Spanwise Optimum Normal Force Distribution of a Hemi-Circular Wing-in-Ground*¹

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Abstract

The aerodynamic characteristics of a hemi-circular wing-in-ground¹⁾ are studied analytically. The exact expression of a spanwise optimum normal force distribution and the minimum induced drag are obtained from Trefftz plane flow field analysis, where Söhngen's inversion integral formula and the elliptic integrals and Jacobian elliptic functions are used. The minimum induced drag obtained in the present study is coincident with that of Mamada and Ando.²⁾

Nomenclature

- A_R : aspect ratio of wing ($=4(Rv)^2/S$)
 C : arbitrary constant of Söhngen's inversion integral formula
 C_L : lift coefficient ($=L/[(1/2)\rho U^2 S]$)
 C_{Di} : induced drag coefficient ($=D_{\min}/[(1/2)\rho U^2 S]$)
 $D, D_{\min}$: induced drag or minimum induced drag
 e : span efficiency factor ($C_{Di}=C_L^2/(\pi A_R e)$)
 K, E : complete elliptic integral of first or second kind, modulus v
 $E(v)$: incomplete elliptic integral of the second kind, modulus v
 KE : kinematic energy of flow in upper half Trefftz plane
 k, k' : modulus or complementary modulus of elliptic integral/function
 L : total lift
 R : radius of hemi-circular wing
 snv, cnv, dnv : Jacobian elliptic function, modulus v
 U : uniform flow speed
 u, v : incomplete elliptic integral of first kind, modulus v ($\xi = vsnu, x = vsnv$)
 W : normal downwash in midspan of wing vortex trace
 w : normal velocity of wing vortex trace
 w_θ : tangential velocity on hemi-circular wing vortex trace
 Γ : bound vortex circulation
 γ : trailing vortex circulation
 ε : gap angle parameter
 θ, θ_1 : angular variable
 $Z(v)$: Jacobian zeta function, modulus v ($=E(v) - (E/K)v$)

- $v = \cos \varepsilon$: modulus of elliptic integral/function
 ρ : air density
 ψ : stream function of Trefftz plane flow field

1. Introduction

The spanwise optimum normal force distribution and the minimum induced drag of a hemi-circular wing-in-ground^{1,2)} are studied by using Trefftz plane flow field analysis. Throughout the present study, the flow is assumed inviscid and incompressible.

The wing-in-ground effect vehicles, such as ram-wings, channel-flow air-cushion vehicles, and winged-hulls have been of interest to engineers,¹⁻⁹⁾ because the induced drag could be drastically reduced by the ground effect. The hemi-circular wing-in-ground is one of the simplest channel-flow air-cushion vehicles, since the minimum induced drag is easily estimated by the analysis.

Ando¹⁾ studied the hemi-circular wing-in-ground analytically, γ - w relation was obtained, and the minimum induced drag and the optimum spanwise lift distribution for the very small tip gap case were studied numerically. Mamada and Ando²⁾ obtained the exact solution of the minimum induced drag of this wing from the analysis by using Söhngen's integral formula.¹⁰⁻¹³⁾

In the present study, Jacobian elliptic functions are introduced, and the exact solutions of a spanwise optimum normal force distribution and the minimum induced drag of this wing are obtained by the analysis.

2. Normal Velocity and Aerodynamic Forces

Consider a Trefftz plane flow field of a hemi-circular wing-in-ground. The wing vortex trace might be the same form in the wing front view. This assumption shall be valid

Fig. 1. Wing vortex trace in Trefftz plane of hemi-circular front view wing-in-ground and some relevant notations.

if the wing lift coefficient is small.

The flow field might be a two-dimensional potential flow, and the ground surface corresponds to one of the stream line. The wing vortex trace has an image vortex, which has the same magnitude but opposite sign (see Fig. 1). The relation between the actual wing vortex trace normal velocity w and the trailing vortex strength γ are examined by using Biot-Savart's law ;

$$dw = \frac{\gamma(\theta_1)d\theta_1}{4\pi} \left(\cot \frac{\theta - \theta_1}{2} - \cot \frac{\theta + \theta_1}{2} \right) \\ = \frac{\gamma(\theta_1)d\theta_1}{2\pi} \frac{\sin \theta_1}{\cos \theta_1 - \cos \theta}$$

the following w - γ relation¹⁾ is found by the integral of the upper relation ;

$$w = \frac{1}{2\pi} \oint_{-v}^v \frac{\gamma(\xi)d\xi}{\xi - x} \quad \dots (1)$$

where $x = \cos \theta$, $\xi = \cos \theta_1$, and $v = \cos \epsilon$. Using Söhnngen's inversion integral formula,¹⁰⁻¹³⁾ we obtain²⁾

$$\sqrt{v^2 - x^2} \gamma(x) = -\frac{2}{\pi} \oint_{-v}^v \frac{\sqrt{v^2 - \xi^2}}{\xi - x} w(\xi) d\xi + C \dots (2)$$

As we consider the symmetrical loading case,

$$w(x) = w(-x), \quad \gamma(x) = -\gamma(-x)$$

then the arbitrary constant C might be $C=0$.

The bound vortex circulation $\Gamma(\theta)$ at a station θ should be equal to the total trailing vortices strength between the point θ and the wing tip ϵ . Then we have

$$\Gamma(\theta) = \Gamma(x) = \int_{\epsilon}^{\theta} \gamma(\theta_1) d\theta_1 = \int_x^v \frac{\gamma(x_1)}{\sqrt{1-x_1^2}} dx_1 \dots (3)$$

From Kutta-Joukowski theorem

$$dL = \rho U \Gamma(\theta) R d\theta \sin \theta = \rho U R \Gamma(x) dx$$

Then, we have

$$L = \rho U R \int_{-v}^v \Gamma(x) dx \quad \dots (4)$$

Induced drag is the kinematic energy¹⁴⁾ of wing vortex trace in the upper half of a Trefftz plane flow field. Then

$$D = KE = \frac{1}{2} \rho \oint_{-v}^v \psi w_{\theta} R d\theta$$

where

$$d\psi = w R d\theta, \quad \gamma = w_{\theta}(\text{upper}) - w_{\theta}(\text{lower})$$

and where ψ is the stream function in Trefftz plane flow field.

$$D = \frac{1}{2} \rho \int_{-v}^v \psi \gamma R d\theta \\ = \frac{1}{2} \rho [\psi \Gamma]_{-v}^v - \frac{1}{2} \rho \int_{-v}^v \frac{d\psi}{R d\theta} \Gamma(\theta) d\theta$$

As $d\psi/(R d\theta) = -w$, $x = R \cos \theta$, the following expression is obtained ;

$$D = \frac{1}{2} \rho R \int_{-v}^v w(x) \Gamma(x) \frac{dx}{\sqrt{1-x^2}} \quad \dots (5)$$

3. Minimum Induced Drag Case

In what follows our formulation will be made, being based on the optimum normal force distribution. Munk's theorem¹⁵⁾ implies that

$$w(x) = W \sqrt{1-x^2} \quad \dots (6)$$

$$UD_{\min} = (W/2)L \quad \dots (7)$$

From Eqs. (2) and (6), we have

$$\gamma(x) = -\frac{2W}{\pi} \oint_{-v}^v \frac{\sqrt{v^2 - \xi^2}}{\sqrt{v^2 - x^2}} \frac{\sqrt{1 - \xi^2}}{\xi - x} d\xi \quad \dots (8)$$

We put $\xi = v \operatorname{sn} u$ and $x = v \operatorname{sn} v$, then

$$\gamma(x) = -\frac{2W}{\pi} \oint_{-K}^K \frac{cn^2 u dn^2 u}{cn v (sn u - sn v)} du \\ = \frac{4W}{\pi} \frac{1}{sn v cn v} \oint_0^K \frac{dn^2 u (1 - sn^2 u)}{1 - (1/sn v)^2 sn^2 u} du \\ = \frac{4W}{\pi} \frac{1}{sn v cn v} \oint_0^K \left(\frac{cn^2 v dn^2 u}{1 - (1/sn v)^2 sn^2 u} + sn^2 u dn^2 u \right) du$$

Ref. 16 314.02 and 415.04 are used to estimate $\gamma(x)$;

$$\gamma(x) = \frac{4W}{\pi} \left(v^2 K sn v cn v - dn v K Z(v) + \frac{sn v}{cn v} E \right) \dots (9)$$

A trailing vortex strength $\gamma(x)$ in the minimum induced

drag case is obtained. In the classical wing theory in which the wing tip vortex separation is not considered like the present study, there exists the following characteristic of the trailing vortex γ :

$$\gamma \rightarrow \pm \infty \quad \text{at} \quad x \rightarrow \pm v$$

3.1. Bound vortex circulation distribution According to Eq. (3), the bound vortex circulation $\Gamma(x)$ is obtained as follows:

$$\Gamma(x) = R \int_x^v \frac{\gamma(x_1)}{\sqrt{1-x_1^2}} dx_1$$

Put $x_1 = v \sin v'$, and from the upper equation and Eq. (9)

$$\begin{aligned} \Gamma(x) &= \int_v^K v \sin v' \gamma(v') dv' \\ &= \frac{4W}{\pi} (Rv) \int_v^K (v^2 K \sin v' \cos^2 v' - \sin v' \cos v' KZ(v') \\ &\quad + E \sin v') dv' \end{aligned}$$

From Ref. 16 360.04

$$\begin{aligned} &K \int_v^K v^2 \sin v' \cos^2 v' dv' \\ &= \frac{K}{2v} (1-v^2) \cosh^{-1} \left(\frac{dv}{\sqrt{1-v^2}} \right) + \frac{1}{2} K \cos v \sin v \end{aligned}$$

From Ref. 16 360.01 and integration by parts

$$\begin{aligned} &-\int_v^K \sin v' \cos v' KZ(v') dv' \\ &= \sin v KZ(v) + K \int_v^K \sin v' \cos^2 v' dv' - E \int_v^K \sin v' dv' \end{aligned}$$

And from Ref. 16 360.06

$$\begin{aligned} &K \int_v^K \sin v' \cos^2 v' dv' \\ &= -\frac{K}{2v} (1-v^2) \cosh^{-1} \left(\frac{dv}{\sqrt{1-v^2}} \right) + \frac{K}{2} \cos v \sin v \end{aligned}$$

From these relations, we obtain a following spanwise optimum normal force distribution expression;

$$\Gamma(x) = \frac{4W(Rv)}{\pi} [K \cos v \sin v + \sin v KZ(v)] \quad \dots (10)$$

And the spanwise optimum normal force distribution shape is

$$\frac{\Gamma(x)}{\Gamma(0)} = \cos v \sin v + \sin v KZ(v) \quad \dots (11)$$

When the tip gap parameter ε reaches $\pi/2$, the approximate solution of the spanwise optimum normal force distribution shape is

Fig. 2. Variation of spanwise optimum normal force distribution with tip gap parameter $\sin \varepsilon$.

$$\left. \frac{\Gamma(v)}{\Gamma(0)} \right|_{\varepsilon \rightarrow \pi/2} = \sqrt{1-x^2} + \frac{v^4}{16} x^2 \sqrt{1-x^2} + O(v^5) \quad \dots (12)$$

where

$$\varepsilon \rightarrow \pi/2 \quad \text{then} \quad v = \cos \varepsilon \rightarrow 0$$

and where Ref. 16 902.00, 903.00 and etc. are used.

If the tip gap parameter ε reaches 0, the approximate solution of the spanwise optimum normal force distribution shape is

$$\left. \frac{\Gamma(v)}{\Gamma(0)} \right|_{\varepsilon \rightarrow 0} = 1 - \frac{1}{2} \frac{x}{\ln(4/\sin \varepsilon)} \ln \frac{1+x}{1-x} + O(\varepsilon) \quad \dots (13)$$

where Ref. 16 141.60 and etc. are used.

See Fig. 2, which shows the relation of the spanwise optimum normal force distribution shape of hemi-circular wing-in-ground to the gap parameter $\sin \varepsilon$.

3.2. Total lift and minimum induced drag The total lift is

$$L = \rho U R \int_{-v}^v \Gamma(x) dx$$

Then the lift coefficient C_L and the span efficiency factor e are

$$\begin{aligned} C_L &= \frac{L}{(1/2)\rho U^2 S} = \frac{2R}{US} \int_{-v}^v \Gamma(x) dx \\ e &= \frac{C_L^2}{\pi A_R C_{Di}} = \frac{R}{\pi W(Rv)^2} \int_{-v}^v \Gamma(x) dx \end{aligned}$$

From Eq. (10), Ref. 16 361.03, 360.11, 314.04, and 314.02 the following expressions are obtained

$$C_L = \frac{2}{\pi} \frac{W A_R}{U} \left[K^2 - \frac{1}{v^2} (K-E)^2 \right] \quad \dots (14)$$

$$e = \frac{4}{\pi^2} \left[K^2 - \frac{1}{v^2} (K-E)^2 \right] \quad \dots (15)$$

Fig. 3. Span-efficiency-factor vs. tip gap parameter.

where K or E is the complete elliptic integral of the first or the second kind where modulus is v . These results are coincident with those of Mamada and Ando.²⁾

When the gap parameter ε reaches $\pi/2$, the approximate solution of C_L and e are as follows :

$$C_L|_{\varepsilon \rightarrow \pi/2} = \frac{\pi}{2} \frac{W}{U} A_R \left[1 + \frac{1}{4} v^2 + \frac{5}{32} v^4 + O(v^5) \right] \dots (16)$$

$$e|_{\varepsilon \rightarrow \pi/2} = 1 + \frac{1}{4} v^2 + \frac{5}{32} v^4 + O(v^5) \dots (17)$$

where Ref. 16 900.00 and 900.07 are used.

If the gap parameter $\sin \varepsilon$ reaches 0, C_L and e are

$$C_L|_{\varepsilon \rightarrow 0} = \frac{4}{\pi} \frac{W}{U} A_R \left[\ln \left(\frac{1}{\sin \varepsilon} \right) + 2 \ln 2 - 0.5 + O\{\varepsilon \ln(1/\varepsilon)\} \right] \dots (18)$$

$$e|_{\varepsilon \rightarrow 0} = \frac{8}{\pi^2} \left[\ln \left(\frac{1}{\sin \varepsilon} \right) + 2 \ln 2 - 0.5 + O\{\varepsilon \ln(1/\varepsilon)\} \right] \dots (19)$$

where Ref. 16 900.05 and 900.10 are used.

The relation of the span efficiency factor e to $\sin \varepsilon$ is examined by the digital computer and shown in Fig. 3.

4. Conclusions

The closed forms of both the spanwise optimum normal force distribution and the minimum induced drag of the hemi-circular wing-in-ground are obtained from Trefftz plane flow field analysis.

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Appendix The Note of Three Formulas Involving the Term $\cosh^{-1}(dnu/k')$ in Ref. 16

Ref. 16 310.01, 360.04 and 360.06 are as follows ;

$$310.01 \quad \int snudu = -\frac{1}{k} \cosh^{-1} \left(\frac{dnu}{k'} \right) \dots (A.1)'$$

$$360.04 \quad \int snucn^2udu = -\frac{1}{2k^3} \left[k'^2 \cosh^{-1} \left(\frac{dnu}{k'} \right) + kc nudnu \right] \dots (A.2)$$

$$360.06 \quad \int dn^2 usnudu = \frac{1}{2k} \left[k'^2 \cosh^{-1} \left(\frac{dnu}{k'} \right) - kc nudnu \right] \dots (A.3)$$

where k and k' are the modulus and the complementary modulus of Jacobian elliptic functions. And where there exists the following relations;

$$k^2 \operatorname{sn} u \operatorname{cn}^2 u - \operatorname{dn}^2 u \operatorname{sn} u = -k'^2 \operatorname{sn} u \quad \dots (A.4)$$

Integral the left hand side of Eq. (A.4) using Eqs. (A.2) and (A.3), then

$$k^2 \int \operatorname{sn} u \operatorname{cn}^2 u du - \int \operatorname{dn}^2 u \operatorname{sn} u du = -k'^2 \frac{1}{k} \cosh^{-1} \left(\frac{\operatorname{dn} u}{k'} \right)$$

From the upper relation and Eq. (A.4), the following expression is obtained:

$$\int \operatorname{sn} u du = \frac{1}{k} \cosh^{-1} \left(\frac{\operatorname{dn} u}{k'} \right) \quad \dots (A.1)$$

Comparing Eq. (A.1)' with Eq. (A.1), the sign is opposite from each other.

The derivation of $\cosh^{-1}(\operatorname{dn} u/k')$ with u is

$$\frac{d}{du} \cosh^{-1} \left(\frac{\operatorname{dn} u}{k'} \right) = \mp k \operatorname{sn} u \quad \dots (A.5)$$

where the double sign $\mp$ shows that

or $-$ sign is the case of the positive branch of

$+$ sign is the case of the negative branch of

the arccosinehyperbolic function that is the two-valued function.

When the following equations are introduced

$$\int \operatorname{sn} u \operatorname{cn}^2 u du = -\frac{1}{2k^3} \left[-k^2 \cosh^{-1} \left(\frac{\operatorname{dn} u}{k'} \right) + k \operatorname{cn} u \operatorname{dn} u \right] \quad \dots (A.2)'$$

$$\int \operatorname{dn}^2 u \operatorname{sn} u du = \frac{1}{2k} \left[-k'^2 \cosh^{-1} \left(\frac{\operatorname{dn} u}{k'} \right) - k \operatorname{cn} u \operatorname{dn} u \right] \quad \dots (A.3)'$$

then a consistent formulas set is as follows:

$$\text{or } \left. \begin{array}{l} \text{Eqs. (A.1)', (A.2)', and (A.3)' set} \\ \text{Eqs. (A.1), (A.2), and (A.3) set} \end{array} \right\} \quad \dots (A.6)$$

Take care as mentioned above to use the formulas of Ref. 16 involving $\cosh^{-1}(\operatorname{dn} u/k')$.