A New Optimal Guidance Law for Air-to-Air Missiles against Evasive Targets*¹

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Abstract

A new optimal guidance law (OPG) for air-to-air missiles, based on the design concept of decreasing the acceleration requirement commanded in the final phase of engagement, is presented. The closed-form solution of the OPG is derived analytically from the time-varying linear state equations composed of the line-of-sight angle and line-of-sight rate. The optimal proportional navigation law and augmented proportional navigation law, where both the navigation constants are 3, have been proven to be simplified versions of the OPG. The performance of the OPG is evaluated and compared with that of the true proportional navigation law (TPN) and augmented proportional navigation law (APN) in terms of the miss distance, interception time and energy expenditure.

1. Introduction

Proportional navigation law has been widely implemented for the terminal guidance in most of tactical homing missiles. It is well-known that the proportional navigation law gives the best results against non-maneuvering targets. However, while dealing with smart targets, the performance of the proportional navigation scheme may degrade sharply. Therefore, various optimal guidance laws for missiles have been suggested extensively for many years in an attempt to replace proportional navigation law in order to intercept highly maneuverable targets. Most optimal guidance laws have been derived from linear quadratic optimal control theory to obtain feedback-form solutions, and many of them take target maneuvering into account to cope with highly maneuverable targets.¹⁻⁶⁾ However, the main drawback of the linearized approach is that the results only apply to a relatively small region of the stated space where trajectory linearization is valid. In addition, many optimal guidance laws of proportional navigation have been proposed to enlarge the capture area based on employing different control variables.^{7,8)}

In a significantly different approach, an optimal solution to the guidance problem for a maneuvering target, using the exact nonlinear equations of motion in relative polar coordinates, was derived by Guleman and Shinar.⁹⁾ The optimality criterion is a weighted combination of the time of interception for a given miss distance and the expenditure of maneuvering energy. The equations of motion were solved in a closed form in terms of elliptic integrals. However, the turn direction of the

resulting control law must be determined by the final angular values of the engagement. Unfortunately, the turn direction is not clear in advance. Hence, they defined the sign of the control as a positive direction in order to solve the optimal guidance problem. Obviously, the definition limited the scope of the guidance law application.

The purpose of the present study is to derive an analytical solution for an optimal guidance law based on the design concept of decreasing the acceleration requirement commanded in the final phase of engagement. Different from the conventional approach, we avoid handling the nonlinear equations of motion and try to solve the guidance problem from the equation of line-of-sight (LOS) rate. It is assumed that a perfect knowledge of the target motion is available to the missiles. The performance index is a weighted combination of the time of interception and the control energy expenditure. The capture criterion is defined by nullifying the LOS rate in the final time of engagement (i.e., the missile can go straight to strike the target without any control energy expenditure).

The rest of the paper is organized as follows: In Section 2, the optimal guidance control problem is formulated by linear time-varying state equations. The optimization problem is then solved in Section 3 and the implementation of the resulting optimal guidance scheme is analyzed. In Section 4, the results of the simulation studies conducted to evaluate the performance of the OPG and a comparison with that of the TPN and APN schemes are presented. Discussions based on the simulated results are also included. Finally, we draw a

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Fig. 1. Intercept geometry.

conclusion in Section 5.

2. Problem Formulation

The intercept geometry of a missile intercepting a target is shown in Fig. 1. M and T represent the actual position of the missile and target. The two-dimensional engagement model can be expressed by the following equations:

$$\dot{\theta} = [V_M \sin \beta_M - V_T \sin \beta_T] / R, \quad \theta(0) = \theta_0 \quad \dots (1)$$

$$\dot{R} = -V_M \cos \beta_M + V_T \cos \beta_T, \quad R(0) = R_0 \quad \dots (2)$$

$$\beta_M = \theta - \gamma_M, \quad \dots (3)$$

$$\beta_T = \theta - \gamma_T, \quad \dots (4)$$

$$\dot{\gamma}_M = \frac{u}{V_M}, \quad \gamma_M(0) = \gamma_{M0} \quad \dots (5)$$

$$\dot{\gamma}_T = \frac{a_T}{V_T}, \quad \gamma_T(0) = \gamma_{T0} \quad \dots (6)$$

where R is the relative distance between the missile and target, V_M and V_T are the velocities of the missile and the target, respectively, θ is the angle of the line-of-sight, β_M and β_T are the angles between the line-of-sight and the current heading of the missile and target, and a_T and u are the lateral accelerations of the target and missile, while γ_M and γ_T are the flight directions of the missile and the target with respect to the inertial reference system. It is assumed that V_T and a_T are known in this study.

If we formulate the optimal guidance problem from Eqs. (1)–(6), we have to handle the exact nonlinear differential equations, and that will be difficult in solving the problem analytically. Hence, we find a new approach to treat the optimal guidance problem via Eq. (1). By differentiating Eq. (1), we obtain the time-varying linear differential equation as

$$\dot{\theta} = A(t)\theta + B(t)u + C(t) \quad \dots (7)$$

where

$$A(t) \triangleq -2 \frac{\dot{R}}{R}$$

$$B(t) \triangleq -\frac{\cos \beta_M}{R}$$

$$C(t) \triangleq \frac{a_T \cos \beta_T}{R}.$$

Defining the states as $\theta = x_1$ and $\dot{\theta} = x_2$, the state equations can be written as follows.

$$\dot{x}_1 = x_2, \quad x_1(0) = x_{10} \quad \dots (8)$$

$$\dot{x}_2 = A(t)x_2 + B(t)u + C(t), \quad x_2(0) = x_{20} \quad \dots (9)$$

Hence, the problem which needs to be solved is to find a control u such that the capture condition is achieved by

$$x_2(t_f) = 0 \quad \dots (10)$$

while minimizing the performance index J

$$J = t_f + \rho \int_0^{t_f} u^2 dt. \quad \dots (11)$$

The first term in J is a measure of the interception time. The second term in J is a measure of the required acceleration command, and ρ is the weighting factor reflecting the relative importance of the commanded acceleration with respect to the interception time. In general, for short ranges, where the time is paramount, the value of ρ has to be zero; for long ranges, where the control energy should be saved, a larger ρ can be chosen. The capture condition in Eq. (10) means that the missile is driven on a constant bearing course in the final phase of the engagement; i.e., the missile can fly straight to hit the target without any control effort. Therefore, such a problem leads to a time-varying linear optimization problem.

3. Optimal Control Solution

The optimization problem formulated in the previous section can be solved by applying the maximum principle.⁶⁾ Therefore, we now write the Hamiltonian H for the system in Eqs. (8)–(11):

$$H = 1 + \rho u^2 + \lambda_1 x_2 + \lambda_2 (A x_2 + B u + C) \quad \dots (12)$$

where λ_1 and λ_2 are the adjoint variables. These adjoint variables are defined by the equations

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \quad \text{and} \quad \dots (13)$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -A \lambda_2 - \lambda_1, \quad \dots (14)$$

while their boundary values are derived from their transversality conditions:

$$\lambda_1(t_f) = 0 \quad \dots (15)$$

$$\lambda_2(t_f) = \psi, \quad \dots (16)$$

where ψ is an unknown constant.

By solving Eqs. (13) and (14), we get:

$$\lambda_1 = 0 \quad \dots (17)$$

$$\lambda_2 = \psi \exp \left[\int_t^{t_f} A dt \right] \quad \dots (18)$$

Defining $f(t) \triangleq \exp \left[\int_t^{t_f} A dt \right] = R^2/R_f^2$, we then have

$$\lambda_2 = \psi f(t). \quad \dots (19)$$

Next, the control function $\partial H/\partial u = 0$ gives

$$u = \frac{-B\lambda_2}{2\rho}. \quad \dots (20)$$

Substituting Eq. (19) into Eq. (20) yields

$$u = \frac{-\psi B f(t)}{2\rho}. \quad \dots (21)$$

Thus, Eq. (9) becomes

$$\dot{x}_2 = Ax_2 - \psi B^2 f(t)/2\rho + C. \quad \dots (22)$$

Then, by solving Eq. (22), we get the following result.

$$x_2(t) = \frac{f(0)}{f(t)} \left\{ \int_0^t [C - \psi B^2 f(t)/2\rho] \frac{f(0)}{f(0)} dt + x_{20} \right\}. \quad \dots (23)$$

Equation (23) has to satisfy the capture criterion in Eq. (10); thus, we derive

$$x_2(t_f) = \frac{f(0)}{f(t_f)} \left\{ \int_0^{t_f} [C - \psi B^2 f(t)/2\rho] \frac{f(t)}{f(0)} dt + x_{20} \right\} = 0. \quad \dots (24)$$

After some arrangements, we obtain

$$\psi = \frac{x_2(0) + \int_0^{t_f} C \frac{f(t)}{f(0)} dt}{\frac{1}{2\rho} \int_0^{t_f} B^2 \frac{f(t)^2}{f(0)} dt}. \quad \dots (25)$$

By substituting Eq. (25) into Eq. (21) and changing the variable from 0 to t , we then derive the resulting control law as follows.

$$u(t) = \frac{R^3 \cos \beta_M \dot{\theta} + R \cos \beta_M \int_t^{t_f} R(a_T \cos \beta_T) dt}{\int_t^{t_f} R^2 \cos^2 \beta_M dt}. \quad \dots (26)$$

Surprisingly, weighting factor ρ in Eq. (26) disappears. Now, we interpret the resulting control law. First, if the missile intercepts a nonmaneuvering target, Eq. (26) becomes

$$u(t) = \frac{R^3 \cos \beta_M \dot{\theta}}{\int_t^{t_f} R^2 \cos^2 \beta_M dt} = -K(t) \dot{R} \dot{\theta} \quad \dots (27)$$

where

$$k(t) \triangleq \frac{-R^3 \cos \beta_M}{\dot{R} \int_t^{t_f} R^2 \cos^2 \beta_M dt}.$$

Equation (27) shows that, under such a condition, the OPG is nothing but a proportional navigation law with a time-varying navigation gain $K(t)$. Second, while against a maneuvering target, the OPG must make use of the extra information, namely, the knowledge of target maneuvers and its flight direction. This explicit knowledge enables the missile to maneuver in a more efficient manner. Third, since the OPG is independent of value ρ , the OPG is not only a time optimal guidance law but also an energy optimal one.

Moreover, for guidance implementation, the values of the time to go t_{go} and integration parts in Eq. (26) remain to be determined. First, the time to go t_{go} is calculated approximately from the following equation.

$$t_{go} = t_f - t \cong -\frac{R(t)}{\dot{R}(t)}. \quad \dots (28)$$

Second, the values of the integration parts in Eq. (26) are for predicting the future trajectory of the target and missile. These values can be obtained by solving together Eqs. (1)–(6), (26) and (28) with a numerical method, and then the optimal control law can be implemented.

In order to gain an insight into the characteristics of the OPG, we now simplify Eq. (26). Assuming that $\beta_M \cong 0$, $\beta_T \cong 0$ and relative closing velocity V_c is constant, we then obtain

$$\cos \beta_M \cong 1 \quad \dots (29)$$

$$\cos \beta_T \cong 1 \quad \dots (30)$$

$$R = -\dot{R}(t_f - t) = V_c(t_f - t). \quad \dots (31)$$

Substituting Eqs. (29)–(31) into Eq. (26) yields the surprising result as

$$u = \frac{V_c^3(t_f - t)^3 + V_c(t_f - t) \int_t^{t_f} V_c(t_f - t) a_T dt}{\int_t^{t_f} V_c^2(t_f - t)^2 dt} \dot{\theta} = 3V_c \dot{\theta} + \frac{3}{2} a_T. \quad \dots (32)$$

In fact, Eq. (32) is the optimal augmented proportional navigation law with the navigation constant of 3. If the target does not execute any maneuver, Eq. (32) becomes

Table 1. Simulation data.

R	5000 m
θ	0 rad
V_M	600 m/s ²
V_T	400 m/s ²
γ_T	2.09 rad
γ_M	0.96 rad

$$u = 3V_c\dot{\theta}. \quad \dots (33)$$

Equation (33) is the optimal proportional navigation law with the navigation constant of 3. The results are similar to those proposed by Bryson and Zarchan.^{6,10)} Therefore, the optimal augmented proportional navigation law and the optimal proportional navigation law are the simplified versions of the OPG and only valid for the trajectory linearization condition.

4. Simulation Results and Discussion

4.1. Simulation scenarios In order to evaluate the performance of the proposed optimal guidance law encountering highly maneuverable targets, simulation studies were conducted and are presented here for three different engagement scenarios. In Scenario 1, the target always performs a level flight. In Scenario 2, the target flies straight at the beginning of the interception and, 2 s later, executes a 9 G left turn and maintains the maneuver till the end of the interception. In Scenario 3, the target first initiates a 9 G right turn and, 3 s later, takes a sustained 9 G left turn till the end of the interception. In this deterministic study, we assume no lag in the guidance system and all data needed for the implementation of the guidance laws are available without any measurement errors. The remaining simulation data are listed in Table 1.

For the purpose of comparison, the following versions of the TPN and APN guidance laws are employed in the simulation studies.

$$u_{TPN} = -N\dot{R}\dot{\theta} \quad \dots (34)$$

$$u_{APN} = -N\dot{R}\dot{\theta} + \frac{N}{2}a_T \quad \dots (35)$$

Here, u_{TPN} and u_{APN} denote the commanded accelerations with the TPN and APN, respectively. To carry out a comparative study, the optimal navigation constants N of 3 are selected in both schemes. In addition, the maximum normal acceleration of the missile is limited to 30 G (1G = 9.8 m/s²).

4.2. Simulation results The simulation results in terms of the miss distance, interception time and energy expenditure data for the three engagement scenarios are listed in Table 2. It is seen that the interception time taken by the OPG is the shortest among the three

Table 2. Simulation results.

Scenario	Performance	TPN	APN	OPG
1	Miss distance (m)	0.47	0.47	0.08
	Interception time (s)	7.51	7.51	7.34
	Energy expenditure (m ² /s ³)	46305	46305	16691
2	Miss distance (m)	114.90	0.61	0.26
	Interception time (s)	6.67	6.44	6.28
	Energy expenditure (m ² /s ³)	230100	103270	60618
3	Miss distance (m)	95.44	0.56	0.33
	Interception time (s)	8.89	8.67	8.24
	Energy expenditure (m ² /s ³)	162990	157970	18316

Fig. 2. The missile-target trajectories of Scenario 1.

Fig. 3. The missile acceleration history of Scenario 1.

guidance schemes. Also, the miss distance and energy expenditure of the OPG are the least. Here, the energy expenditure is defined by integrating the square of the commanded acceleration during the elapsed time. Figures 2, 4 and 6 describe the engagement trajectories of the missile and target for three simulation scenarios, re-

Fig. 4. The missile-target trajectories of Scenario 2.

Fig. 6. The missile-target trajectories of Scenario 3.

Fig. 5. The missile acceleration history of Scenario 2.

Fig. 7. The missile acceleration history of Scenario 3.

spectively. Figures 3, 5 and 7 show the commanded acceleration histories of the missile for corresponding to the three scenarios, respectively.

4.3. Discussion In Scenario 1, the missile intercepts a nonmaneuvering target. It is seen clearly that the OPG makes a hard turn at the beginning of the engagement and then travels straight to hit the target. Against a nonmaneuvering target, the OPG is, in fact, the proportional navigation law with a time-varying navigation constant. This enables the missile to maneuver in a more efficient way to hit the target than the TPN with the fixed navigation constant. Therefore, the OPG yields the least miss distance, energy expenditure and shortest interception time among the three guidance schemes.

Scenario 2 can be described that the target makes a hard turn strategy to evade the missile attack. In this study, we verify that the strategy can avoid the attack of the missile guided by the proportional guidance law. The reason is that the sudden hard turn always saturates the commanded acceleration of the missile and then

results in a large miss distance. Hence, it gives the target a chance to escape from the destruction. On the other hand, the APN can respond instantaneously as it senses the target maneuver; thus, the APN drives the missile to intercept the evasive target in a more efficient manner. Hence, the APN achieves less miss distance than the TPN. Next, we see that the OPG demonstrates its advantage against a hard turn target. This is because the OPG not only has an extra term to account for the current target maneuver but can also predict the future target status. Moreover, the OPG ensures a decreasing acceleration demand which almost reduces to zero as the missile approaches the target.

In Scenario 3, the missile intercepts a jink-like maneuvering target. Figure 6 shows that even against the smart target maneuver, the OPG still drives the missile heading into the collision course as rapidly as possible; thereafter, the geometry is nearly a straight-line course to hit the target. In this case, Fig. 7 shows that the required acceleration of the OPG decreases monotonic-

ally with time in the final phase of engagement and approaches zero at the time of impact. The result justifies the capture criterion of the proposed guidance law. Furthermore, it is seen that the TPN requires an increasing normal acceleration and always leads to saturation in the final phase of engagement. However, in a realistic environment, missiles often run out of fuel in the duration and performing such a large normal acceleration is usually not feasible. This is the reason why the TPN scheme gives poor performance in the presence of target maneuvers.

5. Conclusion

Based on minimizing the linear combination of the interception time and energy cost, a time-varying linear optimal guidance problem has been solved analytically in a closed form. The proposed guidance law is expressed as an explicit function of the current and future behavior of both the missile and target. The resulting guidance law is independent of the value of weighting factor ρ ; thus, it is not only a minimum time guidance law but also a minimum energy expenditure law. In addition, we obtained that the optimal TPN and APN, where both the navigation constants are 3, are the special cases of the OPG.

The performance of the OPG is demonstrated by three engagement scenarios in comparison with the APN and TPN. Simulation results verify that the OPG, against a nonmaneuvering target or even a highly evasive target, yields the best performance in terms of the miss distance, homing time duration and energy expenditure among the three schemes. Besides, the OPG shows a desirable property needed for the guidance law design, wherein, command acceleration decreases monotonically with

time.

From the mathematical point of view, the OPG is a promising candidate for future guidance design. However, for the purpose of guidance implementation, the missile has to have additional requirements to estimate: a) the lateral acceleration of the target and b) the value of time to go. Furthermore, the OPG has to be carried out step by step, based on the real-time estimation of the target's and missile's current state and the prediction of their future behavior. This not only demands excessive computational effort, but also makes the guidance law more complex. This is the most common disadvantage of all existing optimal guidance laws and remains to be improved in the future.

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