Novel Finite Particle Method for Gyrodynamics Analysis*1

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Abstract

Traditionally, Euler's equations are commonly employed for the analyses of interesting gyrodynamic problems. Nevertheless, they generally give no idea how these complicated gyroscopic forces are mutually interacting. Thus, we may risk missing some insights regarding working forces. Therefore, in this study, we present a so-called "finite particle method," which simulates a gyroscope by a small number of dynamically equivalent particles (usually 2 to 8 only) rigidly connected. This method largely degenerates complicated 3-D gyrodymanics to a particle dynamics problem in a rotating frame. The finite particle method has elegantly demonstrated its validity by successfully deriving the same steady gyrodynamic equations as that derived from Euler's approach, yet only in amazingly minimal steps for some cases. Surprisingly enough, by this method, one can swiftly understand some delicate gyrodynamic phenomena more deeply by merely inspecting centripetal and Coriolis forces particle by particle, without even knowing what angular momentum is. In short, this finite particle method is characterized by its simpler concept, succinct derivation, and possibly insightful understanding of intrinsic force interactions for some gyrodynamic problems as demonstrated in the retrograding phenomenon of disk-shaped satellites and other examples.

Nomenclature

\[ \begin{align*}
q & \quad \text{gravitational acceleration} \\
I_x & \quad \text{mass moment of inertia about } x \text{ axis} \\
I_y & \quad \text{mass moment of inertia about } y \text{ axis} \\
I_z & \quad \text{mass moment of inertia about } z \text{ axis} \\
k_c & \quad \text{radius of gyration of disk with respect to } cg \\
k_1 & \quad \text{radius of gyration of disk with respect to } m_1 \\
l & \quad \text{distance between } m_1 \text{ and } m_2 \\
l_1 & \quad \text{distance from } cg \text{ to } m_1 \\
l_2 & \quad \text{distance from } cg \text{ to } m_2 \\
l' & \quad \text{corresponding new distance from } cg \text{ to specified mass} \\
M & \quad \text{total system mass} \\
M_x & \quad \text{Moment about } x\text{-axis} \\
M_y & \quad \text{Moment about } y\text{-axis} \\
M_z & \quad \text{Moment about } z\text{-axis} \\
m & \quad \text{numbered masses} \\
p & \quad \text{rate of spinning} \\
r & \quad \text{position vector of a point} \\
R & \quad \text{position vector of the disk center } O' \\
r_{cg} & \quad \text{distance from fixed coordinate center to disk center} \\
\theta & \quad \text{angle of nutation} \\
\rho & \quad \text{position vector in rotating frame } e_i \\
\psi & \quad \text{rate of precession} \\
\omega & \quad \text{angular velocity of rotating frame } e_i
\end{align*} \]

1. Introduction

Gyroscopic motion is one of the most interesting behaviors in dynamics, and is well mentioned in literature1,2) from classic3,4,7) to modern8,9) as examples. A general survey10,11 shows that the traditional angular momentum approach for rigid bodies by solving associated Euler's equations is generally employed to address gyrodynamic problems. Although, this Euler's approach is highly valued for attaining equations for analyzing the characteristics of gyroscopic motion, it generally gives no clue as to how these complicated forces involved in gyrodynamics interact with each other. Thus, in some cases, we may miss the opportunity of seeing the insights of working forces. For example, taking the well-known problem of a disk-type satellite with steady precession under zero external moment,12) we have the following equation relating \( \dot{\psi} \) and \( p \) from Euler's analysis

\[ I_2 \ddot{\psi} p + (I_{22} - I_{33}) \dot{\psi}^2 \cos \theta = 0 \quad \ldots \quad (1) \]

This equation clearly shows that for \( I_{33} > I_{22} \) of a prolate pencil-shaped body, the precession will be directed in the sense that \( p \) and \( \dot{\psi} \) have the same sign. Otherwise, for \( I_{33} < I_{22} \) of an oblate disk-shaped satellite, the precession will be retrograde with respect to spinning angular velocity \( p \). However, from the mathematics of Eq. (1), one cannot physically understand why an oblate body should precess in a retrograde sense under no external moment effect. Alternately, in
the following study, by the so-called finite particle method (FPM) developed herein, we can clearly see why an oblate disk rotor must have retrograde precession to balance interacting forces, thus warranting a steady gyrodynamic motion with zero moment.

2. Finite Particle Method for Gyrodynamics

In contrast to the commonly learned method of deriving a gyroscopic equation of motion by the traditional Euler's approach, here we intend to derive the same dynamics equation differently by the finite particle method described in the following. Firstly, the thin disk in Fig. 1 is discretized into four dynamically equivalent particles connected by massless rods with a radius equal to $k_{cg}$ as shown in Fig. 2. At any instant, the four particles (named $m_1$ to $m_4$ respectively) with equal mass $m$ are always representing the original disk dynamically equivalent. The disk has rotating frame $e_i$ with its origin attached to disk center $O'$. Further, the disk has spinning angular velocity $\mathbf{\omega}$ with respect to rotating frame $e_i$, and the rotating frame has an angular velocity of $\mathbf{\omega} = \mathbf{\psi}$ with respect to the inertial frame. Also note that, in most figures of this paper, although for the purpose of generalization, all the moment signs $M_{xx}$, $M_{yy}$, and $M_{zz}$ are shown; in fact, they are commonly assigned zeros unless specially specified. Then the well-known acceleration equation with rotating reference axes can be expressed as

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \mathbf{\omega} \times (\dot{\mathbf{\omega}} \times \mathbf{r}) + 2\ddot{\mathbf{\omega}} \times (\dot{\mathbf{\omega}})_{rel} + \dot{\mathbf{\omega}} \times \ddot{\mathbf{r}} + \ddot{\mathbf{r}}_{rel}$$ (2)

where the first term on the right-hand side (rhs) $\ddot{\mathbf{R}}$ represents the absolute acceleration of disk center $O'$ and subscript rel means "relative to rotating frame $e_i".

For steady precession as assumed for this case, it is horizontal centripetal acceleration always toward precession center axis (Fig. 3). This acceleration can be expressed as

$$\ddot{\mathbf{R}} = (r_{cg}\sin \theta)\omega^2$$ (3)

Since the mass center of the four representative particles is also $O'$, the corresponding moment about the $x$-axis through inertial center $O$ needed for such an acceleration will be

$$- [4m(r_{cg}\sin \theta)\omega^2] (r_{cg}\cos \theta)
= - [(2mr_{cg}^2 + 2m(r_{cg}^2 + k_{cg}^2))
- 2mk_{cg}^2\omega^2 \sin \theta \cos \theta
= - I_{xx}\omega^2 \sin \theta \cos \theta + 2mk_{cg}^2\omega^2 \sin \theta \cos \theta$$ (4)
Fig. 4. Acceleration $\hat{\omega} \times (\hat{\omega} \times \hat{\rho})$ for four particles.

Notably, in the above mathematical manipulation, we have applied the well-known "transfer theorem for mass moment of inertia between parallel axes" for the pair of masses of $m_1$ and $m_2$ in term $2m(r_{eq}^2 + k_{eq}^2)$ together with $2mr_{eq}^2$ for $m_3$ and $m_4$ to obtain $I_{xx}$ in the last row according to its basic definition.

For the second rhs term of Eq. (2)

$$\hat{\omega} \times (\hat{\omega} \times \hat{\rho}), \quad \ldots \ldots (5)$$

the forces needed for centripetal acceleration for $m_3$ and $m_4$ particles will balance each other exactly as shown in Fig. 4. On the other hand, the force needed for the centripetal acceleration of $m_1$ or $m_2$ is

$$m(k_{eq} \cos \theta)\omega^2 \quad \ldots \ldots (6)$$

and the corresponding couple needed can be expressed as

$$2[m(k_{eq} \cos \theta)\omega^2](k_{eq} \sin \theta)$$

$$= 2mk_{eq}^2 \omega^2 \sin \theta \cos \theta. \quad \ldots \ldots (7)$$

Finally, the third rhs term of Eq. (2) is the Coriolis acceleration. The centripetal Coriolis forces for $m_3$ and $m_4$ will be colinear along the horizontal diameter and eliminate each other. However, the force needed for the centripetal Coriolis acceleration of $m_1$ or $m_2$ as shown in Fig. 5 is

$$2m\omega (pk_{eq}) \quad \ldots \ldots (8)$$

and the corresponding couple needed can be written as

$$2[2m\omega (pk_{eq})](k_{eq} \sin \theta)$$

$$= (4mk_{eq}^2)\omega \cos \theta$$

$$= I_{xx} \omega \cos \theta \quad \ldots \ldots (9)$$

where $I_{xx}$ is the polar mass moment of inertia with respect to the rotor spinning axis. It is worthy to note that the fourth rhs term of Eq. (2)

$$\hat{\omega} \times \hat{\rho} \quad \ldots \ldots (10)$$

will be nil since $\omega$ is constant for the steady case. Moreover, concerning the last rhs term of Eq. (1) $\vec{p}_{rel}$, the force needed for the acceleration of any one particle representing the disk rotor can be expressed as

$$mk_{eq}p^2. \quad \ldots \ldots (11)$$

However, the conjugate centripetal forces for $m_1$ and $m_2$ will cancel each other out as shown in Fig. 6. Likewise, the conjugate centripetal forces for the $m_3$ and $m_4$ pair will cancel each other as well. Thus, the last two terms of Eq. (2) disappear.

Summing up the last terms of Eq. (4) and Eq. (7), we can attain

Fig. 5. Coriolis acceleration $2\hat{\omega} \times (\hat{\rho})_{rel}$ for four particles.

Fig. 6. Acceleration $\vec{p}_{rot}$ for thin-disk gyroscope.
\[ 2m k_{c_g}^2 \omega^2 \sin \theta \cos \theta + 2m k_{c_g}^2 \omega^2 \sin \theta \cos \theta \]
\[ = 4m k_{c_g}^2 \omega^2 \sin \theta \cos \theta \]
\[ = I_{xx}^c \omega \sin \theta \cos \theta . \quad \ldots \quad (12) \]

With this understanding, we may add Eqs. (4), (7), and (9) to obtain
\[ \sum M_x = 4mg r_{d_g} \sin \theta \]
\[ = I_{zz} \omega \sin \theta - I_{xx} \omega^2 \sin \theta \cos \theta + I_{zz} \omega \sin \theta \cos \theta \]
\[ = I_{zz} \psi \sin \theta + (I_{zz} - I_{xx}) \psi^2 \sin \theta \cos \theta \quad \ldots \quad (13) \]

whereas \( \sum M_z \) and \( \sum M_y \) are zeros, Eq. (13) can be further simplified by clearing the common \( \sin \theta \) to become
\[ 4mg r_{d_g} = I_{zz} \psi - (I_{zz} - I_{xx}) \psi^2 \sin \theta . \quad \ldots \quad (14) \]

It is worthwhile to note that the rhs of Eq. (14) is the same as Eq. (1), which is derived from the traditional Euler's approach. This interesting result can be obtained only through delicate re-grouping terms as exemplified in Eq. (4). Although for the sake of a detailed debut of the finite particle method, the above explanations are somewhat lengthy, the true power of this novel method will be demonstrated in the following.

3. Finite Particle Method for Oblate Satellite with Steady Precession

In fact, the concepts involved in the finite particle method are very simple. Consider a satellite orbiting in space with zero external moment and retrograding precession. The origin of rotating frame \( O' \) is conveniently attached to its cg and coincides with inertial center \( O \). This disk-type satellite can be discretized into four dynamically equivalent particles as before. Referring to Eq. (2), we can readily remove the first rhs term \( \mathbf{R} \) due to the coincidence of origins \( O' \) and \( O \). As before, with constant \( \omega \), the fourth rhs term of Eq. (2) \( \dot{\mathbf{\hat{r}}} \times \mathbf{\hat{r}} \) will be zero, and the fifth rhs term of Eq. (2) \( \mathbf{\hat{r}} \times \dot{\mathbf{\hat{r}}} \) can be eliminated for like reasons following Eq. (11) and Fig. 4. Now, three of the five rhs terms are out, only the couple
\[ 2m k_{c_g}^2 \omega^2 \sin \theta \cos \theta \]
\[ = I_{xx} \psi \sin \theta \cos \theta \]
\[ = 4m k_{c_g}^2 \omega^2 \sin \theta \cos \theta - 2m k_{c_g}^2 \omega^2 \sin \theta \cos \theta \]
\[ = (I_{zz} - I_{xx}) \psi^2 \sin \theta \cos \theta . \quad \ldots \quad (15) \]

needed for centripetal acceleration \( \ddot{\mathbf{r}} \times (\ddot{\mathbf{r}} \times \mathbf{r}) \) for \( m_1 \) and \( m_2 \) as shown in Fig. 7, and the following couple needed for Coriolis acceleration \( 2m \omega (p k_{c_g}) \) as shown in Fig. 8 still exist:
\[ 2[2m \omega (p k_{c_g})] (k_{c_g} \sin \theta) \]
\[ = (4m k_{c_g}^2 \omega) \sin \theta \cos \theta \]
\[ = I_{xx} p \psi \sin \theta . \quad \ldots \quad (16) \]

With only Eqs. (15) and (16) for centripetal and Coriolis accelerations, respectively, we can swiftly obtain the following well-known expression as usually derived by Euler's approach through lengthy and complicated physical and mathematical processes.
\[ \dot{\psi} = \frac{I_{zz} \psi}{(I_{zz} - I_{xx}) \cos \theta} . \quad \ldots \quad (17) \]

It can be clearly seen from Figs. 7 and 8, physically, that a steady precession for the zero-moment satellite can be maintained only if precession \( \dot{\psi} \) is in a retrograde sense with respect to spinning angular velocity \( \psi \), otherwise the forces needed for centripetal and Coriolis accelerations (think \( m_1 \) for example) have no chance to cancel each other. In fact, if not for the purpose of deriving Eq. (17) by the finite particle method formally, we do not have to bother the moment aspects and can immediately understand the retrograding phenomenon simply by inspecting the directions of centripetal and Coriolis accelerations of \( m_1 \) and \( m_2 \) particle by particle (Figs. 7 and 8).

4. Finite Particle Method for Two-Bladed Airplane Propeller

Furthermore, consider an even simpler case as shown in Fig. 9, where a thin two-bladed propeller has constant spinning velocity \( \dot{\psi} \), precession rate \( \psi \) and nutation angle \( \theta = (\pi/2) \). This thin propeller can be conveniently simulated by two dynamically equivalent particles as commonly expected for a typical slender rod. The two particles are designated \( m_1 \) and \( m_2 \), and located at distances \( l_1 \) and \( l_2 \) from \( O' \), respectively.

The propeller has rotating frame \( e_i \), with its origin attached to propeller center \( O' \) and rotating at an angular velocity of \( \dot{\omega} = \dot{\psi} \) with respect to the inertial frame. Here, the \( x-y-z \) coordinate system is assumed to be attached on the propeller for convenience. The mass moment of inertia \( I_{xx} \) of a thin propeller can be easily obtained by
compound pendulum experiments. If $I_{xx}$ is equal to $I_{yy}$, $I_{zz}$ can be assumed negligible for the thinness of the propeller. In order to discretize the propeller into two dynamically equivalent particle masses $m_1$ and $m_2$, it is generally necessary to satisfy the following conditions:

1. The total mass remains the same.
   \[ m_1 + m_2 = 2m = M \]  \hspace{1cm} \ldots (18)

2. The system center of gravity remains in the same position.
   \[ m_1 l_1 = m_2 l_2 \]  \hspace{1cm} \ldots (19)

3. The mass moment of inertia remains the same (according to definition).
   \[ m_1 l_1^2 + m_2 l_2^2 = I_{xx} = I_{zz} = Mk_{cg}^2 = 2mk_{cg}^2 \]  \hspace{1cm} \ldots (20)

To satisfy Eqs. (18) and (19), Eq. (20) shall be expressed as

\[ \frac{l_2}{l_1 + l_2} M l_1^2 + \frac{l_1}{l_1 + l_2} M l_2^2 = I_{xx} = 2mk_{cg}^2 \]  \hspace{1cm} \ldots (21)

After simplifying, we obtain the following important conditional equation for dynamical equivalence

\[ l_1 l_2 = k_{cg}^2 \]  \hspace{1cm} \ldots (22)

with

\[ m_1 = \frac{l_2}{l_1 + l_2} 2m \]  \hspace{1cm} \ldots (23)

and

\[ m_2 = \frac{l_1}{l_1 + l_2} 2m \]  \hspace{1cm} \ldots (24)

Above, interestingly, Eq. (22) is actually the common percussion equation but only in a different format. This point can be quickly proven by the transfer theorem for mass moment of inertia between parallel axes in the following:

\[ M l_1 l_2 = Mk_{cg}^2 + M(l_1)^2 - M(l_2)^2 \]
\[ M l_1 l_2 = Mk_1^2 - M(l_2)^2 \]
\[ M l_1(l_2 + l_1) = Mk_1^2 \]
\[ l_1 l_2 = k_{cg}^2. \]  \hspace{1cm} \ldots (25)

Equation (25) is the commonly seen equation relating distance from the center of oscillation to system $cg$, radius of gyration, and the center of percussion.\(^ {15,16}\) It shows that the locales of $m_1$ and $m_2$ are the center of percussion and center of oscillation mutually and interchangeably. A detailed background is clearly explained in Ref. 16, thus not repeated here. As a reminder, $l$ in Eq. (25) is the distance between $m_1$ and $m_2$.

Perhaps, the simplest way to satisfy Eq. (22) is by assigning

\[ l_1 = l_2 = k_{cg}. \]  \hspace{1cm} \ldots (26)

Thus, the two particles have equal mass and distance $k_{cg}$ from $O'$, respectively. Next, let us study the moment about the $x$-axis. An examination of Eq. (2) shows that all rhs terms are out except the Coriolis term as shown in Fig. 10. Then, the corresponding moment about the $x$-axis can be expressed as

\[ \sum M_x = 2(2m \omega \phi_0 k_{cg} \sin \phi)k_{cg} \]
\[ = (4mk_{cg}^2) \psi \phi \sin \phi \]
\[ = (I_{xx} + I_{zz}) \psi \phi \sin \phi. \]  \hspace{1cm} \ldots (27)

While $\sum M_x$ is zero by an inspection of Eq. (2), the only term left for $\sum M_x$ is due to centripetal acceleration $\vec{a} \times (\vec{a} \times \vec{b})$ for $m_1$ and $m_2$ as shown in Fig. 11:
5. Finite Particle Method for More General Cases

For a relatively thick disk-type gyroscope (Fig. 12), or a more general one as shown in Fig. 13, the finite particle method can be applied as easily as for the thin-disk gyroscope discussed in section 2. The general axisymmetric gyroscope of Fig. 13 can be discretized into a dynamically equivalent five-particle system on the eg plane connected by massless rods as shown in Fig. 14. These five particles (with four on the circle assuming equal mass) are necessary to satisfy the following three conditions:

1. The mass moment of inertia \( I_{zz} \) remains the same.
   \[
   4m_1I_1^2 = I_{zz} \quad \ldots (29)
   \]

2. The mass moment of inertia \( I_{xx} \) remains the same.
   (Applying the familiar transfer theorem as explained following Eq. (4) again.)

Again, by the finite particle method, with only two simple steps, we immediately obtain Eqs. (27) and (28). They are the same set of equations as those obtained by Euler’s approach as expressed on p. 588 of Ref. 14, except that \( \sum M \) in the reference can be assumed negligible for a thin blade. Incidentally, it is worthwhile to note that the constantly varying \( \sin \phi \) involved in the above two equations is primarily responsible for the familiar buzzing of warplanes when they take swift maneuvers of rolling and yawing before attacking a ground target. This kind of scenes is often seen in war movies of WWII and earlier.
\[2m_1l_1^2 + (4m_1 + m_3)r_{cg}^2 = I_{xx} \quad \ldots \quad (30)\]

3. The total mass remains the same.
\[4m_1 + m_3 = M \quad \ldots \quad (31)\]

Noticably, the condition for \( cg \) remaining at the same point will be automatically satisfied for the above situation since the five particles are on the \( cg \) plane, where three unknowns, \( m_1, l_1, \) and \( m_3 \), are to be solved for the above three equations. Once solved, we can subsequently apply the finite particle method for these five particles to obtain \( M_x \) with the aid of Eq. (2) as usual; the only difference is the additional consideration of \( m_5 \).

Alternately, referring to Fig. 15 with \( m_5 \) being moved to fixed origin point \( O \) and renamed \( m_6 \) for distinguishing purposes, one may prefer such a slightly different approach by satisfying the following four dynamically equivalent conditions:

1. The mass moment of inertia \( I_{zz} \) remains the same.
\[4m_1l_1^2 = I_{zz} \quad \ldots \quad (32)\]

2. The mass moment of inertia \( I_{xx} \) remains the same. (Transfer theorem again.)
\[2m_1l_1^2 + 4m_1(r_{cg} + \rho_O)^2 = I_{xx} \quad \ldots \quad (33)\]

3. The system center of gravity remains in the same position.
\[4m_1\rho_O = m_6r_{cg} \quad \ldots \quad (34)\]

4. The total mass remains the same.
\[4m_1 + m_6 = M \quad \ldots \quad (35)\]

Above, \( \rho_O \) is the distance between \( O' \) and \( cg \), and \( r_{cg} \) is the distance from \( O \) to \( cg \) as shown in Fig. 15, whereas four unknowns, \( m_1, l_1, m_6, \) and \( \rho_O \), are to be solved for the above four equations. Once solved, we can discard \( m_6 \) since it is assigned to fixed point \( O \), and will thus not have any moment effect. This means a general gyroscope as shown in the example of Fig. 13 can still be dynamically and equivalently simulated by four equal particles (Fig. 15), but generally not on a \( cg \) plane, as the thin-disk case. Consequently, the exact same procedure as the finite particle method as illustrated in section 2, with reference to Figs. 1–6, can be applied to obtain \( M_x \) with the aid of Eq. (2), as usual. Note that as a prelude for the finite particle method, either of the two sets of Eqs. (29)–(31) or Eqs. (32)–(35) can be solved easily.

Likewise, a drum-type satellite can be simulated by six dynamically equivalent particles, as shown in Fig. 16, by satisfying the following three conditions:

1. The mass moment of inertia \( I_{zz} \) remains the same.
\[ 4m_1l_1^2 = I_{zz} \quad \ldots \quad (36) \]

2. The mass moment of inertia \( I_{xx} \) remains the same.

\[ 2m_1l_1^2 + 2m_3l_3^2 = I_{xx} \quad \ldots \quad (37) \]

3. The total mass remains the same.

\[ 4m_1 + 2m_3 = M \quad \ldots \quad (38) \]

For Fig. 16, we assume that \( m_1 \) to \( m_4 \) are equal, \( m_4 = m_6 \), and \( l_4 = l_6 \). Then, with assumed \( l_4 \) appropriately, there are three unknowns \( m_1, l_1 \) and \( m_6 \), to be solved for the above three equations. Similarly, the finite particle method can be conveniently applied to obtain \( M_{xx} \) with the aid of Eq. (2), as usual.

In fact, no matter whether the rigid body under study is a gyroscope, a satellite, or any other different shape, we can always replace it by the appropriate finite particle model with six equal masses grouped in three diametrical pairs along three principal axes of the body, respectively. In that regard, any general rigid body can be simulated by six dynamically equivalent particles with equal masses similar to Fig. 16 by satisfying the following four conditions:

1. The mass moment of inertia \( I_{zz} \) remains the same.

\[ 2ml_1^2 + 2ml_3^2 = I_{zz} \quad \ldots \quad (39) \]

2. The mass moment of inertia \( I_{xx} \) remains the same.

\[ 2ml_1^2 + 2ml_5^2 = I_{xx} \quad \ldots \quad (40) \]

3. The mass moment of inertia \( I_{yy} \) remains the same.

\[ 2ml_2^2 + 2ml_4^2 = I_{yy} \quad \ldots \quad (41) \]

4. The total mass remains the same.

\[ 6m = M \quad \ldots \quad (42) \]

There are four unknowns, \( m, l_1, l_3, \) and \( l_5 \), to be solved for the above four equations to set up the particle model. Subsequently, the finite particle method can be conveniently applied to obtain the complete set of moment equations with the aid of Eq. (2), as usual. Note that each diametrical pair of particles may be conveniently assumed equally located from \( cg \), although this condition of equal locale for diametrical pairs is not necessary as long as each pair of particles can satisfy a condition corresponding to percussion Eq. (22). Taking the \( m_5 \) and \( m_6 \), pair, for example, as long as the total mass and \( cg \) of the pair remains unchanged and the following equation is satisfied

\[ l_5^* l_6^* = l_5 l_6 \quad \ldots \quad (43) \]

the new locates of this pair are still acceptable; notably in Eq. (43), \( l_5^* \) and \( l_6^* \) represent the new distances from \( cg \) to \( m_5 \) and \( m_6 \), respectively.

6. Conclusion

Traditional Euler’s equations for the analysis of gyrodyamics generally reveal no information as to how those complicated gyrodyamic forces mutually interact. Therefore, we may risk missing some insights regarding working forces. Alternately, giving the example of an oblate disk-shaped satellite, if processed by the novel finite particle method presented in this study, we can clearly see why its precession should be retrograde with respect to its spinning motion in order to balance the interacting centrifugal and Coriolis couples, thus warranting a steady gyrodynamic motion with zero moment. In a touch, the finite particle method has elegantly demonstrated its validity by successfully deriving the same gyrodyamics Eq. (14) as that derived from Euler’s approach. Notably, Eq. (14) is obtainable only through delicate re-grouping terms as exemplified in Eq. (4) and other associated equations. These explanations are somewhat lengthy for the sake of a detailed debut. In fact, if not for demonstrating the same Eq. (17) by FPM, we don’t have to bother deriving the associated couples, and can immediately understand the retrograding phenomenon by simply inspecting directions of centripetal and Coriolis accelerations of \( m_1 \) and \( m_2 \), particle by particle, as shown in Figs. 7 and 8. Although somewhat outlandish, this crystal clear insight of the relatively complicated retrograde phenomenon may be more appropriately described as gyrodyamics “anatomy,” rather than gyrodyamics “analysis.”

Except for their academic significance, the amazingly clear insight and simplicity, with only two steps for understanding this retrograding phenomenon, have also demonstrated their greatly welcomed educational values in the classroom. Surprisingly, students can swiftly understand some delicate gyrodyamics phenomena more deeply without even knowing what angular momentum is. Therefore, the finite particle method and Euler’s approach can be viewed as complementing each other for better understanding complicated gyrodyamics problems. Likewise, the phenomenon of “must-be-direct” procession for a pencil-shaped satellite can be swiftly explained by a similar approach with an 8-particle model for instance (left to readers for having some fun).

This FPM method can be thought to be conceptually extended from the highly useful two-particle dynamically equivalent model for planar dynamics of an engine connecting rod to 3-D gyrodyamics motion. Through this innovative methodology, we can discretize a thin-disk gyroscope into a dynamically equivalent four-particle model, and a propeller into a two-particle model, a general gyroscope into a five-particle model, and a drum-type satellite into a six-particle model. Naturally, for these models, there is great freedom for
choosing convenient particle locales as explained in Eqs. (22) and (43). In the future, for structural optimization, the finite particle method may somehow provide better information for internal stress analyses of spacecraft and other objects in gyroscopic motion while being viewed as a system of more particles. In conclusion, the finite particle method, by simulating a rigid body through a small number of dynamically equivalent particles rigidly connected, largely degenerates complicated 3-D rigid body gyrodynamics to a particle dynamics problem in a rotating coordinate frame. This finite particle method is characterized by its simpler concept, succinct derivation, and possibly an insightful understanding of intrinsic force interactions for some gyrodynamics problems as demonstrated.

References