On the Persistence of Brain-Waves.

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It is desirable for scientific studies of brain-waves to express them quantitatively with some measures. Motokawa¹⁾ has proposed to use three statistical measures, i.e. mean amplitude $\bar{A}$, mean period $\bar{T}$ and persistence P . Among them the two measures $\bar{A}$ and $\bar{T}$ have been used for description of electroencephalograms (eeg), and their meaning is so clear that there is no need to add any explanation, but the other measure P has been newly introduced by one of the authors, and so there are many points to be elucidated, especially with regard to its theoretical meaning and applicability in practice.

1. The Law of Amplitude.

In order to understand the meaning of the persistence, we must know the law of amplitude after which individual waves appear. Motokawa and Mita²⁾ have shown that the α -waves of man vary their amplitude from moment to moment according to a very simple statistical law:

$$w = 2h^2 A e^{-h^2 A^2} dA, \quad (1)$$

where w is the probability that the amplitude lies between A and $A + dA$, h being a constant characteristic of the given eeg.

This law holds for any sort of brain-waves, e.g. for the α -waves of man, δ -waves of epileptic patients, the main rhythm of various animals etc., provided that the periods of individual waves are not too greatly different from one another.

The true mean amplitude $\tilde{A}$ by which the arithmetic mean of infinitely great number of waves is meant can be obtained from the law of amplitude in the following manner:

$$\tilde{A} = \int_0^\infty A 2h^2 A e^{-h^2 A^2} dA = 2h^2 \int_0^\infty A^2 e^{-h^2 A^2} dA = \frac{\sqrt{\pi}}{2h}. \quad (2)$$

2. The Persistence of Brain-Waves.

The law of amplitude predicts how frequently the wave of a given amplitude will appear in an eeg of sufficient length, but can tell us nothing about the order of appearance of waves.

As far as the frequency of appearance is concerned the following two series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. and 5, 6, 7, 3, 2, 8, 10, 9, 4, 1. are equivalent to each other, but they are quite different with respect to the order of appearance. The order of appearance is a very important factor determining the pattern of an eeg, and the persistence P is a measure chosen to express this property quantitatively. P is defined as the mean of the number of waves which are greater than $\bar{A}/2$ and follow one after another without cessation. If waves greater than $\bar{A}/2$ and those smaller than it are expressed by the marks $\bigcirc$ and $\times$ respectively, the persistence of the eeg in which both sorts of waves are distributed in the following order:

$$\times \times \underbrace{\bigcirc \bigcirc \bigcirc}_3 \times \times \underbrace{\bigcirc \bigcirc}_2 \times \times \times \times \underbrace{\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc}_5 \times \times \underbrace{\bigcirc}_1 \times \underbrace{\bigcirc \bigcirc \bigcirc}_3 \times \times$$

must be equal to $(3+2+5+1+3)/5=2.8$. Generally, the persistence equals the total number of waves greater than $\bar{A}/2$ divided by the number of sequences of such waves. The measurement of persistence is carried out in the following way. In the first place, the mean amplitude $\bar{A}$ is obtained by means of Motokawa's method¹⁾ which is very convenient to use and yields comparatively accurate results. After the determination of $\bar{A}$, the amplitudes of waves are compared with the distance between the two sharp points of a divider taken equal to $\bar{A}/2$ one after another, to see how many successive waves remain greater than $\bar{A}/2$. The number of waves of such a sequence is noted down, and the same procedure is repeated up to the end of the eeg. The so noted numbers are added together and divided by the number of sequences to obtain persistence.

The value of persistence is, in general, different from record to record even if they are taken from one and the same subject under the same external conditions. Fig. 1 shows the frequency-distribution of P at six normal subjects. For the construction of each histogram 50–250 records were used, each of which contained about 200 α -waves.

The range of variation seems to be much wider for P than for any other measures like $\bar{A}$ and $\bar{T}$. We know that the greatest variation of $\bar{T}$ is about 10%, and that $\bar{A}$ is less stable than $\bar{T}$, but not so unstable as P . To get the idea of the error in the measurement of P , we measured the persistence of one and the same record repeatedly twenty times and found

the variance $\frac{1}{N} \sum_{i=1}^N (P_i - \bar{P})^2$ to be as small as 0.059. This value is obviously too small to account for the marked fluctuations of P shown in Fig. 1.

Fig. 1. Frequency-distribution (histograms) of the persistence of eeg of six normal subjects. Abscissa: Persistence, Ordinate: Frequency of appearance. (f): Histogram for the standard passive state of consciousness (full line) and that in sensory excitation (broken line).

The values of the variance of P are as large as 20.7, 4.44, 4.07, 4.24 and 3.837 for (a), (b), (c), (d) and (e) in Fig. 1 respectively.

These marked fluctuations of P may lessen the reliability of persistence as a statistical measure of eeg, but it can be said that the greater variability of P might be due to its greater sensibility to delicate changes in the functional state of the brain. We cannot remain in the same state of consciousness even under constant external conditions, and such delicate changes in the brain might express themselves in the fluctuations of P . We sometimes obtained eeg of very different pattern from one and the same subject,

and in such cases the persistence of eeg was found to be very different from one another, the differences in $\bar{A}$ and $\bar{T}$ being not always so marked. This suggests that the persistence can be a better measure of the pattern of eeg.

3. The Theoretical Meaning of Persistence.

The persistence has its own value as an experimental measure suitable for the quantitative description of eeg, but it is desirable for the interpretation of data to elucidate the theoretical meaning of this measure.

We designate the state in which the amplitude is greater than $\bar{A}/2$ as n , and the state in which it is smaller than $\bar{A}/2$ as Nn . We denote with $W(n, n)$ the probability that n is followed by n , and with $W(n, Nn)$ the probability that n is followed by Nn . Then, the probability that n occurs twice in succession is given by the product of $W(n, n)$ and $W(n, Nn)$, that is $W^{2-1}(n, n) \cdot W(n, Nn)$, because two successive occurrences of n and appearance of Nn after n are independent phenomena of each other, and so the probability must be given by the product of both probabilities. For the same reason, the probability that n occurs k times successively is equal to $W^{k-1}(n, n) \cdot W(n, Nn)$. If N is the total number of groups containing only n -states, the number of the groups containing k such waves must be equal to $NW^{k-1}(n, n)W(n, Nn)$. Hence, the total number of waves contained in such groups is equal to $kNW^{k-1}(n, n)W(n, Nn)$, and the total number of waves contained in all sorts of groups can be obtained as the sum of the following series:

$$\sum_{k=0}^{\infty} kNW^{k-1}(n, n)W(n, Nn).$$

From the definition of persistence as the total number of waves greater than $\bar{A}/2$ divided by the number of groups follows at once such an expression of P as:

$$\begin{aligned} P &= \frac{1}{N} \sum_{k=0}^{\infty} kNW^{k-1}(n, n)W(n, Nn) = \sum_{k=0}^{\infty} kW^{k-1}(n, n)W(n, Nn) \\ &= W(n, Nn) [1 + 2W(n, n) + 3W^2(n, n) + 4W^3(n, n) + \dots]. \end{aligned}$$

The infinite series in the parenthesis reduces to $1/[1 - W(n, n)]^2$ by simple calculation of the sum of the series, taking account of the relation

$$W(n, n) < 1.$$

Since $W(n, Nn)$ is the probability that n is followed by Nn , it must be equal to $1 - W(n, n)$, so that P is expressed in terms of $W(n, n)$ or $W(n, Nn)$ as follows:

$$P = \frac{1}{1 - W(n, n)} = \frac{1}{W(n, Nn)}. \quad (3)$$

As seen from this relation, the persistence is the reciprocal of the probability that a wave greater than $\bar{A}/2$ is followed by a wave smaller than $\bar{A}/2$.

It is important from the theoretical as well as practical point of view to evaluate the persistence for the ideal case in which all the waves are independent of one another. In such an ideal case the probability $W(n, n)$ that n is followed by n , must be equal to the probability for the independent occurrence of n , which will be denoted by $W(n)$ in the following. $W(n)$ can be obtained from the law of amplitude (1), for it gives the probability for the independent occurrence of a wave of any amplitude. The probability for the appearance of all the waves greater than $\bar{A}/2$ can be obtained by integrating (1) from $\bar{A}/2$ to infinite as follows:

$$W(n) = 2h^2 \int_{\bar{A}/2}^{\infty} A e^{-h^2 A^2} dA = e^{-\frac{1}{4} h^2 \bar{A}^2}. \quad (4)$$

By eliminating h from (4) and (2) we obtain

$$W(n) = e^{-\frac{\pi}{16} \left(\frac{\bar{A}}{\check{A}} \right)^2}, \quad (5)$$

and consequently

$$P = \frac{1}{1 - W(n)} = \frac{1}{1 - \exp. \left[-\frac{\pi}{16} \left(\frac{\bar{A}}{\check{A}} \right)^2 \right]}. \quad (6)$$

This relation indicates that the persistence depends upon the deviation of the measured mean amplitude $\bar{A}$ from the true mean amplitude $\check{A}$. If a sufficiently great number of waves is taken for the measurement of the mean amplitude the ratio of $\bar{A}$ to $\check{A}$ can be put equal to unity, and the persistence reduces to a constant value 5.6, since

$$\lim_{\bar{A} \rightarrow \check{A}} P = \frac{1}{1 - \exp. \left(-\frac{\pi}{16} \right)} = 5.6.$$

When the number of waves is not sufficiently great as is the case in our experiments, the persistence can fluctuate owing to the fluctuation of $\bar{A}$. But the fluctuation of P of real eeg cannot entirely be due to the fluctuation of the measured mean amplitude. The latter is nothing but one of many factors influencing the persistence. If the fluctuation of the measured mean amplitude were the only factor involved, the value of P of real eeg should fluctuate round the theoretical value 5.6. As seen from Fig. 1, the most frequently appearing value of P are in general greater than the ideal value 5.6. This fact indicates that the assumption that all the waves are independent of one another does not hold good for real brain waves. The real brain waves, therefore, must be dependent of one another, at least, in such a manner that the preceding wave has some after-effect on the succeeding one. If the after-effect facilitates the appearance of a wave of the same sort the persistence must be greater than that in the ideal case.

4. The Persistence of a Wave-Complex.

In some cases the maximum of the histogram of P lay at a smaller value than 5.6, as seen from (e) in Fig. 1. Such small values of P are usually obtained in the excited state of the brain. A decrease of P can easily be realized by stimulating the subject in some way or other.

The effect of visual stimulation upon the persistence is illustrated in (f) of Fig. 1; the histogram in full line is for a subject with his eyes closed, and that in broken line is for the same subject with his eyes open. The histogram shifts as a whole to the left on visual stimulation, indicating that the persistence is reduced by sensory excitation.

Such a decrease of P far below the theoretical value 5.6 could be explained by assuming an inhibitory after-effect, but we feel some difficulty to answer the question why the after-effect is usually facilitation, but in excited states inhibition. Another explanation of small values of P in the excited states would be found in the composite nature of eeg taken under these conditions.

Eeg in the excited states are generally composed of sequences of waves which are too different from one another to regard as belonging to one and the same population, and such eeg yield usually very small values of P .

The ideal value of P , 5.6 has been obtained from the theory of a single population, and it is this very theory that has necessitated the assumption of an inhibitory after-effect in order to explain the appearance of smaller values of P than 5.6. However, it is inadequate at least for eeg in the excited states to rely upon this way of explanation, because they consist of obviously different sorts of waves, e.g. α - and β -waves which belong apparently to different populations. These two sorts of waves may co-exist in any eeg obtained under the standard condition of consciousness or in the passive state of the mind, but we usually measure only the predominating α -waves, neglecting the β -waves superposed upon them. Hence, it is justified to regard such a standard eeg as a single population of waves. On the contrary, an eeg taken in excited states consists usually of various sorts of waves arranged in such a manner that some portions of the eeg are covered by a certain kind of waves, but other portions by other sorts of waves. Such an eeg consisting of two or more populations may be designated a wave-complex.

We can prove that the persistence of a wave-complex is always smaller than that of the waves of a single population. Suppose that an eeg be composed of two kinds of waves x and y , the mean amplitudes of which are $\bar{A}_x$ and $\bar{A}_y$ respectively, and that the probability for the appearance of x and y in the eeg be m and n respectively, then the mean amplitude of the eeg $\bar{A}$ is given by a linear combination of $\bar{A}_x$ and $\bar{A}_y$ as follows:

$$\bar{A} = m\bar{A}_x + n\bar{A}_y. \quad (7)$$

When x and y appear alternatively in the eeg the sum of the probabilities for appearance of both sorts of waves must be equal to unity, namely:

$$m + n = 1. \quad (8)$$

It will be shown in the following that the persistence of this eeg is smaller than that of any eeg of a single population x or y . For simplicity, let the waves be independent of one another or free from an after-effect, then the probability that any wave of the x -population is greater than $\bar{A}/2$ is $\exp. \left[-\frac{\pi}{16} \left(\frac{\bar{A}}{\bar{A}_x} \right)^2 \right]$ after the relation (5). Since the probability for the appearance of a wave of the x -population is m , the product of these two probabilities gives that for the appearance of any x -wave which is greater than $\bar{A}/2$. Similarly the probability for the appearance of any y -wave which is greater than $\bar{A}/2$ must be $n \exp. \left[-\frac{\pi}{16} \left(\frac{\bar{A}}{\bar{A}_y} \right)^2 \right]$. Hence the total probability that any wave is greater than $\bar{A}/2$ is given by the sum of the probabilities for both populations, namely:

$$W = m \exp. \left[-\frac{\pi}{16} \left(\frac{\bar{A}}{\bar{A}_x} \right)^2 \right] + n \exp. \left[-\frac{\pi}{16} \left(\frac{\bar{A}}{\bar{A}_y} \right)^2 \right], \quad (9)$$

TABLE I.

The persistence of an eeg consisting of two kinds of waves x and y whose mean amplitudes are $\bar{A}_x$ and $\bar{A}_y$ respectively. The probability of appearance in the eeg for both sorts of waves were assumed to be equal to each other. The statistical after-effect was neglected. W is the probability for the appearance of waves greater than one half mean amplitude of the eeg.

$\bar{A}_x : \bar{A}_y$	W	P
1	0.882	5.6
2	0.804	5.1
4	0.749	3.98
6	0.698	3.33
8	0.655	3.1
10	0.619	2.62

TABLE II.

The persistence of a wave-complex. m and n represent the probability for the appearance of any x -wave and of any y -wave respectively. The ratio $\bar{A}_x : \bar{A}_y$ was assumed equal to 1:4. W is the probability for the appearance of waves greater than one half mean amplitude of the wave-complex.

$m:n$	W	P
0 : 1	0.822	5.60
0.1 : 0.9	0.799	4.98
0.2 : 0.8	0.779	4.53
0.3 : 0.7	0.765	4.29
0.4 : 0.6	0.754	4.06
0.5 : 0.5	0.749	3.98
0.6 : 0.4	0.749	3.98
0.7 : 0.3	0.756	4.09
0.8 : 0.2	0.770	4.34
0.9 : 0.1	0.791	4.78
1 : 0	0.882	5.60

and the persistence of the eeg in question can be evaluated according to the formula $P=1/1-W$, and the relations (7), (8) and (9), if the ratios $m:n$ and $\bar{A}_x:\bar{A}_y$ are given.

It will be numerically proven that the persistence so calculated is smaller than the value for a single population 5.6. Table I shows how the persistence is altered as the ratio $\bar{A}_x:\bar{A}_y$ changes from 1 to 10 when the ratio $m:n$ is fixed to a constant value 1. From this table it can be seen that the persistence of a wave-complex is always smaller than that of waves of a single population, and that the greater the difference in the mean amplitude of the component wave-populations, the smaller the value of persistence.

Table II shows the effect of the ratio $m:n$ on the persistence when the ratio $\bar{A}_x:\bar{A}_y$ is fixed to a constant value 1:4. It is to be noted that P is minimum when two sorts of waves appear with the same probability. In general, the persistence is least when two sorts of waves are greatly different in their amplitude and mixed in equal proportion.

The above argument applies also to cases in which more than two sorts of waves are involved. We have already shown that the persistence is generally small in the excited state of the brain. In normal man, small values of P are in most cases accompanied by small values of the mean amplitude. This is clearly caused by β -waves which appear frequently in the excited state. We studied the correlation between persistence and mean amplitude on 63 normal subjects and found 0.43 as the correlation-coefficient. This positive correlation is evidently due to dominant β -waves which cause both P and $\bar{A}$ to decrease simultaneously.

As the β -waves are a sign of cerebral excitation, a small value of P associated with a small value of $\bar{A}$ is to be regarded as an indication of a high excitation-level of the brain. A small value of P alone, however, does not always indicate any cerebral excitation, for a decrease of P ensues from the appearance of δ -waves which are no sign of cerebral excitation. In such cases a small value of P is accompanied by a large value of $\bar{A}$, because the δ -waves usually have great amplitudes. A good example of such cases is the eeg of epileptic patients in the interval between seizures. In Table III mean values of P at five normal subjects and those at five epileptic patients are shown. The mean values are in all cases but one of the normal subjects greater than the ideal value 5.6, whereas it is smaller in all cases of epileptic patients. The exceptional case was a β -rich eeg with a small mean amplitude suggesting the high cerebral excitation-level of the subject.

It has been shown that in the ideal case of a single population without any after-effect the persistence is wholly determined by the mean amplitude. Moreover, there exists some positive correlation between per-

TABLE III.

Mean values of persistence $\bar{P}$ in 5 normal subjects and 5 epileptic patients. n : number of records measured for each subject.

Normal		Epileptic	
$\bar{P}$	n	$\bar{P}$	n
9.5	226	3.79	38
8.24	66	4.92	47
7.65	51	5.31	22
7.1	52	3.72	22
4.98	51	3.87	38

sistence and mean amplitude. These circumstances might make doubtful whether the persistence can be an independent statistical measure. But in reality, both measures P and $\bar{A}$ go parallel in some cases, however, vary independently in other cases, so that the one cannot be represented by the other. It must be emphasized that at least three statistical measures, $\bar{T}$, $\bar{A}$ and P are necessary to characterize eeg numerically.

SUMMARY.

It was proposed to analyse electroencephalograms (eeg) and to express them quantitatively with three statistical measures, mean amplitude $\bar{A}$, mean period $\bar{T}$ and persistence P . The persistence is the mean of the number of waves greater than $\bar{A}/2$ following successively without interruption.

1. 50-250 eeg were taken at each subject and analysed. The histogram of P had its maximum at the value of P 4-8, and the mean values of P were 9.5, 8.24, 7.65, 7.1 and 4.98 in the standard passive state of consciousness of normal subjects.

2. The theoretical meaning of persistence was discussed. The value of P 5.6 was calculated from the theory for the ideal eeg in which all waves are independent of one another and belong to one and the same population of waves.

3. In general, the persistence is greater than 5.6, and this relation was regarded as expression of dependence among individual waves.

4. The persistence is smaller than 5.6 in cerebral excitation and under other conditions that more than a single sort of waves appear in eeg. It was theoretically proven that the persistence can decrease below the ideal value 5.6 by mixing more than one sort of waves.

5. Mixing of β -waves with normal α -waves reduces both P and $\bar{A}$, whereas mixing of δ -waves with normal α -waves decreases P , but increases $\bar{A}$.

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