows:

\[
\frac{d}{dt} \left[ \begin{array}{c}
q_{B1} \\
q_{B1}
\end{array} \right] = \left[ \begin{array}{cc}
-\dot{g}_0 + (J_{B1})^{-1} \dot{g}_0 & 0 \\
0 & \dot{g}_0
\end{array} \right] \left[ \begin{array}{c}
q_{B1} \\
q_{B1}
\end{array} \right]
\]

(15)

where

\[
\dot{g}_0 = \frac{h_w}{2} (J_{B1})^{-1} U_{B1} (z_{tot} - z_{tot})
\]

\[
\dot{g}_1 = h_w U_{B1} (z_{tot} - z_{tot}).
\]

Let \( A(t) \) denote the coefficient matrix of Eq. (15), and
let \( \Phi(t) \) denote the fundamental solution of Eq. (15) under
\( \Phi(0) = I_4 \). Consider the following perturbative expansion:

\[
A(t) \simeq A_0(t) + \delta A(t) + \delta^2 A(t)
\]

\[
\Phi(t) \simeq \Phi_0(t) + \delta \Phi(t) + \delta^2 \Phi(t).
\]

(16)

Since \( z_{tot} - z_{tot} \) does not contain zeroth order term as shown
in Eq. (12), it follows that \( A_0(t) = 0 \) and \( \Phi_0(t) = I_4 \) for
all values of \( t \). Substituting Eq. (16) into

\[
\Phi = A(t) \Phi,
\]

it follows that

\[
\delta \Phi(t) = \delta A(t)
\]

\[
\delta^2 \Phi(t) = \delta^2 A(t) + \delta A(t) \delta \Phi(t).
\]

(17)

Consequently, \( \delta A(t) \) and \( \delta^2 A(t) \) can be calculated from the small cyclic motion \( z_{tot} \), and then the attitude change can be
approximated as follows:

\[
\left[ \begin{array}{c}
q_{B1} \\
q_{B1}
\end{array} \right] \simeq \left[ I_4 + \delta \Phi(T) + \delta^2 \Phi(T) \right] \left[ \begin{array}{c}
0 \\
1
\end{array} \right]
\]

(18)

where \( \delta \Phi(T) \) and \( \delta^2 \Phi(T) \) are given by

\[
\delta \Phi(T) = \int_0^T \delta A(t) dt
\]

\[
\delta^2 \Phi(T) = \int_0^T \delta^2 A(t) + \delta A(t) \left( \int_0^t \delta A(s) ds \right) dt.
\]

Given the values of initial attitude and the desired atti-
dude, it is possible to determine the trajectory of small cyclic
motion \( z_{tot} \) using Eq. (18). This procedure is specifically
discussed in the next section.
4. Attitude Control Scheme

In this section, a control scheme utilizing nutational motion is derived for the attitude control problem of the spacecraft body. In the sequel, it is assumed that the inertia property of the entire spacecraft is known and that the attitude and the angular velocity of the spacecraft body can be measured.

4.1. Motion planning

Consider $z_1(t)$ and $z_2(t)$ in Eq. (11) as trigonometric functions of the form

$$z_i(t) = a_i + b_i \cos \frac{2\pi t}{T} + c_i \cos \frac{4\pi t}{T} + d_i \sin \frac{2\pi t}{T} + e_i \sin \frac{4\pi t}{T}$$

for $i = 1, 2$, where the coefficients are sufficiently small constants, i.e., $|a_i|, |b_i|, |c_i|, |d_i|, |e_i| < 1$. Since $z_i'$ is cyclic, $z_1(t)$ and $z_2(t)$ satisfy

$$z_i(0) = z_i(T) = 0$$

for $i = 1, 2$. In addition, $z_1(t)$ and $z_2(t)$ are specified to satisfy

$$\dot{z_i}(0) = \dot{z_i}(T) = 0$$

for $i = 1, 2$ such that the spacecraft takes off smoothly from the stationary state. Then, the coefficients satisfy

$$a_i = -b_i - c_i, \quad e_i = -\frac{1}{2}d_i$$

for $i = 1, 2$.

Let us calculate the attitude change in terms of the coefficients $b_1, b_2, c_1, c_2, d_1, d_2$. The sign of $q_{Bu;Br}$ remains the same during the small cyclic motion of $z'_i$; therefore, the scalar part $q_{Bu;Br}$ is uniquely determined from the vector part $q_{Bu;Br}$. Hence, we only evaluate $q_{Bu;Br}$. For simplicity, assume that the inertial tensor of spacecraft body is given by

$$J^B = \text{diag}(J_1, J_2, J_3),$$

where diag() denotes a diagonal matrix. Then, it follows from Eq. (18) that $q_{Bu;Br}$ can be approximated as follows:

$$q_{Bu;Br} = \begin{bmatrix} \alpha_1 + \beta_1 + \gamma_1 + \sigma_1 \alpha_2 \alpha_3 \\ \alpha_2 + \beta_2 + \gamma_2 + \sigma_2 \alpha_3 \alpha_1 \\ \alpha_3 + \beta_3 + \gamma_3 + \sigma_3 \alpha_1 \alpha_2 \end{bmatrix}$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \frac{h_w T}{2} (J^B)^{-1} U_B^T U_{I,B_0} \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ 0 \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{h_w T}{2} (J^B)^{-1} U_B^T U_{I,B_0} \begin{bmatrix} 0 \\ 0 \\ 12(b_1^2 + b_2^2 + c_1^2 + c_2^2) + 16(h_1 c_1 + b_2 c_2) + 5(d_1^2 + d_2^2) \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \frac{5h_w^2 T^2}{32 \pi J_1 J_2 J_3} \begin{bmatrix} -J_1 + J_2 + J_3 & 0 & 0 \\ 0 & J_1 - J_2 + J_3 & 0 \\ 0 & 0 & J_1 + J_2 - J_3 \end{bmatrix} \times U_B^T U_{I,B_0} \begin{bmatrix} 0 \\ 0 \\ 2(-b_1 d_2 + b_2 d_1) - c_1 d_2 + c_2 d_1 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{J_2 - J_3}{J_1}, \quad \sigma_2 = \frac{J_3 - J_1}{J_2}, \quad \sigma_3 = \frac{J_1 - J_2}{J_3}.$$

Given a desired attitude change $q_{Bu;Br}$, it is possible to calculate the coefficients $b_1, b_2, c_1, c_2, d_1, d_2$ using Eq. (21). In the following subsections, we find the approximate solution and discuss the fundamental difficulty of this problem. Then, we propose a numerical algorithm to adjust the coefficients.

4.2. Analytical algorithm

Let us define $\delta$ as follows:

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \frac{2}{h_w^2} U_{I,B_0} J^B q_{Bu;Br}.$$

(22)

Although the desired attitude change $q_{Bu;Br}$ is scaled by $\frac{1}{h_w^2} J^B$, $\delta$ denotes the desired attitude change in $F_I$.

On the right-hand side of Eq. (21), $\alpha$ is the first-order approximation and the other elements are higher-order approximations. In the case of a small attitude change, $\gamma$ can be approximated as follows:

$$\gamma = \frac{p h_w T}{2} (J^B)^{-1} U_{I,B_0} \begin{bmatrix} 0 \\ 0 \\ 2(-b_1 d_2 + b_2 d_1) - c_1 d_2 + c_2 d_1 \end{bmatrix}$$

(23)

where

$$p = \frac{5h_w^2 T}{16 \pi J_1 J_2 J_3} \{ J_1 (-J_1 + J_2 + J_3) u_{1,1}^2 \\ J_2 (J_1 - J_2 + J_3) u_{2,2}^2 + J_3 (J_1 + J_2 - J_3) u_{3,3}^2 \}$$

(24)

and $u_{ij}$ denotes the $(j, i)$ element of $U_{I,B_0}$. The last components of Eq. (21) can be neglected since they are multilization of the elements of $\alpha$ and the conditions $|\sigma_i| < 1$ are satisfied for $j = 1, 2, 3$. It turns out that $\delta$ can be approximated as follows:

$$\delta_1 = b_1 + c_1$$

(25)

$$\delta_2 = b_2 + c_2$$

(26)

$$\delta_3 = \frac{12(b_1^2 + b_2^2 + c_1^2 + c_2^2) + 16(h_1 c_1 + b_2 c_2) + 5(d_1^2 + d_2^2)}{16} + p \{ 2(-b_1 d_2 + b_2 d_1) - c_1 d_2 + c_2 d_1 \}.$$  

(27)
Let us determine the coefficients $b_1, b_2, c_1, c_2, d_1, d_2$ such that they satisfy Eqs. (25)–(27) for given $\delta_1, \delta_2, \delta_3$. Note that it is not clear whether the coefficients exist, and even if they do exist, they cannot always be uniquely determined. In this subsection, we describe two algorithms that are classified by the value of $\delta_3$.

Algorithm 1: Select

$$b_1 = c_1 = \frac{\delta_1}{2}, \quad b_2 = c_2 = \frac{\delta_2}{2}$$

(28)

such that Eqs. (25) and (26) are satisfied and that the expression $b_1^2 + b_2^2 + c_1^2 + c_2^2$ is minimized. Substituting Eq. (28) in Eq. (27), it can be seen that

$$\delta_3 = \left(\frac{5}{2} - \frac{8}{5}p^2\right)(b_1^2 + b_2^2) + \frac{8}{15}(d_1 + \frac{2}{3}pb_2)^2 + \frac{8}{15}(d_2 - \frac{2}{3}pb_1)^2.$$  

(29)

Since the second and third terms on the right-hand side of Eq. (29) are nonnegative, the condition

$$\delta_3 \geq \frac{4}{5}\left(\frac{5}{2} - \frac{8}{5}p^2\right)(\delta_1^2 + \delta_2^2)$$

(30)

is necessary to select $d_1$ and $d_2$. If Eq. (30) is satisfied, select

$$d_1 = \left[-\frac{24}{5}p + \frac{8}{5}\sqrt{\frac{5\delta_3}{b_1^2 + b_2^2} + 36p^2 - \frac{25}{4}}\right] b_2$$

(31)

$$d_2 = \left[\frac{24}{5}p - \frac{8}{5}\sqrt{\frac{5\delta_3}{b_1^2 + b_2^2} + 36p^2 - \frac{25}{4}}\right] b_1$$

(32)

such that Eq. (29) is satisfied and that the expression $d_1^2 + d_2^2$ is minimized. Note that, in order to specify negative $\delta_3$, the condition

$$p^2 > \frac{25}{72}$$

(33)

is necessary.

Algorithm 2: If Eq. (30) is not satisfied, it is not possible to find the values of $d_1$ and $d_2$ that satisfy Eq. (29). In this case, three-axis attitude control is difficult. Therefore, we separate $z$-axis attitude control from $x, y$-axis attitude control. In the case of $x, y$-axis attitude control, i.e. $\delta_3 = 0$, the conditions in Eqs. (33) and (30) are satisfied by assuming sufficiently large values of $T$, and it is possible to apply Algorithm 1. Now, we consider $z$-axis attitude control, i.e., $\delta_1 = \delta_2 = 0$. The condition $\delta_1 = \delta_2 = 0$ is equivalent to

$$b_1 = -c_1, \quad b_2 = -c_2.$$  

(34)

On substituting Eq. (34) into Eq. (27), we obtain

$$\delta_3 = \left(\frac{5}{2} - \frac{8}{5}p^2\right)(b_1^2 + b_2^2) + \frac{8}{15}(d_1 + \frac{2}{3}pb_2)^2 + \frac{8}{15}(d_2 - \frac{2}{3}pb_1)^2.$$  

(35)

Assuming that that $p$ satisfies

$$p^2 > \frac{5}{8}$$

(36)

which is also satisfied for sufficiently large values of $T$, it is then possible to determine the coefficients satisfying Eq. (35). Select

$$d_1 = \frac{8}{5}pb_2, \quad d_2 = \frac{8}{5}pb_1$$

(37)

such that the second and third terms of Eq. (35) are 0, and select

$$b_1 = b_2 = \sqrt{\frac{5\delta_3}{5 - 8p^2}}$$

(38)

such that the expression $b_1^2 + b_2^2$ is minimized.

The conditions in Eqs. (25)–(27) show the difficulty in rotating the spacecraft body in the same or opposite directions as that of the entire spacecraft, i.e., $\delta_3 \neq 0$. As shown in these equations, $\delta_1$ and $\delta_2$ are of the first order but $\delta_3$ is of the second order. Hence, the magnitude of the cyclic motion needs to be large when $\delta_3$ is not close to 0.

Further, the conditions in Eqs. (30), (33), and (36) show the additional difficulty in rotating the spacecraft body in the opposite direction to that of the entire spacecraft, i.e., $\delta_3 < 0$, especially in a short period $T$. In the case of $\delta_3 > 0$, the conditions in Eqs. (33) and (36) are satisfied by choosing a sufficiently large value of $T$, so that Algorithm 1 can be applied. In contrast, in the case of $\delta_3 < 0$, the conditions in Eqs. (33) and (36) are not suitable for small values of $T$. Algorithm 2 provides an alternative approach to separate three-axis attitude control into $z$-axis attitude control and $x, y$-axis attitude control.

4.3. Numerical algorithm

In this subsection, the numerical Newton method to adjust the coefficients is described. Let $x_0$ denote a vector consisting of a set of coefficients $b_1, b_2, c_1, c_2, d_1, d_2$ derived by using Algorithms 1 or 2. Let $x_i$ denote a vector consisting of a set of calculated coefficients in the $i$-th iteration where $i = 1, 2, \cdots$. In the $i$-th iteration, a numerical integration of Eq. (15) with the coefficients $x_{i-1}$ is carried out to calculate the vector part of the Euler parameters. Let $q_{B_0B_T}(x_i)$ denote the vector part of the numerically calculated Euler parameters in the $i$-th iteration. In order to carry out iterations of the numerical Newton method, the following calculation is repeated

$$x_{i+1} = x_i - \nabla f(x_i)^T f(x_i)$$

$$f(x_i) = q_{B_0B_T}(x_i) - q_{B_0B_T},$$

where $q_{B_0B_T}$ denotes the vector part of the desired Euler parameters and $\nabla f(x_i)^T$ is a pseudoinverse of $\nabla f(x_i)$. Note that the $n$-th component of $\nabla f(x_i)$ is numerically computed from $(f(x_i + e_n) - f(x_i))/\epsilon$, where $|\epsilon| < 1$ and $e_n$ denotes the unit vector whose components are zero except for $n$-th component.
4.4. Repetitive control

The proposed control scheme is based on the small cyclic motion of the wheel axis, and it can be used to cause a small change in the attitude of the spacecraft body. In case of a large change in the attitude of the spacecraft body, the proposed control law cannot be applied to achieve in one step. However, there exists a possible alternative approach; a large change can be achieved by repeating the proposed control scheme in several steps. An outline of the repetitive attitude control scheme is shown in Fig. 2.

Let the pair \((\hat{q}_{BF}, \dot{q}_{BF})\) denote the desired final attitude of the spacecraft body in terms of the Euler parameters. Instead of applying the proposed control scheme to achieve the desired attitude \((\hat{q}_{BF}, \dot{q}_{BF})\) in one step, we divide the total attitude change \((\hat{q}_{BF}, \dot{q}_{BF})\) into \(k_{\text{max}}\) steps:

\[
\begin{align*}
\dot{q}_{BF} &= \dot{q}_{BF_{\text{max}}} + T\,q_{BF_{\text{max}}} \\
\dot{q}_{BF} &= \dot{q}_{BF_{\text{max}}},
\end{align*}
\]

where \(T\) denotes the time interval for each step of the attitude change. The time trajectory of the wheel axis is divided into \(k_{\text{max}}\) steps, the desired attitude change for each step is selected such that the sum of these attitude changes generates the desired attitude change, i.e., the following recursive relations are satisfied:

\[
\begin{align*}
\dot{q}_{BF_{kT}} &= \dot{q}_{BF_{(k-1)T}} + T\,q_{BF_{(k-1)T}} \\
\dot{q}_{BF_{kT}} &= \dot{q}_{BF_{(k-1)T}},
\end{align*}
\]

for \(k = 1, \ldots, k_{\text{max}}\). One possible way of dividing the total attitude change is to internally divide the arc that connects the initial attitude to the final attitude on the unit sphere; however, there are other possible ways depending on the requirements of the mission. By selecting a sufficiently large number of steps \(k_{\text{max}}\), the desired attitude change in each step can be reduced such that the proposed control scheme can be applied in a single step.

In the \(k\)-th step, the following calculation is performed to achieve the attitude change \(\hat{q}_{BF_{(k-1)T}}\): If the condition in Eq. (30) is satisfied, Algorithm 1 is applied to compute the approximate values of the coefficients of \(z_1\) and \(z_2\) in Eq. (19). Otherwise, Algorithm 2 is firstly applied to compute the approximate values of the coefficients of \(z_1\) and \(z_2\) in Eq. (19); then, the numerical Newton method is used to adjust the coefficients. In practice, this numerical adjustment can be omitted to avoid computational complexity. The time trajectory of the wheel axis \(z_{wF}\) in Eq. (11) is computed by substituting the numerically computed coefficients in Eq. (19). Substituting Eq. (19) in Eq. (6), the control input to the spacecraft is obtained in terms of the time derivative of the gimbal angles \(\theta_x\) and \(\theta_y\).

5. Numerical Examples

In this section, the numerical simulations carried out in this study to demonstrate the effectiveness of the proposed control scheme are described.

The spacecraft’s inertia tensor \(J^B\) and the wheel angular momentum \(h_w\) are selected as

\[
J^B = \text{diag}(13, 15, 20) \text{[kgm}^2\text{]}], \quad h_w = 10 \text{[Nms]}.
\]

The period \(T\) is selected as \(T = 25\text{s}\) so that the conditions in Eqs. (33) and (36) are satisfied. An initial wheel axis is selected such that it satisfies \(z_{w0}^{(0)} = z_{\text{init}}^{(0)}\) at \(t = 0\); therefore, the spacecraft body is stationary. Then initial angles of the gimbals are selected such that they satisfy \(\theta_x = \theta_y = 0\), and therefore \(\hat{F}_{B0} = \hat{F}_I\):

\[
q_{IB0} = 0, \quad q_{IF0} = 1.
\]

In the sequel, two numerical examples are provided to demonstrate the proposed attitude control scheme.

Case 1: The final attitude of the spacecraft body is given as follows:

\[
\begin{align*}
\hat{q}_{IF} &= z_{IF} \sin \frac{\theta_f}{2}, \quad \dot{q}_{IF} = \cos \frac{\theta_f}{2}
\end{align*}
\]

where the rotational axis and the angle of rotation are given by

\[
z_{IF} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \quad \theta_f = 10 \text{[deg]}.
\]
Fig. 3. Simulation result of case 1.

Fig. 4. Simulation result of case 2.
Since the proposed attitude control scheme is based on the small cyclic motion of the wheel axis, the attitude change is divided into several steps, which are illustrated by five steps. At each step \((k = 1, \cdots, 5)\), the desired attitude is given by
\[
q_{IB_{kT}} = z_1 F \sin \frac{k \theta t}{10}, \quad \dot{q}_{IB_{kT}} = \cos \frac{k \theta t}{10}.
\]
In the \(k\)-th step, the desired attitude change is given by
\[
\dot{q}_{B_{(k-1)T}B_{kT}} = z_1 F \sin \frac{\theta f}{10}, \quad \dot{q}_{B_{(k-1)T}B_{kT}} = \cos \frac{\theta f}{10}
\]
and the proposed attitude control scheme is implemented during the time interval \((k-1)T \leq t \leq kT\).

In each step, the condition in Eq. (30) is satisfied. Algorithm 1 is applied to compute the approximate values of the coefficients of \(z_1\) and \(z_2\) in Eq. (19). The numerical Newton method is used to adjust the coefficients. The time trajectory of the wheel axis \(z_1^w\) in Eqs. (11) is computed by substituting the numerically computed coefficients in Eq. (19). By substituting Eq. (19) in Eq. (6), the control input to the spacecraft is obtained in terms of the time derivative of the gimbal angles \(\dot{\theta}_B\) and \(\dot{\theta}_w\).

At time \(t = 5T = 125\,[s]\), the attitude of the spacecraft is brought to the desired value as follows:
\[
q_{IB_{5T}} = q_{IF}, \quad \dot{q}_{IB_{5T}} = \dot{q}_{IF}.
\]

The simulation results are shown in Fig. 3. Figure 3 (a) shows the path traced by \(z_1^w\) on the unit sphere for \(0 \leq t \leq 25\,[s]\). Figure 3 (b) shows the inner and outer gimbal angles for \(0 \leq t \leq 300\,[s]\). Figure 3 (c) shows the vector part of the attitude error:
\[
q_{FB_{kT}} = -q_{IB_{kT}} \dot{q}_{IF} + \dot{q}_{FB_{kT}} \dot{q}_{IB_{kT}},
\]
for \(0 \leq t \leq 300\,[s]\). Using Eqs. (39)–(41), it can be shown that the attitude error satisfies
\[
q_{FB_{5T}} = \frac{1}{\sqrt{3}} \sin \frac{(k-5)\pi}{180} \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right],
\]
\[
\leq 0.01(k-5) \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \leq 0.01(k-5) \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right],
\]
for \(k = 1, \cdots, 5\), and the this expected attitude error is shown in Fig. 3 (c) at \(t = kT\) for \(k = 1, \cdots, 5\). At time \(t = 5T = 125\,[s]\), the spacecraft body is brought to the desired attitude and control action is terminated.

Case 2: The final attitude of the spacecraft body is specified as follows:
\[
q_{IF} = -z_1 F \sin \frac{\theta f}{2}, \quad \dot{q}_{IF} = \cos \frac{\theta f}{2},
\]
namely, the spacecraft body is rotated in a direction opposite to that in Case 1. In this case, Eq. (30) is not satisfied; therefore, the attitude change is separated into \(z\)-axis control and \(x\), \(y\)-axis control. Firstly, \(z\)-axis control is applied for \(0 \leq t \leq 5T\,[s]\) and is divided into several steps, which are illustrated by five steps. In each step \((k = 1, \cdots, 5)\), Algorithm 2 is applied at first and then, the numerical Newton method is applied. Thereafter, \(x\), \(y\)-axis control is applied for \(5T \leq t \leq 10T\,[s]\) and is divided into several steps, which are illustrated by five steps. In each step \((k = 6, \cdots, 10)\), Algorithm 1 is applied at first and then, the numerical Newton method is applied. The simulation results are shown in Fig. 4. Figures 4 (a) and (b) show the path traced by \(z_1^w\) on the unit sphere for \(0 \leq t \leq 25\,[s]\) and for \(125 \leq t \leq 150\,[s]\), respectively. The magnitude of cyclic motion for \(x\), \(y\)-axis control \((125 \leq t \leq 250\,[s])\) is significantly smaller than that for \(z\)-axis control \((0 \leq t \leq 125\,[s])\). This leads to the difficulty in \(z\)-axis control, as discussed in Sec. 4.2. Figure 4 (c) shows the inner and outer gimbal angles for \(0 \leq t \leq 300\,[s]\), and Figure 4 (d) shows the vector part of the attitude error \(q_{FB_{kT}}\) for \(0 \leq t \leq 300\,[s]\). During \(z\)-axis control, the \(x\)-component of \(q_{FB_{kT}}\) converges to 0 but the \(y\)-components of \(q_{FB_{kT}}\) are constant at \(t = kT\) for \(k = 1, \cdots, 5\). During \(x\)-, \(y\)-axis control, the \(z\)-component of \(q_{FB_{kT}}\) is 0 and the \(x\), \(y\)-components of \(q_{FB_{kT}}\) converge to 0 at \(t = kT\) for \(k = 6, \cdots, 10\). At time \(t = 10T = 250\,[s]\), the spacecraft body is brought to the desired attitude and control action is terminated.

6. Conclusion

We have developed a novel three-axis attitude control scheme for spacecraft that is based on the nutational motion generated by a double-gimbal CMG. We suppose that the inertia property of the entire spacecraft is known and that the attitude and the angular velocity of the spacecraft body can be measured. Repetition of a small cyclic motion of the wheel axis causes the desired attitude of spacecraft body. During each cyclic motion, the attitude change is carried out without measuring the present attitude. It is also possible to control the spacecraft attitude by measuring the present attitude at each beginning of cyclic motion so that the disturbance and modeling errors can be compensated for.

References