Attitude Control of Spacecraft with Mobile Antenna by a Gain-Scheduled H_∞ Controller*¹

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Abstract

This paper studies the attitude control problem of a class of data relay satellite which has a large mobile antenna for tracking target spacecraft in low orbit. The movement of the antenna disturbs the satellite attitude motion through dynamic coupling of the motions of the antenna and the satellite main body. In the operation, control of the spacecraft attitude must be maintained even when the antenna rotates at a high angular velocity. In this paper, we apply robust gain-scheduled controller synthesis to this problem. The controller is designed as a robust stabilizing controller based on the H_∞ controller synthesis of linear parameter varying system. The capability of the method is investigated by a numerical study.

1. Introduction

Data relay satellite (DRS) is used to relay inter-communications between ground stations and orbiting spacecraft for the development of an advanced communication means for the future space communications infrastructure. The satellite has a large mobile antenna which is capable of rotating at high angular velocity within a wide range of angles, in order to accurately track target spacecraft in orbit. In such a situation, the movement of the antenna highly disturbs satellite attitude motion. The attitude of the satellite main body must be controlled to ensure that the nadir is pointed properly under such antenna disturbance to maintain the communication link with the ground station. For this problem, a feed-forward control, which cancels the disturbance torque, with a conventional attitude controller is considered to be practical, based on the assumption that the antenna dynamics is precisely known. However, it is not easy to obtain accurate physical parameters to calculate the complex dynamical coupling terms for the feed-forward control. Yamada and Yoshikawa¹⁾ used an adaptive scheme to estimate parameters in the dynamics of the antenna in order to overcome this drawback.

This paper discusses solving such attitude control problems by a feedback control which attenuates the influence of the antenna disturbance torque to the spacecraft attitude. However, since the attitude of the DRS is nonlinear, the existing linear control theory could not be successfully applied. Therefore, we describe the dynamics as a linear parameter varying (LPV) model

with the varying parameters of angles and angular velocities of the antenna. Then, we apply robust gain-scheduled controller synthesis based on a H_∞ control method by making affine parameter dependent plants from the LPV model.²⁻⁵⁾ Using the synthesis process, provided that some limited numbers of parameters are measured in real time, the LPV controller is scheduled by linear time invariant (LTI) vertex controllers which are designed at each operating point of the affine parameter dependent plants. This controller guarantees robust stability based on the small gain theorem, and it is expected to achieve higher performance than existing LTI controllers which are designed using fixed parameter values. The problem is described by linear matrix inequalities (LMIs) when a plant exists in the specified polytope for all admissible varying parameters, and its solution is obtained by a convex optimization problem.⁶⁾ In order to verify the above design procedure, we make simulation studies using a simple DRS numerical model.

2. Modeling

The satellite configuration we consider in this paper is shown in Fig.1. It is composed of a main body and a rotating antenna, both of which are assumed as rigid bodies. The antenna rotates around the roll and pitch axes, respectively, which correspond to scanning directions of the antenna in the North-South (NS) and East-West (EW) directions in the inertial frame. The dynamical equation is derived from a conservation law of angular momentum as follows^{7,8)} (details are in the

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Fig. 1. Spacecraft configuration.

Appendix).

$$M(\phi)\ddot{\theta} + H(\phi, \dot{\theta})\dot{\theta} + T(\phi, \dot{\phi}, \ddot{\phi}) + U(\dot{\theta} \times \dot{\theta}) = u \dots (1)$$

where vector $\theta(t) \in \mathbf{R}^3$ is the attitude angle of the main body and vector $\phi(t) \in \mathbf{R}^2$ is the angle of the antenna. The matrices $M(\phi) \in \mathbf{R}^{3 \times 3}$ and $H(\phi, \dot{\phi}) \in \mathbf{R}^{3 \times 3}$ are satellite inertia matrix and damping matrix, respectively, that depend on the angle $\phi(t)$ and angular velocity $\dot{\phi}(t)$ of the antenna. Vector $T(\phi, \dot{\phi}, \ddot{\phi}) \in \mathbf{R}^3$ is the torque disturbance which is induced by the rotation of the antenna, and vector $U(\dot{\theta} \times \dot{\theta}) \in \mathbf{R}^3$ has nonlinear elements which depend on the vector product of angular velocity $\dot{\theta}(t)$. Vector $u \in \mathbf{R}^3$ is the control torque input. Moreover, the measurement output which is used for the attitude control is supposed to be the attitude angle described by the following output equation.

$$y = \theta \dots (2)$$

3. Gain-Scheduled Controller Synthesis

Our objective is to control the attitude angle $\theta(t)$ under the influence of the disturbance torque induced by antenna rotation by taking the varying parameters of the plant Eq. (1) into consideration. For this purpose, we design a robust gain-scheduled controller which has disturbance attenuation capability. Here, we describe the generalized plant and controller design procedure.

3.1. Generalized plant Based on Eqs. (1) and (2), we develop a generalized plant for robust gain-scheduled controller synthesis. Here, we assume that $U(\dot{\theta} \times \dot{\theta}) = 0$ in Eq. (1), because the vector product $\dot{\theta} \times \dot{\theta}$ is a very small value. Under the assumption, we treat the torque $T(\phi, \dot{\phi}, \ddot{\phi})$ as a disturbance against the attitude motion of the satellite and intend to design a controller that attenuates the disturbance. Toward this end, we first estimate the characteristics of the disturbance torque induced by the antenna. Since it is usually driven by velocity command according to a trapezoid law, its

acceleration becomes constant during certain periods. By supposing the situation, we construct a controller which includes an integral compensator in the loop to achieve zero off-set error. For this purpose, we consider a fictitious disturbance d applied to the plant through a low-pass filter as

$$f(s) = \frac{1}{s + \rho} \dots (3)$$

where ρ is a scalar design parameter. It is noted that $f(s) \rightarrow 1/s$ when $\rho \rightarrow 0$. This filter is expanded to a 3×3 diagonal matrix, corresponding to the disturbance inputs around roll, pitch and yaw axes, whose state space realization is

$$\begin{aligned} \dot{\zeta} &= A_f \zeta + B_f d \\ t &= C_f \zeta + D_f d \end{aligned} \dots (4)$$

Then, variable t is considered to be applied at the input side of the plant.

Next, the following control variable is defined to provide controller disturbance attenuation capability against the disturbance d .

$$z = \lambda y \dots (5)$$

Finally, the varying parameters are defined as an affine model because the plant is not LTI but LPV. The gain-scheduled controller synthesis in this paper does not admit parameter change in the mass matrix $M(\phi)$ because of the limitation in solving the LMIs. Fortunately, the influence of the parameter variations in the matrix $M(\phi)$ is negligibly small from the satellite structure, as described later. Hence, we use the constant matrix M_0 instead of $M(\phi)$ when parameter $\phi(t)$ is the nominal value. Therefore, in our problem, it is sufficient to consider only the parameter change of $H(\phi, \dot{\phi})$. For obtaining an affine model, the varying parameter vector to describe an affine parameter dependent plant is defined as

$$p(t) = [\phi^T(t) \quad \dot{\phi}^T(t)]^T \dots (6)$$

because the matrix $H(\phi, \dot{\phi})$ depends on vectors $\phi(t)$ and $\dot{\phi}(t)$. Thus, we obtain the following equation, from Eqs. (1), (2) and (6), and augmented dynamics Eq. (4).

$$\begin{aligned} M_0 \ddot{\theta} &= -H(p)\dot{\theta} - C_f \zeta - D_f d + u \\ \dot{\zeta} &= A_f \zeta + B_f d \\ z &= \lambda \theta \\ y &= \theta \end{aligned} \dots (7)$$

For the controller synthesis, from Eq. (7), the affine parameter dependent plant $S(p)$ is defined as follows.

$$\begin{aligned} \dot{x} &= A(p)x + B_1 d + B_2 u \\ z &= C_1 x + D_{11} d + D_{12} u \\ y &= C_2 x + D_{12} d + D_{12} u \end{aligned} \dots (8)$$

Fig. 2. Generalized plant.

where the state variable vector is $x = [\theta^T \ \dot{\theta}^T \ \zeta^T \ \dot{\zeta}^T]^T$ and each coefficient matrix is as follows.

$$A(p) = \begin{bmatrix} 0 & I & 0 \\ 0 & -M_0^{-1}H(p) & -M_0^{-1}C_f \\ 0 & 0 & A_f \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -M_0^{-1}D_f \\ B_f \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ M_0^{-1} \\ 0 \end{bmatrix} \quad \dots (9)$$

$$C_1 = [\lambda I \ 0 \ 0] \quad D_{11} = 0 \quad D_{12} = 0$$

$$C_2 = [I \ 0 \ 0] \quad D_{21} = 0 \quad D_{22} = 0$$

This is the generalized plant; a block diagram is shown in Fig. 2. Hereafter, the controller design procedure is discussed.

3.2. Controller design When the parameter vector $p(t)$ takes values in a box of $\mathbf{R}^n$ with corners $\{\Pi_i\}$, $i=1, \dots, N$ ($N=2^n$) in the parameter space, the plant $S(p)$ ranges in a matrix polytope with vertexes $S(\Pi_i)$. Therefore, provided that any convex decomposition of $p(t)$ over the corners of the parameter box is given as

$$p(t) = \sum_{i=1}^N \alpha_i \Pi_i \quad \alpha_i \geq 0 \quad \sum_{i=1}^N \alpha_i = 1 \quad \dots (10)$$

the plant $S(p)$ satisfies the following relation.

$$S(p) = \sum_{i=1}^N \alpha_i S(\Pi_i) \quad \dots (11)$$

Thus the parameter-dependent controller $K(p)$ which is defined as

$$\begin{aligned} \dot{x}_k &= A_k(p)x_k + B_k(p)y \\ u &= C_k(p)x_k + D_k(p)y \end{aligned} \quad \dots (12)$$

can be constructed by convex interpolation of the LTI vertex controllers $K(\Pi_i)$.

$$K(p) = \sum_{i=1}^N \alpha_i K(\Pi_i) \quad \dots (13)$$

Calculations of each controller $K(\Pi_i)$ are performed by a H_∞ -like synthesis for parameter values at Π_i , $i=1, \dots, N$. Because the closed-loop system of the plant and controller is a polytopic system, the condition that

the H_∞ norm from disturbance d to control variable z does not exceed the prescribed value $\gamma > 0$ is equivalent to the following LMIs from the extended bounded real lemma

$$\mathcal{N}_R^T \begin{bmatrix} A_i R + R A_i^T & R C_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \mathcal{N}_R < 0$$

$$\mathcal{N}_Q^T \begin{bmatrix} A_i^T Q + Q A_i & Q B_1 & C_1^T \\ B_1^T Q & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \mathcal{N}_Q < 0 \quad \dots (14)$$

$$\begin{bmatrix} R & I \\ I & Q \end{bmatrix} \geq 0$$

where $A_i = A(\Pi_i)$, ($i=1, \dots, N$), the two matrices R, Q are symmetric, and $\mathcal{N}_R$ and $\mathcal{N}_Q$ are bases of the null spaces of $[R_2^T \ D_{12}^T \ 0]$ and $[C_2 \ D_{21} \ 0]$. Then, R, Q and γ can be obtained by solving a convex optimization problem applied to the above LMIs (14). From matrices R and Q , matrix X_{cl} is defined as follows.

$$VW^T = I - RQ$$

$$X_{cl} = \begin{bmatrix} Q & I \\ W^T & 0 \end{bmatrix} \begin{bmatrix} I & R \\ 0 & V^T \end{bmatrix}^{-1} \quad \dots (15)$$

The vertex controller $K(\Pi_i)$ is designed from LMIs (16) for the closed-loop system $G(s, \Pi_i) = D_{cl}(\Pi_i) + C_{cl}(\Pi_i)(sI - A_{cl}(\Pi_i))B_{cl}(\Pi_i)$.

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \quad \dots (16)$$

Thus, the parameter-dependent controller $K(p)$ is made to schedule those controller parameters by the measurements of ϕ and $\dot{\phi}$ from Eq. (13). The robust stability of the closed-loop system is guaranteed by the small gain theorem when the norm of the exogenous input does not exceed $1/\gamma$.

4. Numerical Example

Finally, we show the results of a numerical simulation using Eqs. (1) and (2) of the gain-scheduled controller described so far. In order to evaluate its capability, we design two other types of controllers for comparison: a fixed LTI controller with a low-pass filter, and a gain-scheduled H_∞ controller without a low-pass filter. The fixed controller is designed as a standard H_∞ controller by using nominal values of the parameter vectors $p(t)$ of the plant (8). The gain-scheduled H_∞ controller without a low-pass filter is designed for another generalized plant constructed by directly inserting the disturbance input d without the filter dynamics of Eq. (4). The generalized plant is, by deleting the term of ζ from Eq. (7), as follows:

Fig. 3. Antenna driving pattern.

Table 1. Physical parameters of the satellite.

Symbol	Unit	Value
m_b, m_a	kg	1050, 155
d_a	m	3.6
l_1, l_2, l_3	m	2, 7, 2
J_{b1}, J_{b2}, J_{b3}	kg m ²	5400, 1600, 5400
J_{a1}, J_{a2}, J_{a3}	kg m ²	90, 90, 90
r_{01}, r_{02}, r_{03}	m	-1, 0, 1
r_{11}, r_{12}, r_{13}	m	0, 0, 0.6
r_{21}, r_{22}, r_{23}	m	-0.1, 0, 0
r_{31}, r_{32}, r_{33}	m	0, 0, 0.4

$$\begin{aligned}
 A(p) &= \begin{bmatrix} 0 & I \\ 0 & -M_0^{-1}H(p) \end{bmatrix} \\
 B_1 &= \begin{bmatrix} 0 \\ -M_0^{-1}\Sigma \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ M_0^{-1} \end{bmatrix} \quad \dots (17) \\
 C_1 &= [\lambda I \ 0] \quad D_{11} = 0 \quad D_{12} = 0 \\
 C_2 &= [I \ 0] \quad D_{21} = 0 \quad D_{22} = 0
 \end{aligned}$$

by defining state variable $x = [\theta^T \ \dot{\theta}^T]^T$, where the matrix Σ is

$$\Sigma = \text{diag}[\sigma, \sigma, \sigma] \quad \sigma = \max |T(\phi, \dot{\phi}, \ddot{\phi})|. \quad \dots (18)$$

For the numerical study, we use the driving pattern of the antenna shown in Fig. 3. This is supposed to make the antenna trace the target spacecraft and return to a initial position. In this case, the velocity command is generated by a trapezoid law. In our design, it is also assumed that the rotation of the antenna is made around roll and pitch axes concurrently according to the same driving pattern mentioned above.

The physical parameters of the satellite are set as shown in Table 1. These parameters are values of the neutral state where the antenna points toward the nadir, the center of the Earth, as in Fig. 1, and are used as nominal values of fixed LTI controller synthesis.

Fig. 4. Parameter variations of $H(\phi, \dot{\phi})$. Fig. 5. Parameter variations of $M(\phi)$.

The low-pass filter is designed as $\rho = 0.001$ to approximate integral function under the solvability condition of a convex optimization problem.

The vertexes of parameter vectors are the maximum and minimum values of the element of the matrix $H(p)$. The parameter variations of matrix $H(p)$ are shown in Fig. 4. The parameter variations of matrix $M(\phi)$ are shown in Fig. 5 as a reference. From this figure, the influence of the parameter variations by attitude motion is considered sufficiently small because the nominal values are much larger than the ranges of the parameter variations.

Two simulation results when the antenna is driven according to the pattern in Fig. 3 are shown in Figs. 6 and 7. In Fig. 6, we compare the performance of the proposed controller with that of a fixed controller. On the other hand, in Fig. 7, it is compared with the

Fig. 6. Time responses of spacecraft attitude 1: gain-scheduled controller (solid) and fixed controller (dashed).

Fig. 7. Time responses of spacecraft attitude 2: gain-scheduled controller with filter (solid) and without filter (dashed).

gain-scheduled controller without a low-pass filter in the generalized plant. Apparently, it is observed that the closed-loop system is stabilized by all controllers. Moreover, we also observed that the proposed gain-scheduled controller achieves higher performance than the fixed controller and the gain-scheduled controller without a low-pass filter. This shows the effectiveness of the usage of gain scheduling based on a filter accommodated generalized plant. The obtained controller, needless to say, has the capability of disturbance attenuation for all types of disturbance torque input other than that induced by the antenna rotation. To prove this, a simulation of a case where impulse disturbance is

Fig. 8. Time responses of spacecraft attitude 3: influence of impulse disturbance.

applied to the spacecraft during antenna rotation is performed. The results are shown in Fig. 8. In the simulation, the amplitude of the disturbance torque input is 1[Nm], and it is concurrently applied around three axes at 10[s] after the simulation starts. From the results, the capability of the gain-scheduled controller was confirmed.

5. Conclusion

In this paper, we have proposed the use of a gain-scheduled controller to stabilize a class of DRS. It is designed by H_∞ -like synthesis applying a LPV system. From the numerical results, it is shown that the gain-scheduled controller achieves higher performance than a single LTI controller by using a filter accommodated generalized plant. Since it utilizes the feedback control law, it can efficiently reduce the influence of disturbance torque to the spacecraft attitude even when the antenna rotates at high speed. A similar approach can be applied to other types of spacecraft equipped with rotating observation equipment or mobile appendages.

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Appendix

In the appendix, the derivation of dynamical Eq. (1) is outlined. The major vectors of the satellite are defined as follows (also in Fig. 9).

- $\mathcal{C}$: instantaneous mass center of total satellite system
- $\mathcal{C}_0$: nominal model mass center of total satellite system
- $\{i\}$: inertial frame
- $\{e\}$: frame of principal axes of inertia
- $\{b\}$: frame of main body, $\{b\} = \{e\}$
- $\{a\}$: frame of antenna
- $\{k_1\}$: frame of second link
- $\{k_2\}$: frame of third link, $\{k_2\} = \{a\}$
- m_b, m_a : masses of main body and antenna, $m = m_b + m_a$
- d_a : diameter of antenna
- l_1, l_2, l_3 : sizes of main body
- J_{b1}, J_{b2}, J_{b3} : moments of inertia of main body, $J_b = \{e\}^T J_b \{e\} = \{e\}^T \text{diag}[J_{b1}, J_{b2}, J_{b3}] \{e\}$
- J_{a1}, J_{a2}, J_{a3} : moments of inertia of antenna, $J_a = \{a\}^T J_a \{a\} = \{a\}^T \text{diag}[J_{a1}, J_{a2}, J_{a3}] \{a\}$
- r_{01}, r_{02}, r_{03} : vector components from mass center of main body to first link attachment point, $r_0 = \{e\}^T r_0 = \{e\}^T [r_{01} \ r_{02} \ r_{03}]^T$
- r_{11}, r_{12}, r_{13} : vector components of first link system, $r_1 = \{e\}^T r_1 = \{e\}^T [r_{11} \ r_{12} \ r_{13}]^T$
- r_{21}, r_{22}, r_{23} : vector components of second link system, $r_2 = \{k_1\}^T r_2 = \{k_1\}^T [r_{21} \ r_{22} \ r_{23}]^T$
- r_{31}, r_{32}, r_{33} : vector components of third link system, $r_3 = \{k_2\}^T r_3 = \{k_2\}^T [r_{31} \ r_{32} \ r_{33}]^T$
- $\theta_1, \theta_2, \theta_3$: angles of satellite attitude
- ϕ_1, ϕ_2 : angles of second and third links
- g_b, g_a : vectors from $\mathcal{C}$ to arbitrary points at main body and antenna
- q_b, q_a : vectors from $\mathcal{O}$ to arbitrary points at main body and antenna
- r_b, r_a : vectors from $\mathcal{O}$ to each mass center of main body and antenna, $r_b = \{e\}^T r_b, r_a = \{e\}^T r_a$
- c : vector from $\mathcal{O}$ to $\mathcal{C}$, $c = \{e\}^T c = \{e\}^T m^{-1}(m_b r_b + m_a r_a)$
- ε : vector from $\mathcal{O}$ to first link attachment point
- J : dyadic of inertia of satellite system
- ω : vector of angular velocity of $\{e\}$ with respect to $\{i\}$, $\omega = \{e\}^T \omega$

Fig. 9. Definitions of frames and vectors.

- ω_1 : vector of angular velocity of $\{k_1\}$, $\omega_1 = \{k_1\}^T [\dot{\phi}_1 \ 0 \ 0]^T$
- ω_2 : vector of angular velocity of $\{k_2\}$, $\omega_2 = \{k_2\}^T [0 \ \dot{\phi}_2 \ 0]^T$
- ω_t : $\omega_1 + \omega_2 = \{e\}^T \omega_t$

When the changes in satellite attitude angle and orbital angular velocity are assumed very small, the vector ω can be obtained as

$$\omega = \{e\}^T [\dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T$$

The coordinates transforms are defined as

$$\{k_1\} = R_1 \{e\}$$

$$\{k_2\} = R_2 \{k_1\} = R_2 R_1 \{e\}$$

where, matrices R_1, R_2 are the direction cosine matrices

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_1) & \sin(\phi_1) \\ 0 & -\sin(\phi_1) & \cos(\phi_1) \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \cos(\phi_2) & 0 & -\sin(\phi_2) \\ 0 & 1 & 0 \\ \sin(\phi_2) & 0 & \cos(\phi_2) \end{bmatrix}$$

Using the above notations, the angular momentum h of the satellite system is derived as

$$\begin{aligned} h &= \int_b g_b \times \dot{g}_b dm + \int_a g_a \times \dot{g}_a dm \\ &= \int_b q_b \times (\omega \times q_b) dm + \int_a q_a \times (\omega \times q_a) dm \\ &\quad - m(c \times \dot{c}) + \int_a g_a \times \dot{g}_a dm \\ &= J \cdot \omega - m(c \times \dot{c}) - m(c \times (\omega \times c)) \\ &\quad - m_a r_a \times (r_2 \times \omega_t) + J_a \cdot \omega_t \end{aligned}$$

where, “ $\dot{}$ ” and “ $\ddot{}$ ” are time derivatives in $\{i\}$ and $\{e\}$ frames. From the time derivative of $\mathbf{h}$ in frame $\{i\}$, the dynamical equation of the satellite is obtained as follows:

$$\begin{aligned}\dot{\mathbf{h}} = & \dot{\mathbf{J}} \cdot \boldsymbol{\omega} + \mathbf{J} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J} \cdot \boldsymbol{\omega} \\ & - m(\mathbf{c} \times \ddot{\mathbf{c}}) + m(\ddot{\mathbf{c}} \times (\boldsymbol{\omega} \times \mathbf{c})) \\ & + m(\mathbf{c} \times (\dot{\boldsymbol{\omega}} \times \mathbf{c})) + m(\mathbf{c} \times (\boldsymbol{\omega} \times \dot{\mathbf{c}})) \\ & + m(\boldsymbol{\omega} \times (\mathbf{c} \times \dot{\mathbf{c}})) + m(\boldsymbol{\omega} \times (\mathbf{c} \times (\boldsymbol{\omega} \times \mathbf{c}))) \\ & - m_a \dot{\mathbf{r}}_a \times (\mathbf{r}_2 \times \boldsymbol{\omega}_t) - m_a \mathbf{r}_a \times (\dot{\mathbf{r}}_2 \times \boldsymbol{\omega}_t) \\ & - m_a \mathbf{r}_a \times (\mathbf{r}_2 \times \dot{\boldsymbol{\omega}}_t) - m_a \boldsymbol{\omega} \times (\mathbf{r}_a \times (\mathbf{r}_2 \times \boldsymbol{\omega}_t)) \\ & + \dot{\mathbf{J}}_a \cdot \boldsymbol{\omega}_t + \mathbf{J}_a \cdot \dot{\boldsymbol{\omega}}_t + \boldsymbol{\omega} \times \mathbf{J}_a \cdot \boldsymbol{\omega}_t\end{aligned}$$

By describing all vectors and dyadics in frame $\{e\}$, we obtain the following:

$$\begin{aligned}\dot{\mathbf{h}} = & m_a(((R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\mathbf{r}}_a + \tilde{\mathbf{r}}_a((R_1^T R_2^T r_2) \tilde{\omega}_t)) \boldsymbol{\omega} \\ & + (J_b - m_b \tilde{\mathbf{r}}_b \tilde{\mathbf{r}}_b + R_1^T R_2^T J_a R_2 R_1 - m_a \tilde{\mathbf{r}}_a \tilde{\mathbf{r}}_a) \dot{\boldsymbol{\omega}} - (J \boldsymbol{\omega}) \tilde{\boldsymbol{\omega}} \\ & + m_a \tilde{\mathbf{c}}(R_1^T R_2^T r_2) \dot{\boldsymbol{\omega}}_t - m_a \tilde{\mathbf{c}}((R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\boldsymbol{\omega}}_t \\ & + \tilde{\mathbf{c}}(m_b \mathbf{r}_b + m_a \mathbf{r}_a) \dot{\boldsymbol{\omega}} - 2m_a \tilde{\mathbf{c}}((R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\boldsymbol{\omega}} - \tilde{\mathbf{c}}(\tilde{\mathbf{c}} \boldsymbol{\omega}) \boldsymbol{\omega} \\ & + m_a \tilde{\mathbf{r}}_a((R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\boldsymbol{\omega}}_t - m_a \tilde{\mathbf{r}}_a(R_1^T R_2^T r_2) \dot{\boldsymbol{\omega}}_t \\ & + m_a(\tilde{\mathbf{r}}_a(R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\boldsymbol{\omega}} \\ & + m_a(((R_1^T R_2^T r_2) \tilde{\omega}_t) \tilde{\mathbf{r}}_a + \tilde{\mathbf{r}}_a((R_1^T R_2^T r_2) \tilde{\omega}_t)) \boldsymbol{\omega}_t \\ & + R_1^T R_2^T J_a R_2 R_1 \dot{\boldsymbol{\omega}}_t - (R_1^T R_2^T J_a R_2 R_1 \boldsymbol{\omega}_t) \tilde{\boldsymbol{\omega}}\end{aligned}$$

where, “ $\tilde{}$ ” denotes an operator to make a 3×3 skew-symmetric matrix from a 3×1 matrix. This is compactly described as Eq. (1).