Space Partitioning Evolutionary Many-Objective Optimization: Performance Analysis on MNK-Landscapes

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Summary

This work proposes space partitioning, a new approach to evolutionary many-objective optimization. The proposed approach instantaneously partitions the objective space into subspaces and concurrently searches in each subspace. A partition strategy is used to define a schedule of subspace sampling, so that different subspaces can be emphasized at different generations. Space partitioning is implemented with adaptive ϵ -ranking, a procedure that re-ranks solutions in each subspace giving selective advantage to a subset of well distributed solutions chosen from the set of solutions initially assigned rank-1 in the high dimensional objective space. Adaptation works to keep the actual number of rank-1 solutions in each subspace close to a desired number. The effects on performance of space partitioning are verified on MNK-Landscapes. Also, a comparison with two substitute distance assignment methods recently proposed for many-objective optimization is included.

1. Introduction

Multiobjective evolutionary algorithms (MOEAs) [Deb 01, Coello 02] optimize simultaneously two or more objective functions, aiming to find a set of trade-off solutions in a single run of the algorithm. Recently, there is a growing interest on applying MOEAs to solve *many*-objective optimization problems, where the number of objective functions is four or more. However, high dimensionality imposes several difficulties and in general conventional MOEAs scale up poorly with the number of objectives, particularly Pareto based MOEAs [Purshouse 03, Aguirre 05, Hughes 05]. One of the difficulties is that the number of Pareto non-dominated solutions in the instantaneous population grows rapidly with the number of objectives [Aguirre 04]. In such case, multi-objective evolutionary algorithms that use Pareto-based selection cannot fine-rank solutions. This results in a weak selection pressure that translates into a severe deterioration of their search ability. The increased number of non-dominated solutions also poses a challenge to the density estimators that should favor an even distribution of solutions to balance selection pressure towards the different regions of objective space. Another difficulty is that the number of points required to represent accurately the Pareto front grows exponentially

with the number of objectives. Current algorithms using population sizes in the order of hundreds can only generate very coarse approximations of the Pareto front. In addition, visualization, metrics to analyze performance of the algorithms, and an increased computational cost are other important issues in high dimensional spaces.

To cope with the challenges imposed by many-objective optimization new evolutionary algorithms are being proposed [Ishibuchi 08]. These new methods have focused mainly in the effectiveness of selection, dimensionality reduction, and the incorporation of user preferences. Some methods that focus on the effectiveness of selection try to improve Pareto-based selection by incorporating indicator functions or extensions of Pareto dominance [Zitzler 04, Emmerich 05, Corne 07, Kukkonen 07, Ishibuchi 07b, Koppen 07]. Other methods use aggregation functions in the selection mechanism instead of the Pareto criteria [Ishibuchi 07a, Hughes 05].

Dimensionality reduction approaches try to reduce the number of objectives the evolutionary algorithm uses to solve the problem [Deb 06b, Brockhoff 07, López Jaimes 08]. Here the main challenge consists in how to determine a minimum set of objectives functions that preserves most of the characteristics of the original problem. When such reductions are possible, the (lower dimensional) problem

becomes more amenable to the evolutionary algorithm and to the decision maker as well. However, even if dimensionality reduction is possible, there is no guarantee that the new dimension of the problem will be low enough to overcome the problems of weak selection pressure of conventional MOEAs.

Methods that focus on the incorporation of user preferences aim to provide the MOEA with a reference point so that it can concentrate its search on a small region of the Pareto front [Deb 06a]. Incorporation of user preferences is a very interesting and useful approach, however here the assumptions are that the user has a clear idea about the Pareto front and knows where to look for solutions. When such knowledge does not exist, we first need methods that can search effectively in many-objective spaces and generate an approximation of the Pareto front.

In practice, the optimization of real-world problems is not a single step process. Rather, it often consists of a series of steps in which the above mentioned approaches could be used to complement each other.

This work proposes space partitioning, a new approach to evolutionary many-objective optimization. The proposed approach instantaneously partitions the objective space into lower dimensional subspaces and concurrently searches in each subspace. Partitioning of the objective space into subspaces aims to instantaneously emphasize the search within lower dimensional objective spaces in order to find a good approximation of the true Pareto front in the original high dimensional space. The proposed approach uses a partition strategy to define a schedule of subspace sampling, so that different subspaces can be emphasized at different generations. Space partitioning can serve as a framework to improve the effectiveness of selection in high dimensional spaces, for dynamic dimensionality reduction, and to realize parallel implementations of many-objective optimization algorithms.

In this work we focus on space partitioning and concurrent subspace search as a way to improve the effectiveness of selection in high dimensional spaces. In this paper, we implement space partitioning with adaptive ϵ -ranking [Aguirre 09b]. Adaptive ϵ -ranking re-ranks solutions in each subspace, giving selective advantage to a subset of well distributed solutions chosen from the set of solutions initially assigned rank-1 in the original high dimensional objective space, so that selection can put more emphasizes in exploitation. Adaptation in the re-ranking procedure works to keep the actual number of rank-1 solutions in each subspace close to a desired number. The combination of space partitioning, partitioning strategy, and adaptive ϵ -ranking allows to perform an effective search aiming to

improve convergence and diversity of solutions on many-objective problems. It should be mentioned that methods to divide the search space into regions, without reducing dimensionality of the space, have been proposed to parallelize evolutionary algorithms for two and three objective problems [Branke 04]. However, their extension to larger dimensions is not straight forward.

We verify the effects on performance of space partitioning by comparing adaptive ϵ -ranking with and without space partitioning. We also examine three different partition strategies to investigate the influence of subspace sampling. In addition, we compare results of the space partitioning approach with two other many-objective optimizers [Aguirre 09c]. Namely, Subvector Dominance Assignment (SVDOM) and Epsilon Dominance Assignment (EPSDOM) [Koppen 07].

We test the algorithms on MNK-Landscapes [Aguirre 04, Aguirre 06]. MNK-Landscapes are scalable models of epistatic interactions for multi-objective combinatorial problems, useful to understand the fundamental search mechanisms of the algorithms and their scalability on subclasses of problems by varying the non-linear complexity K , number of objectives M , and size of the search space 2^N of the problem. In this work we analyze the scalability of the algorithms on subclasses of problems of increased complexity ($0 \leq K \leq 50$) and increased dimensionality of the objective space ($4 \leq M \leq 10$).

2. Definitions

Let us consider, without loss of generality, a maximization multiobjective problem with M objectives:

$$\text{maximize } \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_M(\mathbf{x})) \quad (1)$$

where $\mathbf{x} \in \mathcal{X}$ is a vector in the solution space $\mathcal{X}$, and $f_1, f_2, \dots, f_M$ the M objective functions to be optimized.

[Definition 1](Objective space ϕ) The objective space of the problem is determined by the set $\phi = \{f_1, f_2, \dots, f_M\}$ of the M objective functions to be optimized.

Pareto dominance and ϵ -dominance [Laumanns 02] are two domination structures [Yu 74] of special relevance to this work and are defined as follows.

[Definition 2](Pareto dominance) A solution $\mathbf{x}$ is said to dominate other solution $\mathbf{y}$ in the objective space ϕ , denoted $\mathbf{f}(\mathbf{x}) \succeq \mathbf{f}(\mathbf{y})$, if:

$$\begin{aligned} \forall f_m \in \phi \quad f_m(\mathbf{x}) &\geq f_m(\mathbf{y}) \quad \wedge \\ \exists f_m \in \phi \quad f_m(\mathbf{x}) &> f_m(\mathbf{y}). \end{aligned} \quad (2)$$

[Definition 3](ϵ -dominance) A solution $\mathbf{x}$ is said to ϵ -dominate other solution $\mathbf{y}$ in the objective space ϕ for some $\epsilon > 0.0$, denoted $\mathbf{f}(\mathbf{x}) \succeq^\epsilon \mathbf{f}(\mathbf{y})$, if:

$$\begin{aligned} \forall f_m \in \phi \quad (1 + \epsilon)f_m(\mathbf{x}) &\geq f_m(\mathbf{y}) \quad \wedge \\ \exists f_m \in \phi \quad (1 + \epsilon)f_m(\mathbf{x}) &> f_m(\mathbf{y}). \end{aligned} \quad (3)$$

Subspace, non-overlapping subspace, and space partition are important concepts we use to describe the proposed approach and are defined as follows.

[Definition 4](Subspace ψ) A subspace ψ of ϕ is a lower dimension space that includes some of the functions in ϕ , i.e. $\psi \subset \phi$.

[Definition 5](Non-overlapping subspaces) Two subspaces $\psi_1 \subset \phi$ and $\psi_2 \subset \phi$ are said to be non-overlapping if they have no common objectives, i.e. $\psi_1 \cap \psi_2 = \emptyset$.

[Definition 6](Space partition Ψ_{N_S}) A space ϕ is said to be partitioned into N_S subspaces, denoted as Ψ_{N_S} , if all subspaces are non-overlapping and no objective function in ϕ is left unassigned to a subspace, i.e. $\Psi_{N_S} = \{\psi_1, \psi_2, \dots, \psi_{N_S} \mid \psi_1 \cap \psi_2 \dots \cap \psi_{N_S} = \emptyset \wedge \psi_1 \cup \psi_2 \dots \cup \psi_{N_S} = \phi\}$.

The proposed method uses ϵ -dominance to re-rank solutions within a subspace, taking into account only the objective functions assigned to the subspace. To distinguish ϵ -dominance applied on a subspace from conventional ϵ -dominance, subspace ϵ -dominance is defined as follows.

[Definition 7](Subspace ϵ -dominance) A solution $\mathbf{x}$ is said to ϵ -dominate other solution $\mathbf{y}$ in the subspace ψ for some $\epsilon > 0.0$, denoted $\mathbf{f}(\mathbf{x}) \succeq_{\psi}^{\epsilon} \mathbf{f}(\mathbf{y})$, if:

$$\begin{aligned} \forall f_m \in \psi \quad (1 + \epsilon)f_m(\mathbf{x}) &\geq f_m(\mathbf{y}) \quad \wedge \\ \exists f_m \in \psi \quad (1 + \epsilon)f_m(\mathbf{x}) &> f_m(\mathbf{y}). \end{aligned} \quad (4)$$

3. Method

3.1 ϵ R-EMO

In this section, we describe the proposed space partitioning with adaptive ϵ -ranking method, called ϵ R-EMO for short. ϵ R-EMO first ranks solutions by Pareto dominance calculated in the original high-dimensional objective space. Then, it instantaneously partitions the objective space into subspaces, re-ranks solutions for each subspace using an adaptive subspace- ϵ -ranking procedure, and concurrently performs one generation of the evolutionary search within each subspace. During the next cycle of the algorithm, parents and offspring from all subspaces will be joined together so that they will be ranked again in the original high dimensional objective space.

By partitioning the objective space into subspaces, we aim to instantaneously emphasize the search within lower dimensional objective spaces. At each generation, we don't search in all possible subspaces. Instead, we define a schedule of subspace sampling by using a partition strategy. Re-ranking of solutions by the adaptive subspace- ϵ -ranking gives selective advantage to a subset of well distributed solutions chosen from the usually too many solutions assigned rank-1 in a high dimensional space, so that selec-

tion can put more emphasizes in exploitation while balancing selection pressure towards the different regions of objective space. Adaptation in the re-ranking procedure works to keep the actual number of rank-1 solutions in each subspace close to a desired number. The combination of space partitioning, partitioning strategy, and adaptive subspace- ϵ -ranking, aims to effectively search on many-objective problems.

Procedure 1 ϵ R-EMO

Input: N_S , number of subspaces at each generation. α , desired number of rank-1 solutions in each subspace (as a fraction of the entire parent population)

Output: $\mathcal{F}_1$, set of Pareto non-dominated solutions

```

1:  $\mathcal{P} \leftarrow \emptyset, \mathcal{Q} \leftarrow \text{random}$ 
   // initialize parent  $\mathcal{P}$  and offspring  $\mathcal{Q}$  populations
2:  $\epsilon_1, \epsilon_2, \dots, \epsilon_{N_S} \leftarrow 0.0$ 
3:  $\Delta_1, \Delta_2, \dots, \Delta_{N_S} \leftarrow \Delta_0$ 
4: repeat
5:   evaluation( $\mathcal{Q}, \phi$ ) //  $\phi = \{f_1, f_2, \dots, f_M\}$ 
6:    $\mathcal{F} \leftarrow \text{non-domination-sorting}(\mathcal{P} \cup \mathcal{Q})$ 
   //  $\mathcal{F} = \{\mathcal{F}_i\} \ (i = 1, 2 \dots, N_F)$ 
7:   crowding-distance( $\mathcal{F}$ )
8:    $\Psi_{N_S} \leftarrow \text{space-partitioning}(\phi, N_S)$ 
   //  $\Psi_{N_S} = \{\psi_1, \psi_2, \dots, \psi_{N_S}\}$ 
9:    $\mathcal{P} \leftarrow \emptyset, \mathcal{Q} \leftarrow \emptyset$ 
10:  for  $s = 1$  to  $N_S$  do
11:     $\mathcal{F}^{\epsilon_s} \leftarrow \text{subspace-}\epsilon\text{-ranking}(\psi_s, \epsilon_s, \mathcal{F})$ 
   //  $\mathcal{F}^{\epsilon_s} = \{\mathcal{F}_j^{\epsilon_s}\} \ (j = 1, 2 \dots, N_F^{\epsilon_s})$ 
12:     $\epsilon_s, \Delta_s \leftarrow \epsilon\text{-adaptation}(\epsilon_s, \Delta_s, \alpha, |\mathcal{F}_1^{\epsilon_s}|)$ 
   // adapt  $\epsilon_s$  and  $\Delta_s$  for the next generation
13:     $\mathcal{P}_s \leftarrow \text{truncation}(\mathcal{F}^{\epsilon_s})$ 
   //  $|\mathcal{P}_s| = |\mathcal{P}|/N_S, |\mathcal{F}^{\epsilon_s}| = |\mathcal{P}| + |\mathcal{Q}|$ 
14:     $\mathcal{Q}_s \leftarrow \text{recombination and mutation}(\mathcal{P}_s)$ 
   //  $|\mathcal{Q}_s| = |\mathcal{Q}|/N_S$ 
15:     $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{P}_s, \mathcal{Q} \leftarrow \mathcal{Q} \cup \mathcal{Q}_s$ 
16:  end for
17: until termination criterion is met
18: return  $\mathcal{F}_1$ 
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ϵ R-EMO implemented in NSGA-II's framework [Deb 00] is illustrated in **Procedure 1**. In the following we explain in detail its distinctive features.

3.2 Objective Space Partitioning

In our approach, we partition the M dimensional space $\phi = \{f_1, f_2, \dots, f_M\}$ into N_S non-overlapping subspaces $\Psi_{N_S} = \{\psi_1, \psi_2, \dots, \psi_{N_S}\}$ (**Procedure 1**, line 8). All subspaces have the same dimension $M_S = M/N_S$ in case $r = (M \bmod N_S)$ is zero. Otherwise, r of the N_S subspaces have dimension $M_S = \lfloor M/N_S \rfloor + 1$ and the rest $M_S =$

Fig. 1 Space partitioning strategies.

$\lfloor M/N_S \rfloor$. The number of all possible ways to partition ϕ into subspaces of dimension M_S is given by $\frac{M!}{M_S!(M-M_S)!}$ and becomes very large with M . For example, for $M = \{4, 6, 8, 10\}$ objectives and $N_S = 2$ subspaces there are $\{6, 20, 70, 252\}$ ways to partition the space, respectively. In our approach, we don't explicitly search in all possible subspaces at each generation. Instead, we set N_S to a small value and define a schedule of subspace sampling by using a partitioning strategy. We investigate three strategies to partition ϕ . Namely, *random*, *shift*, and *fixed* partitioning strategies as illustrated in **Figure 1**.

Random strategy randomly assigns objectives $f_i \in \phi$ to subspaces $\psi_s \in \Psi_{N_S}$. With this strategy, any possible M_S dimensional subspace of ϕ could be formed. However, it does not seek to correlate the s -th subspace ψ_s from generation t to the next.

Shift strategy, at the first generation, assigns deterministically objectives $f_i \in \phi$ to subspaces $\psi_s \in \Psi_{N_S}$, so that objectives assigned to a given ψ_s are ordered by objective index i . Then, in subsequent generations, the objective with highest index in the s -th subspace is shifted to the $((s+1) \bmod N_S)$ -th subspace, $\forall \psi_s \in \Psi_{N_S}$. This strategy correlates the s -th subspace from generation t to the next. In fact, subspace ψ_s at generation t overlaps with ψ_s at generation $t-1$ in all but one objective. However, only M of all possible M_S dimensional subspaces of ϕ could be formed.

Fixed strategy assigns deterministically objectives $f_i \in \phi$ to subspaces $\psi_s \in \Psi_{N_S}$ and keeps the same assignment throughout the generations. With this strategy only $N_S = M/M_S$ subspaces of ϕ could be formed.

Note that when a subspace is created by one of the three partitioning strategies we do not linearly aggregate objective functions nor apply other transformation to the individual objectives. We just indicate which functions in the

space ϕ will be included in the subspace $\psi_s \in \Psi_{N_S}$.

The above strategies would allow us to verify the impact of subspace sampling on the quality of solutions and the effect of subspace correlation from one generation to the next on the adaptation of ϵ for subspace- ϵ -ranking.

3.3 Adaptation of ϵ

ϵ R-EMO re-ranks solutions in each subspace by using a subspace- ϵ -ranking procedure (**Procedure 1**, line 11) in which the number of rank-1 solutions depends on the value set to ϵ (≥ 0) and on the instantaneous distribution of solutions in objective space. The adaptive procedure adapts ϵ (**Procedure 1**, line 12) at each generation in order to keep in each subspace the actual number of rank-1 solutions close to a desired number specified by $\alpha \times |\mathcal{P}|$, where α is a parameter in the range $[0.0, 1.0]$ set by the user and $|\mathcal{P}|$ is the size of the entire parent population. In addition to adapting ϵ , the step of adaptation Δ is also adapted to properly follow the dynamics of the evolutionary process on a given problem. Basically, if the number of solutions assigned rank-1 by subspace- ϵ -ranking is above the desired number $\alpha \times |\mathcal{P}|$ then Δ is updated by $\Delta \leftarrow \min(\Delta \times 2.0, \Delta_{max})$ and ϵ by $\epsilon \leftarrow \epsilon + \Delta$. Otherwise, $\Delta \leftarrow \max(\Delta \times 0.5, \Delta_{min})$ and $\epsilon \leftarrow \epsilon - \Delta$, where the range of the step of adaptation is $[\Delta_{min}, \Delta_{max}]$.

ϵ R-EMO adapts one ϵ_s with its corresponding step of adaptation Δ_s for each one of the N_S subspaces ψ_s , instead of using one ϵ for all subspaces. Note that the actual combination of objectives that define ψ_s change from one generation to the next, depending on the partition strategy. So, adaptation of ϵ_s reacts to the characteristics of the different instantaneous subspaces (actual combinations of f_i) assigned to ψ_s . Since the dimension of the subspace is strongly correlated to the value of ϵ that renders the desired number of rank-1 solutions $\alpha \times |\mathcal{P}|$, when the space

ϕ is partitioned we make sure that the dimension of the subspace ψ_s remains the same throughout the generations.

3.4 Subspace- ϵ -ranking

Subspace- ϵ -ranking fine grains ranking of solutions initially ranked by Pareto dominance in the objective space ϕ , using a randomized ϵ -sampling procedure in the subspace $\psi \subset \phi$ that favors a good distribution of solutions based on dominance regions wider than conventional Pareto dominance. Subspace- ϵ -ranking extends ϵ -ranking [Aguirre 08], where ϵ -sampling acts on ϕ instead of ψ . In the following, we first explain ϵ -sampling and then subspace- ϵ -ranking.

ϵ -sampling assumes that there is a set of equally ranked solutions from which a subset should be chosen to give them selective advantage in order to proceed further with the evolutionary search. That is, ϵ -sampling acts as a decision making procedure, not to find a final solution, but to help selection of the evolutionary algorithm. Hence, the sampling heuristic must reflect criteria that favor an effective search. Here, the sample of solutions to be given selective advantage are obtained with the following criteria,

- Extreme solutions are always part of the sample.
- Each (not extreme) sampled solution is the sole sampled representative of its area of influence. The area of influence of the sampled solutions is determined by a domination region wider than Pareto dominance, i.e. ϵ -dominance.
- Sampling of (not extreme) solutions follows a random schedule.

The first criterion tries to push the search towards the optimum values of each fitness function, aiming to find non-dominated solutions in a wide area of objective space. The second criterion assures that only one solution in a given zone of objective space is given rank-1, trying to distribute the search effort more or less uniformly among the different zones represented in the actual population. The third criterion dynamically establishes the zones that are represented in the sample. Also, in the case that there are several solutions within each zone, it increases the likelihood that the sampled solutions that will be given higher rank are different from one generation to the next, increasing the possibility of exploring wider areas of objective and variable space.

Procedure 2 illustrates ϵ -sampling algorithm. Let us denote $\mathcal{A}$ the set of solutions that have been assigned the same rank based on conventional Pareto dominance, for example by applying non-domination sorting [Deb 00]. ϵ -sampling returns the sampled solutions $\mathcal{S} \subset \mathcal{A}$ that will be

given selective advantage as well as the set of solutions $\mathcal{D}^\epsilon$ to be demoted. See that extreme solutions are the first to be assigned to the sample $\mathcal{S}$ (lines 1,2). Then, one by one, solutions are randomly chosen and included in $\mathcal{S}$ (lines 4-6), whereas solutions that lie in the wider domination region of the randomly picked solution are assigned to $\mathcal{D}^\epsilon$ (lines 7,8). Note that subspace ϵ -dominance $\mathbf{f}(z) \succeq_\psi^\epsilon \mathbf{f}(y)$ is used. **Figure 2 (a)** illustrates the application of ϵ -sampling on the set of solutions $\mathcal{A} = \mathcal{F}_1$. The numbers close to the solutions represents the random schedule in which solutions are sampled (0 means extreme solutions, which are all selected at once).

Procedure 2 ϵ -sampling ($\psi, \epsilon, \mathcal{A}, \mathcal{S}, \mathcal{D}^\epsilon$)

Input: Subspace ψ , ϵ -dominance factor ϵ and a set of solutions $\mathcal{A}$

Output: $\mathcal{S}$ and $\mathcal{D}^\epsilon$ ($\mathcal{S} \cup \mathcal{D}^\epsilon = \mathcal{A}$). $\mathcal{S}$ contains the sample of solutions from $\mathcal{A}$, whereas $\mathcal{D}^\epsilon$ contains ϵ -dominated solutions in ψ

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1:  $\mathcal{X} \leftarrow \{x \in \mathcal{A} \mid f_m(x) = \max(f_m(\cdot)), \forall f_m \in \psi\}$ 
2:  $\mathcal{S} \leftarrow \mathcal{X}, \mathcal{A} \leftarrow \mathcal{A} \setminus \mathcal{X}, \mathcal{D}^\epsilon \leftarrow \emptyset$ 
3: while  $\mathcal{A} \neq \emptyset$  do
4:    $r \leftarrow \text{rand}() \quad // 1 \leq r \leq |\mathcal{A}|$ 
5:    $z \leftarrow r\text{-th solution} \in \mathcal{A}$ 
6:    $\mathcal{S} \leftarrow \mathcal{S} \cup \{z\}$ 
7:    $\mathcal{Y} \leftarrow \{y \in \mathcal{A} \mid \mathbf{f}(z) \succeq_\psi^\epsilon \mathbf{f}(y), z \neq y\}$ 
8:    $\mathcal{D}^\epsilon \leftarrow \mathcal{D}^\epsilon \cup \mathcal{Y}$ 
9:    $\mathcal{A} \leftarrow \mathcal{A} \setminus \{\{z\} \cup \mathcal{Y}\}$ 
10: end while
11: return
```

The ϵ -sampling procedure works on a set of equally ranked solutions, however within a population there could be several sets of such solutions (each set with a different rank). To re-rank all possible sets of equally ranked solutions, we use subspace- ϵ -ranking.

Subspace- ϵ -ranking is applied at each generation for each subspace after non-domination sorting to reclassify the sets $\mathcal{F}_i$ ($i = 1, \dots, N_F$). **Procedure 3** describes subspace- ϵ -ranking algorithm. See that the reclassified sets $\mathcal{F}_j^{\epsilon_s}$ ($j = 1, \dots, N_F^{\epsilon_s}$) now contains only the sample of solutions $\mathcal{S} \subset \mathcal{F}_i$ found by ϵ -sampling (lines 8,9). Also, see that solutions $\mathcal{D}^\epsilon$, which are not part of the sample (line 8) are demoted by joining them with solutions of a lower ranked set in the next iteration of the loop (line 4). Thus, $\mathcal{F}_1^{\epsilon_s}$ contains some of the solutions initially ranked first, but $\mathcal{F}_j^{\epsilon_s}$, $j > 1$, can contain solutions that initially were assigned to sets with different ranks. This gives chance to lateral diversity present in the initial ranking of solutions and can punish highly crowded solutions even if they are initially ranked first by conventional Pareto dominance.

Fig. 2 ϵ -sampling on (a) the set of solutions initially ranked first and (b) on the set of solutions initially ranked second joined with solutions demoted from the first set.

Procedure 3 subspace- ϵ -ranking ($\psi_s, \epsilon_s, \mathcal{F}, \mathcal{F}^{\epsilon_s}$)

Input: Subspace ψ_s , ϵ -dominance factor ϵ_s and solutions $\mathcal{F}$ classified in fronts $\mathcal{F}_i$ ($i = 1, \dots, N_F$) by non-domination sorting

Output: $\mathcal{F}^{\epsilon_s}$, solutions re-classified in groups $\mathcal{F}_j^{\epsilon_s}$ ($j = 1, \dots, N_F^{\epsilon_s}$)

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1:  $\mathcal{D}^\epsilon \leftarrow \emptyset, i \leftarrow 1, j \leftarrow 1$ 
2: repeat
3:   if  $i \leq N_F$  then
4:      $\mathcal{A} \leftarrow \mathcal{F}_i \cup \mathcal{D}^\epsilon, i \leftarrow i + 1$ 
5:   else
6:      $\mathcal{A} \leftarrow \mathcal{D}^\epsilon$ 
7:   end if
8:    $\epsilon$ -sampling( $\psi, \epsilon_s, \mathcal{A}, \mathcal{S}, \mathcal{D}^\epsilon$ )
9:    $\mathcal{F}_j^{\epsilon_s} \leftarrow \mathcal{S}, j \leftarrow j + 1$ 
10: until  $\mathcal{D}^\epsilon = \emptyset$ 
11: return
```

Figure 2 illustrates ϵ -ranking calling on ϵ -sampling to re-rank the set $\mathcal{F}_1$ of solutions initially ranked first and the set $\mathcal{F}_2$ of solutions ranked second joined with the demoted solutions $\mathcal{D}^\epsilon$ from $\mathcal{F}_1$. The example illustrates the application of ϵ -sampling to a 2 dimensional objective space ϕ . When ϵ -sampling is applied to a subspace $\psi \subset \phi$, the non-dominated solutions in ϕ projected in the subspace ψ (assuming a 2 dimensional subspace) would look similar to Figure 2 (b). ϵ -sampling will be applied to all projected solutions.

4. Multiobjective MNK-Landscapes

In this work we use multiobjective MNK-Landscapes as test problems. A multiobjective MNK-Landscape [Aguirre 04, Aguirre 06] is defined as a vector function mapping binary strings into real numbers $\mathbf{f}(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_M(\cdot)) : \mathcal{B}^N \rightarrow \mathbb{R}^M$, where M is the number of objectives, $f_i(\cdot)$ is the i -th objective function, $\mathcal{B} = \{0, 1\}$, and N is the bit string length. $\mathbf{K} = \{K_1, \dots, K_M\}$ is a set of integers where K_i ($i = 1, 2, \dots, M$) is the number of bits

in the string that epistatically interact with each bit in the i -th landscape. Each $f_i(\cdot)$ can be expressed as an average of N functions as follows

$$f_i(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^N f_{i,j}(\mathbf{x}_j, z_1^{(i,j)}, z_2^{(i,j)}, \dots, z_{K_i}^{(i,j)}) \quad (5)$$

where $f_{i,j} : \mathcal{B}^{K_i+1} \rightarrow \mathbb{R}$ gives the fitness contribution of bit x_j to $f_i(\cdot)$, and $z_1^{(i,j)}, z_2^{(i,j)}, \dots, z_{K_i}^{(i,j)}$ are the K_i bits interacting with bit x_j in the string $\mathbf{x}$. The fitness contribution $f_{i,j}$ of bit x_j is a number between $[0.0, 1.0]$ drawn from a uniform distribution. Thus, each $f_i(\cdot)$ is a non-linear function of $\mathbf{x}$ expressed by a NK-Landscape model of epistatic interactions [Kauffman 93].

For a given N , we can tune the ruggedness of the fitness function $f_i(\cdot)$ of the i -th objective by varying K_i . In the limits, $K_i = 0$ corresponds to a model in which there are no epistatic interactions and the fitness contribution from each bit value is simply additive, which yields a single peaked smooth i -th fitness landscape. On the opposite extreme, $K_i = N - 1$ corresponds to a model in which each bit value is epistatically affected by all the remaining bit values yielding a maximally rugged fully random i -th fitness landscape. Varying K_i from 0 to $N - 1$ gives a family of increasingly rugged multi-peaked landscapes.

In addition to M , N , and $\mathbf{K}$, it is also possible to specify the epistatic pattern between bit x_j and the K_i other interacting bits. That is, the distribution $D_i = \{\text{random}, \text{nearest neighbor}\}$ of K_i bits among N . Thus, M , N , $\mathbf{K} = \{K_1, K_2, \dots, K_M\}$, and $\mathbf{D} = \{D_1, D_2, \dots, D_M\}$, completely specify a multiobjective MNK-Landscape.

MNK-Landscapes are easy to construct and can scale to any number of objectives M , number of bits N , and number of epistatic interactions K_i , allowing the creation of sub-classes of combinatorial non-linear problems for discrete search spaces. By varying these parameters we can analyze the properties of the multiobjective landscapes and study the effects of the number of objectives, size of the search space, intensity of epistatic interactions, and epistatic pattern on the performance of multiobjective optimization algorithms on combinatorial discrete search spaces.

5. Performance Measures and Parameters

5.1 Performance measures

In this work, we use the hypervolume $\mathcal{H}$, the set coverage $\mathcal{C}$, and the normalized maximum spread $\mathcal{D}$ measures [Zitzler 99] to evaluate and compare the performance of the algorithms. The measure $\mathcal{H}$ calculates the volume of the M -dimensional region in objective space enclosed by

a set of non-dominated solutions and a dominated reference point. Let $\mathcal{A}$ be a set of non-dominated solutions. The hypervolume of $\mathcal{A}$ can be expressed as

$$\mathcal{H}(\mathcal{A}) = \lambda(\cup_{i=1}^{|\mathcal{A}|} \mathcal{V}_i) \quad (6)$$

where λ is the Lebesgue measure, and $\mathcal{V}_i$ is the hypervolume rendered by the point $\mathbf{x}_i \in \mathcal{A}$ and the reference point. In this work, the reference point is set to $[0.0, \dots, 0.0]$. Given two sets of non-dominated solutions $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{H}(\mathcal{A}) > \mathcal{H}(\mathcal{B})$ then set $\mathcal{A}$ can be considered better on convergence and/or diversity of solutions. To calculate $\mathcal{H}$, we use Fonseca et al. [Fonseca 06] algorithm, which significantly reduces computational time.

The set coverage $\mathcal{C}$ measure [Zitzler 99] provides complementary information on convergence. Let us denote $\mathcal{A}$ and $\mathcal{B}$ the sets of non-dominated solutions found by two algorithms. $\mathcal{C}(\mathcal{A}, \mathcal{B})$ gives the fraction of solutions in $\mathcal{B}$ that are dominated at least by one solution in $\mathcal{A}$. More formally,

$$\mathcal{C}(\mathcal{A}, \mathcal{B}) = \frac{|\{b \in \mathcal{B} \mid \exists a \in \mathcal{A} : f(a) \preceq f(b)\}|}{|\mathcal{B}|}. \quad (7)$$

$\mathcal{C}(\mathcal{A}, \mathcal{B}) = 1.0$ indicates that all solutions in $\mathcal{B}$ are dominated by solutions in $\mathcal{A}$, whereas $\mathcal{C}(\mathcal{A}, \mathcal{B}) = 0.0$ indicates that no solution in $\mathcal{B}$ is dominated by solutions in $\mathcal{A}$. Since usually $\mathcal{C}(\mathcal{A}, \mathcal{B}) + \mathcal{C}(\mathcal{B}, \mathcal{A}) \neq 1.0$, both $\mathcal{C}(\mathcal{A}, \mathcal{B})$ and $\mathcal{C}(\mathcal{B}, \mathcal{A})$ are required to understand the degree to which solutions of one set dominate solutions of the other set.

The maximum spread $\mathcal{D}$ [Zitzler 99] measures the length of the diagonal of a hyperbox formed by the extreme function values observed in the non-dominated set. A normalized version of $\mathcal{D}$ [Deb 01] is expressed by

$$\mathcal{D} = \sqrt{\frac{1}{M} \sum_{m=1}^M \left(\frac{\max_{i \in \mathcal{F}_1} f_m^i - \min_{i \in \mathcal{F}_1} f_m^i}{f_m^{\max} - f_m^{\min}} \right)^2}. \quad (8)$$

Here $\mathcal{F}_1$ is the set of non-dominated solutions, and $f_m^{\max}$ and $f_m^{\min}$ are the maximum and minimum value of the m -th objective. A larger value of $\mathcal{D}$ indicates that a widely spread set of solutions is obtained. In this work we set $f_m^{\max} = 1.0$ and $f_m^{\min} = 0.0$, the maximum and minimum possible values in MNK-Landscapes.

5.2 Parameters

The performance of the algorithms is verified on MNK-Landscapes with $4 \leq M \leq 10$ objectives, $N = 100$ bits, number of epistatic interactions $K = \{0, 1, 3, 5, 7, 10, 15, 25, 35, 50\}$ ($K_1, \dots, K_M = K$), and *random* epistatic patterns among bits in all objectives ($D_1, \dots, D_M = \text{random}$). Results presented below show the average performance of

the algorithms on 50 different problems randomly generated for each combination of M , N and K . That is, in this study 3.500 different problems are used, grouped in subclasses according to the number of epistatic interactions K and number of objectives M . In the plots, error bars show 95% confidence intervals on the mean.

All algorithms used in our study are set with overall parent and offspring populations of size $|\mathcal{P}| = |\mathcal{Q}| = 100$, two point crossover for recombination with rate $p_c = 0.6$, and bit flipping mutation with rate $p_m = 1/N$ per bit. The number of evaluations is set to 3×10^5 .

We study the performance of ϵ R-EMO setting the number of subspaces to $N_S = \{1, 2\}$, varying the parameter α . For $N_S = 1$, no subspace partitioning $\Psi_{N_S} = \{\psi_1 = \phi\}$, we set $\alpha = \{1.0, 0.7, 0.5, 0.3\}$ so that the desired number of solutions with rank-1 after subspace- ϵ -ranking is $\alpha \times |\mathcal{P}| = \{100, 70, 50, 30\}$, respectively. For $N_S = 2$, subspace partition $\Psi_{N_S} = \{\psi_1, \psi_2\}$, we set $\alpha = \{0.5, 0.35, 0.25, 0.15\}$ so that $\alpha \times |\mathcal{P}| = \{50, 35, 25, 15\}$, respectively, in each of the two subspaces. Also, note that for $N_S = 2$ the size of the parent population in each subspaces is $|\mathcal{P}_s| = |\mathcal{P}|/N_S$, as indicated in **Procedure 1**.

For the step of adaptation Δ_s , we set its initial value to $\Delta_0 = 0.005$ (0.5%) and its maximum and minimum values to $\Delta_{\max} = 0.05$ (5%) and $\Delta_{\min} = 0.0001$ (0.01%), respectively.

6. Effects of Space Partitioning

6.1 ϵ R-EMO with no objective space partitioning

In this section, we discuss the performance of ϵ R-EMO when no objective space partitioning is considered ($N_S = 1$), setting the fraction between desired number of rank-1 individuals and population size to $\alpha = \alpha^*$ that achieves maximum hypervolume $\mathcal{H}$. **Figure 3 (a)** shows the average ratio $\frac{\mathcal{H}(E)}{\mathcal{H}(N)}$, where E and N denote the set of solutions found by ϵ R-EMO and conventional NSGA-II, respectively. Thus, a ratio greater than 1.0 indicates better $\mathcal{H}$ by ϵ R-EMO than conventional NSGA-II. As a reference, we include a horizontal line to represent the $\mathcal{H}(N)$ values normalized to 1.0. From this figure, we can see that ϵ R-EMO can significantly improve $\mathcal{H}$ on $4 \leq M \leq 10$ objectives problems, for all values of K (up to 27% improvement). Note that improvements on $\mathcal{H}$ become larger as we increase the number of objectives M from 4 to 6, whereas improvements on $\mathcal{H}$ are similarly high for $8 \leq M \leq 10$. Due to space limitations, we include results for $M = \{4, 6, 8, 10\}$ only and not for $M = \{5, 7, 9\}$.

Improvements on $\mathcal{H}$ can be due to solutions with better convergence, better diversity, or both. To complement

Fig. 3 Normalized $\mathcal{H}$ and $\mathcal{C}$ between NSGA-II and ϵ R-EMO when no objective space division ($S = 1$) is considered. ϵ R-EMO is set to α^* that achieves maximum $\mathcal{H}(E)$.

Fig. 4 Results by NSGA-II and ϵ R-EMO partitioning the objective space in two subspaces ($S = 2$) using *random* strategy and setting α^* that achieves maximum $\mathcal{H}(E)$.

Fig. 5 Results by NSGA-II and ϵ R-EMO partitioning the objective space in two subspaces ($S = 2$) using *shift* strategy and setting α^* that achieves maximum $\mathcal{H}(E)$.

Fig. 6 Results by NSGA-II and ϵ R-EMO partitioning the objective space in two subspaces ($S = 2$) using *fixed* strategy and setting α^* that achieves maximum $\mathcal{H}(E)$.

the analyzes of results on $\mathcal{H}$ we also present results using the $\mathcal{C}$ measure. **Figure 3 (b)** shows the average $\mathcal{C}$ values between conventional NSGA-II and ϵ R-EMO set with α^* . From this figure, we can see that $\mathcal{C}(N, E)$ is close to 0.0 for most K and M . This indicates that there are almost no solutions by conventional NSGA-II that dominate solutions by ϵ R-EMO. On the other hand, the values of $\mathcal{C}(E, N)$ are very high for 4 objectives (in the range 0.60-0.85) and reduce gradually as we increase M up to 10 objectives (in the range 0.01-0.08). This suggests that a better convergence of solutions contributes to the increases of $\mathcal{H}$ by ϵ R-EMO on $M = 4$ problems. As we increase M , gains on diversity gradually become more significant than gains on convergence as the reason for the significant improvement of $\mathcal{H}$ on $6 \leq M \leq 10$.

6.2 ϵ R-EMO with objective space partitioning

In this section we analyze the performance of ϵ R-EMO partitioning instantaneously the objective space into two subspaces using the *random*, *shift*, and *fixed* partition strategies introduced in section **Chapter 3**.

First, we show results by the *random* partitioning strategy in **Figure 4**. Looking at **Figure 4 (a)** and comparing with **Figure 3 (a)**, we can see that ranking on subspaces using a *random* strategy leads to a remarkable improvement on $\mathcal{H}$ for all values of M and K (up to 82.5% improvement). Note that the increase on $\mathcal{H}$ gets bigger with the number of objectives M . Looking at **Figure 4 (b)** and comparing with **Figure 3 (b)**, we can see that $\mathcal{C}(E, N)$ also increases for any value of K and M , whereas $\mathcal{C}(N, E)$ remains close to zero. That is, convergence also improves substantially.

Next, we discuss results by the *shift* partition strategy shown in **Figure 5 (a)** and **(b)**. From these figures note that the *shift* strategy also leads to a remarkable improvement on $\mathcal{H}$ and $\mathcal{C}$. Comparing the *shift* and *random* strategies, the latter leads to slightly better results than the former especially for $M \geq 8$ and $K \leq 15$. As mentioned above, the *random* strategy can sample any possible subspace of ϕ , whereas the *shift* strategy can sample only M of all possible subspaces of ϕ . The number of subspaces unable to sample the *shift* strategy increase with the dimension of the objective space, especially if we keep constant the number of subspaces. Better results by the *random* strategy suggest that sampling more subspaces becomes relevant as the number of objectives increase. However, since results by the *random* and *shift* strategies are not very different, this also suggests that not all subspaces need to be sampled to achieve good performance, which could be highly relevant to dimensionality reduction and

parallelization under a space partitioning approach.

Results by the *fixed* partition strategy are shown in **Figure 6 (a)** and **(b)**. See that the *fixed* strategy leads to smaller $\mathcal{H}$ and $\mathcal{C}(E, N)$ than the *random* and *shift* partition strategies. Comparing to ϵ R-EMO with no subspace partitioning, the *fixed* strategy leads to higher $\mathcal{H}$ on all M but with smaller $\mathcal{C}(E, N)$ on $M \leq 6$. We did not expect to achieve best performance with the *fixed* strategy. However, since this strategy only explores two of all possible subspaces of ϕ , these results are rather surprising and illustrate the gains in performance that could be achieved in high-dimensional spaces by space partitioning even if poor partition strategies are used.

It should be noted that as we increase K (non-linearity of the problem) improvements on both $\mathcal{H}$ and $\mathcal{C}$ reduce regardless of the partition strategy. This suggests that in addition to better selection strategies, we should also look into ways to improve recombination and mutation to achieve better performance on highly non-linear problems.

The timing to reassign objectives to subspaces is an important aspect that could lead to better performance and facilitate parallel implementations of the proposed method. The *fixed* and *random* strategies explored in this work are two extremes scenarios between no reassignment at all and reassignment at each generation, which could be seen as exploitation within a subspace versus exploration through different subspaces, respectively. A reassignment of objectives to subspaces after a certain number of generations may lead to a better balance between these two important issues in selection. This is an important line of research that we would like to pursue in the future.

6.3 Analysis of α

In this section we analyze the parameter α that determines the desired number of rank-1 solution in each subspace. As an example, **Figure 7** shows $\mathcal{H}$ and $\mathcal{C}$ results achieved by different settings of α on $M = 8$ objectives landscapes partitioning the objective space in $N_S = 2$ subspaces using *shift* strategy. Note that, since we use an overall parent population size of $|P| = 100$, the size of the parent population in each one of the two subspaces is $|P_s| = |P|/N_S = 50$. Also, recall that the number of rank-1 solutions in each subspace is calculated as a fraction of the overall parent population by $\alpha \times |P|$.

From **Figure 7**, note that $\alpha \geq 0.25$ (at least 50% of the parent population P_s in each subspace is given rank-1) leads to high performance, whereas results by $\alpha = 0.15$ are clearly lower on $M = 8$ objectives landscapes. Analyzing performance by $\alpha \geq 0.25$, see that setting α to 0.35 or 0.25 leads to best performance for most K , both on $\mathcal{H}$

Fig. 7 Results by NSGA-II and ϵ R-EMO partitioning the objective space in two subspaces ($S = 2$) using *shift* strategy and varying α on $M = 8$ objectives.

Fig. 8 Adaptation of ϵ in ϵ R-EMO by *fixed*, *shift*, and *random* partitioning strategies for $\alpha = 0.25$ on $M = 8$ objectives and $K = 10$ epistatic bits.

Fig. 9 Concurrent adaptation of ϵ on subspaces of different dimensionality by ϵ R-EMO with *shift* partitioning strategy for $\alpha = 0.25$ on $M = 7$ objectives and $K = 10$ epistatic bits.

and $\mathcal{C}$. However, see that setting α to 0.5 could give highest performance especially on small K . Although there is not an absolute winner among $\alpha \geq 0.25$ values, it is important to note that subspace partitioning's lower bound performance ($\min \mathcal{H} \wedge \min \mathcal{C}(E, N), \forall \alpha \in \{0.5, 0.35, 0.25\}$) is significantly better than the performance by no subspace partitioning (see **Figure 7** and compare with **Figure 3**, $M = 8$). Analyzing our data for other values of M , in general, we see that performance by $\alpha = 0.25$ is better than 0.35 when the number of objectives decrease to $M = 6$ and $M = 4$; whereas performance by 0.5 and 0.35 is better than 0.25 when we increase M to 10 objectives. As a rule

of thumb, when the space is partitioned into 2 subspaces, $\alpha = 0.25$ works well on $M = 4$ and $M = 6$, $\alpha = 0.35$ on $M = 8$, and $\alpha = 0.5$ on $M = 10$.

6.4 Adaptation of ϵ

In this section we discuss the adaptation of ϵ_s . **Figure 8 (a)-(c)** illustrate the adaptation of ϵ_s in one of the two four-dimensional subspaces by *fixed*, *shift*, and *random* partitioning strategies for $\alpha = 0.25$ in a $M = 8$ and $K = 10$ landscape. These figures depict results of a single run of the algorithm. The horizontal lines indicate the size of the overall parent population $|P| = 100$, the size of

the parent population in the subspace $|\mathcal{P}_s| = 50$, and the desired number of rank-1 individuals in the subspace $\alpha \times |\mathcal{P}| = 25$. From these figures note that the number of non-dominated individuals $|\mathcal{F}_1|$ (considering all objectives) exceeds $|\mathcal{P}|$ since the initial generations, as shown by the line with triangles. See also that the adaptive mechanism appropriately varies ϵ_s throughout the generations, as illustrated by the line with circles, so that after subspace- ϵ -ranking the number of rank-1 individuals $|\mathcal{F}_1^{\epsilon_s}|$ in the subspace is kept around the desired number $\alpha \times |\mathcal{P}|$ as shown by the line with squares. Note that adaptation works appropriately regardless of the partition strategy.

Note that we adapt a different ϵ_s for each subspace in order to have a general adaptive algorithm that can search concurrently on subspaces of different dimensionality, where very dissimilar distributions of solutions appear. **Figure 9 (a) and (b)** show the concurrent adaptation of ϵ_s in the three and four-dimensional subspaces, respectively, obtained by partitioning a $M = 7$ objectives landscapes into two subspaces using *shift* strategy. From the figures note that very different values of ϵ are required for subspaces of different dimensionality in order to keep the number of rank-1 individuals around the desired number.

When the concurrent search is performed on subspaces of the same dimensionality, the adaptation trend of ϵ for each subspace is similar suggesting that in this particular case one ϵ could be adequate for all subspaces. However, this deserves further investigation. We should recall that in this work landscapes of similar non-linear complexity are used in all objectives (given by the number of epistatic interaction $K_1 = K_2 = \dots = K_M = K$). The complexity of the landscapes associated to each objective function also affects the distribution of solutions [Aguirre 06], which could lead to dissimilar adaptation trends of ϵ even if subspaces of the same dimensionality are used.

7. Comparison with Substitute Assignment Distance Methods

In this section we compare the performance among objective space partitioning with adaptive ϵ -ranking ϵ R-EMO and two substitute distance assignment methods, namely Subvector Dominance Assignment (SVDOM) and Epsilon Dominance Assignment (EPSDOM) [Koppen 07]. These three many-objective methods enhance selection using different strategies to recalculate the primary or secondary ranking of solutions. As explained before, adaptive ϵ -ranking in ϵ R-EMO focuses on the primary ranking of solutions. On the other hand, SVDOM and EPSDOM focus on the secondary ranking of solutions, replacing the diver-

sity estimator with a substitute assignment distance. The motivation for this comparison is to understand better the effects on performance of different strategies that re-rank solutions to improve Pareto selection of many-objective optimizers.

The substitute distance that determines the secondary ranking of solutions in SVDOM and EPSDOM are based on measurement procedures that calculate the highest degree to which a solution is nearly Pareto dominated by any other solution [Koppen 07]. SVDOM re-ranks a non-dominated solution based on the number of objectives that are better in other non-dominated solutions. On the other hand, EPSDOM re-ranks a non-dominated solution based on the magnitude that other non-dominated solutions need to improve in order to dominate it. The reader is referred to [Koppen 07] for details on SVDOM and EPSDOM.

Figure 10 (a)-(c) show the average hypervolume ratios between the many-objective algorithms and conventional NSGA-II. That is, $\frac{\mathcal{H}(S)}{\mathcal{H}(N)}$, $\frac{\mathcal{H}(P)}{\mathcal{H}(N)}$, and $\frac{\mathcal{H}(E)}{\mathcal{H}(N)}$, where S , P , E , and N denote the set of solutions found by SVDOM, EPSDOM, ϵ R-EMO and conventional NSGA-II, respectively. From these figures, it can be seen that ϵ R-EMO achieves better hypervolume than EPSDOM and SVDOM for all M and K . Also note that between the substitute distance assignment algorithms, EPSDOM achieves better hypervolume than SVDOM for all M and K . Note that in this work we use $(0.0, 0.0, \dots, 0.0)$ as reference point to calculate the hypervolume. This point is far away from the Pareto front found by the algorithms and would relatively emphasize solutions located in the edges of the multi-dimensional Pareto front in the calculation of the hypervolume.

Figure 11 (a)-(c) show the average $\mathcal{C}$ values between conventional NSGA-II and the many-objective algorithms. From these figures, it can be seen that $\mathcal{C}$ values by EPSDOM are better than SVDOM for most M and K . Also, note that $\mathcal{C}$ by ϵ R-EMO is better than SVDOM for $M = \{4, 6\}$, but slightly worse for $M = \{8, 10\}$. Comparing EPSDOM and ϵ R-EMO, overall EPSDOM achieves similar or better $\mathcal{C}$ than ϵ R-EMO.

The above results on $\mathcal{H}$ and $\mathcal{C}$, i.e. better $\mathcal{H}$ by ϵ R-EMO and a better $\mathcal{C}$ by EPSDOM, suggest that there is a diversity-convergence trade-off between ϵ R-EMO and EPSDOM. **Figure 12 (a)-(d)** show $\mathcal{C}$ values between ϵ R-EMO and EPSDOM. From these figures, note that some solutions by ϵ R-EMO are dominated by solutions of EPSDOM, especially for $M \geq 6$. These results, confirm that convergence is better by EPSDOM than ϵ R-EMO. However, note that the fraction of dominated solutions in ϵ R-EMO is larger for $M = 6$ and reduces with the number of

Fig. 10 Normalized $\mathcal{H}$ by SVDOM, EPSDOM, and ϵ R-EMO. $M = \{4, 6, 8, 10\}$ objectives.

Fig. 11 $\mathcal{C}$ values by SVDOM, EPSDOM, and ϵ R-EMO. $M = \{4, 6, 8, 10\}$ objectives.

Fig. 12 $\mathcal{C}$ values between EPSDOM and ϵ R-EMO. $M = \{4, 6, 8, 10\}$ objectives.

objectives.

Figure 13 (a)-(c) show the normalized maximum spread $\mathcal{D}$ by SVDOM, EPSDOM and ϵ R-EMO. From these figures, note that $\mathcal{D}$ is substantially better by ϵ R-EMO than EPSDOM and SVDOM for all M and K ; whereas $\mathcal{D}$ is slightly better by EPSDOM than by SVDOM. These results show that ϵ R-EMO can find solutions in a region of objective space broader than the region where substitute assignment distance algorithms find solutions.

Figure 14 (a)-(d) plot the maximum attained fitness values in each objective function by conventional NSGA-II, ϵ R-EMO, SVDOM, and EPSDOM for $M = \{4, 6, 8, 10\}$ objectives and $K = 7$ epistatic interactions. Note that SVDOM and EPSDOM attain high values of $\mathcal{H}$ and $\mathcal{C}$ at $K =$

7. From Figure 14, we can see that conventional NSGA-II achieves in all objectives better maximum-fitness values than SVDOM and EPSDOM. Note also that the gap between NSGAII and SVDOM/ EPSDOM increases with the number of objectives M of the problem. These results confirm that solutions found by SVDOM and EPSDOM tend to concentrate within an increasingly reduced space of objective space as the number of objectives of the problem increase. On the other hand, note that ϵ R-EMO obtains overall better maximum-fitness than conventional NSGA-II in all M . These results show that the good values of hypervolume by SVDOM and EPSDOM are exclusively due to a better convergence of some solutions in a narrow region of objective space at the expense of diver-

Fig. 13 Normalized maximum spread $\mathcal{D}$ by SVDOM, EPSDOM, and ϵ R-EMO. $M = \{4, 6, 8, 10\}$ objectives.

Fig. 14 Maximum achieved fitness in each objective by NSGA-II, SVDOM, EPSDOM, and ϵ R-EMO. $M = \{4, 6, 8, 10\}$ objectives and $K = 7$.

sity. These results show that adaptive ϵ -ranking and substitute distance assignment EPSDOM emphasize different areas of objective space, suggesting that both strategies could complement each other to produce better approximations of the Pareto front.

8. Conclusions

This work has proposed space partitioning for evolutionary many-objective optimization focusing on improving selection of the optimizer. We implemented space partitioning using adaptive ϵ -ranking showing that the performance of a many-objective optimizer can be improved considerable, in terms of both convergence and diversity of solutions, by instantaneously partitioning the objective space into subspaces and concurrently searching within each subspace. Also, we analyzed the relevance of the partition strategy used to define a schedule of subspace sampling in order to emphasize different subspaces at different generations of the algorithm. We showed that a strategy able to sample all possible subspaces, without seeking to keep any correlation between subspaces from one generation to the next, gives better performance than strategies that emphasize the correlation of the subspaces at the expense of not being able to sample all subspaces. In addition, we compared the space partitioning algorithm with

two substitute distance assignment many-objective optimizers; namely SVDOM and EPSDOM. The three many-objective optimizers use different strategies to enhance the effectiveness of selection in high-dimensional spaces. We showed that the adaptive ϵ -ranking strategy and the substitute distance assignment strategies emphasize different areas of objective space and that could be used as complementary strategies to produce a better approximation of the high dimensional Pareto front.

As future works, we would like to extend the space partitioning approach to apply complementary ranking strategies concurrently, include dynamic dimensionality reduction, and realize parallel implementations of many-objective optimizers.

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